6

Gorenstein Algebras, Approximations, Serre Duality

Contents

This chapter discusses the homological theory of modules over Gorenstein rings. A characteristic feature is the decomposition of the module category into two orthogonal subcategories: the Gorenstein projective (or maximal Cohen–Macaulay) modules and the modules of finite projective dimension. These subcategories are glued together via certain approximation sequences. The orthogonality refers to $\operatorname{Ext}^n(-,-)$ for $n > 0$ and this leads to the notion of a cotorsion pair. The stable category of Gorenstein projective modules admits

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a natural triangulated structure and is triangle equivalent to the singularity category, which is obtained from the derived category by forming the quotient modulo the subcategory of perfect complexes.

In the second part we focus on Artin algebras and study Serre functors for the stable category of Gorenstein projective modules and the category of perfect complexes.

6.1 Approximations

We establish the existence of approximations in exact categories. To formulate these results we use the concept of a cotorsion pair. Later on we will take up cotorsion pairs in the context of tilting.

Cotorsion Pairs

Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a class of objects. The right and left *perpendicular categories* are the full subcategories

$$^{\perp}\mathcal{C} = \{X \in \mathcal{A} \mid \operatorname{Ext}^n(X,Y) = 0 \text{ for all } Y \in \mathcal{C},\ n > 0\}$$

and

$$\mathcal{C}^{\perp} = \{Y \in \mathcal{A} \mid \operatorname{Ext}^n(X,Y) = 0 \text{ for all } X \in \mathcal{C},\ n > 0\}.$$

Let $\mathcal{A}$ be an exact category and $\mathcal{X}, \mathcal{Y}$ full subcategories of $\mathcal{A}$. Then $(\mathcal{X}, \mathcal{Y})$ is a (hereditary and complete) *cotorsion pair* for $\mathcal{A}$ if

$$\mathcal{X}^{\perp} = \mathcal{Y} \qquad \text{and} \qquad \mathcal{X} = {}^{\perp}\mathcal{Y}$$

and every object $A \in \mathcal{A}$ fits into admissible exact sequences

$$0 \longrightarrow Y_A \longrightarrow X_A \longrightarrow A \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow A \longrightarrow Y^A \longrightarrow X^A \longrightarrow 0 \tag{6.1.1}$$

with $X_A, X^A \in \mathcal{X}$ and $Y_A, Y^A \in \mathcal{Y}$.

The sequences (6.1.1) are called *approximation sequences*, because every morphism $X \to A$ with $X \in \mathcal{X}$ factors through $X_A \to A$ and every morphism $A \to Y$ with $Y \in \mathcal{Y}$ factors through $A \to Y^A$. One may think of a cotorsion pair as a *decomposition* of the ambient category.

Remark 6.1.2. Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair for $\mathcal{A}$ and set $\mathcal{C} = \mathcal{X} \cap \mathcal{Y}$. We write $\mathcal{A}/\mathcal{C}$ for the additive quotient category which is obtained from $\mathcal{A}$ by annihilating all morphisms that factor through an object in $\mathcal{C}$.

(1) We have $X_A \in \mathcal{C}$ if $A \in \mathcal{Y}$, and $Y^A \in \mathcal{C}$ if $A \in \mathcal{X}$. In particular, any morphism from $\mathcal{X}$ to $\mathcal{Y}$ factors through an object in $\mathcal{C}$.

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(2) The exact sequences in (6.1.1) are uniquely determined up to isomorphism in the quotient category $\mathcal{A}/\mathcal{C}$. In fact, the assignment $A \mapsto X_A$ gives a right adjoint of the inclusion $\mathcal{X}/\mathcal{C} \to \mathcal{A}/\mathcal{C}$, while the assignment $A \mapsto Y^A$ gives a left adjoint of the inclusion $\mathcal{Y}/\mathcal{C} \to \mathcal{A}/\mathcal{C}$.

A Decomposition via Resolutions

Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive subcategory. A *finite* $\mathcal{C}$*-resolution* of an object A in $\mathcal{A}$ is an admissible exact sequence (that is, an acyclic complex)

$$0 \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

such that $X_i \in \mathcal{C}$ for all i. We write $\operatorname{Res}(\mathcal{C})$ for the full subcategory of objects in $\mathcal{A}$ that admit a finite $\mathcal{C}$-resolution.

The following theorem establishes a decomposition for exact categories; it yields a procedure for constructing cotorsion pairs and is the basis for the existence of approximations.

Theorem 6.1.3. *Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive subcategory. Set $\mathcal{X} = {}^{\perp}\mathcal{C}$ and let $\mathcal{Y}$ be the closure under direct summands of* $\operatorname{Res}(\mathcal{C})$. *Suppose that $\mathcal{A} = \operatorname{Res}(\mathcal{X})$ and that $\mathcal{C}$ cogenerates $\mathcal{X}$, that is, every object $X \in \mathcal{X}$ fits into an admissible exact sequence $0 \to X \to Y \to Z \to 0$ with $Y \in \mathcal{C}$ and $Z \in \mathcal{X}$. Then $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair for $\mathcal{A}$.*

Proof Let $A \in \mathcal{A}$ and choose an admissible exact sequence

$$0 \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

with $X_i \in \mathcal{X}$ for all i. We need to construct the sequences (6.1.1) and use induction on r. The case $r = 0$ is clear. Now suppose $r > 0$ and let B denote the image of $X_1 \to X_0$ given by an admissible monomorphism $B \to X_0$. By the inductive hypothesis there is an exact sequence $0 \to B \to Y^B \to X^B \to 0$

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with $X^B \in \mathcal{X}$ and $Y^B \in \mathcal{Y}$. We form the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y^B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & X^B & = & X^B & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and obtain an exact sequence $0 \to Y^B \to X \to A \to 0$ with $X \in \mathcal{X}$ and $Y^B \in \mathcal{Y}$. This gives the first approximation sequence. Now take this sequence and complete the admissible monomorphism $Y^B \to X \to C$. This yields the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^B & \longrightarrow & C & \longrightarrow & Y & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X' & = & X' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and the sequence $0 \to A \to Y \to X' \to 0$ has $X' \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Thus we have constructed the second approximation sequence.

It remains to show that $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{X}^{\perp} = \mathcal{Y}$. The first equality is clear since

$$\mathcal{X} = {}^{\perp}\mathcal{C} = {}^{\perp}\operatorname{Res}(\mathcal{C}).$$

Also, the inclusion $\mathcal{X}^{\perp} \supseteq \mathcal{Y}$ is clear, since $\mathcal{X}^{\perp} \supseteq \mathcal{C}$. For the other inclusion, let $A \in \mathcal{X}^{\perp}$ and consider the sequence $0 \to A \to Y^A \to X^A \to 0$ which splits. Thus $A \in \mathcal{Y}$. □

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6.2 Gorenstein Rings

Let Λ be a ring and suppose that Λ is two-sided noetherian. The ring Λ is called *Gorenstein* (or sometimes *Iwanaga-Gorenstein*) if the injective dimension of Λ is finite as a left and as a right module over itself. In that case one can show that both dimensions coincide. We denote this dimension by d and say Λ is Gorenstein *of dimension* d.

Gorenstein Projective Modules

We begin our discussion with the lemma that justifies the definition of the dimension of a Gorenstein ring. In fact, this numerical invariant admits other descriptions involving weak dimensions.

The *weak dimension* (or *flat dimension*) of a Λ-module X is by definition

$$\operatorname{w.dim} X = \inf\{n \geq 0 \mid \operatorname{Tor}^{\Lambda}_{n+1}(X, -) = 0\}.$$

Lemma 6.2.1. *Let Λ be a two-sided noetherian ring. If* $\operatorname{inj.dim}(\Lambda_\Lambda)$ *and* $\operatorname{inj.dim}({}_\Lambda\Lambda)$ *are both finite, then they coincide.*

Proof Given a finitely generated Λ-module X and an injective Λ^{op}-module I, we have a natural isomorphism

$$\operatorname{Hom}_\Lambda(\operatorname{Ext}^i_\Lambda(X, \Lambda), I) \cong \operatorname{Tor}^{\Lambda}_i(X, I) \qquad (i \geq 0).$$

Thus

$$\operatorname{inj.dim}(\Lambda_\Lambda) = \sup\{\operatorname{w.dim}({}_\Lambda I) \mid {}_\Lambda I \text{ injective}\}.$$

Given a Λ-module X of finite weak dimension, one can test the vanishing of $\operatorname{Tor}^{\Lambda}_{n+1}(X, -)$ on injective modules, since any module embeds into an injective module. Thus

$$\sup\{\operatorname{w.dim}(X_\Lambda) \mid \operatorname{w.dim}(X_\Lambda) < \infty\} \leq \operatorname{inj.dim}(\Lambda_\Lambda)$$

and

$$\operatorname{inj.dim}(\Lambda_\Lambda) \leq \sup\{\operatorname{w.dim}({}_\Lambda Y) \mid \operatorname{w.dim}({}_\Lambda Y) < \infty\}.$$

This symmetry implies $\operatorname{inj.dim}(\Lambda_\Lambda) = \operatorname{inj.dim}({}_\Lambda\Lambda)$. □

Now suppose that the ring Λ is Gorenstein. A Λ-module X is called *Gorenstein projective* (or *maximal Cohen–Macaulay*) if $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for all $i \neq 0$. We set

$$\operatorname{Gproj} \Lambda = \{X \in \operatorname{mod} \Lambda \mid X \text{ is Gorenstein projective}\}.$$

Fix a finitely presented Λ-module X and a projective resolution

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow X \longrightarrow 0$$

with all P_i finitely generated. We set $X^* = \operatorname{Hom}_\Lambda(X, \Lambda)$ and for $n \geq 1$ let $\Omega^n X = \operatorname{Im} d^n$ denote the nth *syzygy* of X.

Lemma 6.2.2. *Let Λ be a Gorenstein ring of dimension d and X a finitely presented Λ-module. Then the following holds.*

(1) *The module $\Omega^n X$ is Gorenstein projective for all $n \geq d$. In particular,*

$$\operatorname{proj.dim} X < \infty \quad \iff \quad \operatorname{proj.dim} X \leq d.$$

(2) *If X is Gorenstein projective, then $\Omega^n X$ is Gorenstein projective for all $n \geq 1$.*

(3) *If X is Gorenstein projective, then the sequence*

$$0 \longrightarrow X^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow P_2^* \longrightarrow \cdots$$

is exact and X^* is Gorenstein projective. Moreover, $X \xrightarrow{\sim} X^{**}$.

(4) *The functor* $\operatorname{Hom}_\Lambda(-, \Lambda)$ *induces an exact duality*

$$(\operatorname{Gproj} \Lambda)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Gproj}(\Lambda^{\mathrm{op}}).$$

Proof We apply the dimension shift formula

$$\operatorname{Ext}^p_\Lambda(\Omega^q X, -) \cong \operatorname{Ext}^{p+q}_\Lambda(X, -) \qquad (p, q \geq 1).$$

Then (1) and (2) are clear. From this we obtain the exactness of

$$0 \longrightarrow X^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots$$

and therefore X^* is a syzygy of arbitrarily high order. Thus X^* is Gorenstein projective by (1). Applying $\operatorname{Hom}_\Lambda(-, \Lambda)$ to this coresolution of X^* gives a resolution of X^{**}, and we have $X \xrightarrow{\sim} X^{**}$ since $P_i \xrightarrow{\sim} P_i^{**}$ for all i. This completes (3) and the assertion in (4) is then a consequence. □

We are now able to give another description of Gorenstein projective modules, which is usually taken as the definition.

Call a complex X in some additive category $\mathcal{A}$ *totally acyclic* if the complexes of abelian groups $\operatorname{Hom}_{\mathcal{A}}(A, X)$ and $\operatorname{Hom}_{\mathcal{A}}(A, X)$ are both acyclic for each object $A \in \mathcal{A}$.

Lemma 6.2.3. *Let Λ be a Gorenstein ring of dimension d. Then a finitely presented Λ-module X is Gorenstein projective if and only if*

$$X \cong \operatorname{Coker}(P_1 \to P_0)$$

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for some totally acyclic complex P of finitely generated projective Λ-modules.

Proof Fix a complex of projective Λ-modules

$$P\colon \qquad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow \cdots$$

and set

$$C_n = \operatorname{Coker}(P_{n+1} \to P_n) \qquad (n \in \mathbb{Z}).$$

We claim that P is totally acyclic when P is acyclic. This is clear since

$$H^n \operatorname{Hom}_\Lambda(P, \Lambda) \cong \operatorname{Ext}^n_\Lambda(C_0, \Lambda) \qquad (n > 0)$$

implies

$$H^n \operatorname{Hom}_\Lambda(P, \Lambda) \cong \operatorname{Ext}^{d+1}_\Lambda(C_{n-d-1}, \Lambda) = 0 \qquad (n \in \mathbb{Z}).$$

If X is Gorenstein projective, then we choose a projective resolution P of X and a projective resolution Q of X^*. Applying Lemma 6.2.2, we have $X^{**} \cong X$ and can splice together P and Q^* giving an acyclic complex

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q_0^* \longrightarrow Q_1^* \longrightarrow Q_2^* \longrightarrow \cdots$$

with cokernel of $P_1 \to P_0$ isomorphic to X. Conversely, if $X = \operatorname{Coker}(P_1 \to P_0)$ for some totally acyclic complex P in $\operatorname{proj} \Lambda$, then

$$\operatorname{Ext}^n_\Lambda(X, \Lambda) = H^n \operatorname{Hom}_\Lambda(P, \Lambda) = 0 \qquad (n > 0)$$

and it follows that X is Gorenstein projective. □

Gorenstein Approximations

For Gorenstein rings there is a good approximation theory. The category of finitely presented modules decomposes into two orthogonal subcategories which are glued together via approximation sequences.

Theorem 6.2.4. *Let Λ be a Gorenstein ring. Set $\mathcal{X} = \operatorname{Gproj} \Lambda$ and write $\mathcal{Y}$ for the category of finitely presented Λ-modules of finite projective dimension. Then $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair for* $\operatorname{mod} \Lambda$ *with* $\mathcal{X} \cap \mathcal{Y} = \operatorname{proj} \Lambda$.

Proof We apply Theorem 6.1.3. Thus we set $\mathcal{A} = \operatorname{mod} \Lambda$ and $\mathcal{C} = \operatorname{proj} \Lambda$. This gives $\mathcal{X} = {}^\perp\mathcal{C}$ and $\mathcal{Y} = \operatorname{Res}(\mathcal{C})$. The assumption on Λ implies that $\mathcal{A} = \operatorname{Res}(\mathcal{X})$ and that $\mathcal{C}$ cogenerates $\mathcal{X}$; this follows from Lemma 6.2.2. More precisely, if Λ is Gorenstein of dimension d, then any Λ-module X admits a resolution

$$0 \longrightarrow \Omega^d X \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

such that $P_0, \ldots, P_{d-1}$ are projective Λ-modules. Thus $X \in \operatorname{Res}(\mathcal{X})$, since

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$\Omega^d X$ is Gorenstein projective. If X is Gorenstein projective, choose an exact sequence $0 \to Y \to P \to X^* \to 0$ in $\operatorname{mod} \Lambda^{\mathrm{op}}$ such that P is projective. This yields an exact sequence $0 \to X \to P^* \to Y^* \to 0$ in $\operatorname{Gproj} \Lambda$, since $X \xrightarrow{\sim} X^{**}$.

It remains to show that $\mathcal{X} \cap \mathcal{Y} = \operatorname{proj} \Lambda$. One inclusion is obvious. Thus consider a module X that is Gorenstein projective and of finite projective dimension. Then an induction on the projective dimension of X shows that X is projective, keeping in mind that $\Omega^n X$ is Gorenstein projective for all $n \geq 1$. □

The Stable Category

For a noetherian ring Λ we consider the derived category $\mathbf{D}^b(\operatorname{mod} \Lambda)$ and obtain the *singularity category* (or *stabilised derived category*) by forming the triangulated quotient

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) = \frac{\mathbf{D}^b(\operatorname{mod} \Lambda)}{\mathbf{D}^b(\operatorname{proj} \Lambda)}.$$

Also, we consider the triangulated category $\mathbf{K}_{\mathrm{ac}}(\operatorname{proj} \Lambda)$ of complexes of finitely generated projective Λ-modules that are acyclic.

An exact category $\mathcal{A}$ is a *Frobenius category* if $\mathcal{A}$ has enough projective objects and enough injective objects, and if projective and injective objects in $\mathcal{A}$ coincide. The *stable category* of $\mathcal{A}$ is obtained by annihilating all morphisms that factor through a projective object. The exact structure of $\mathcal{A}$ induces a triangulated structure for the stable category.

Theorem 6.2.5. *Let Λ be Gorenstein. Then the Gorenstein projective Λ-modules form a Frobenius category. Writing $\underline{\operatorname{Gproj}} \Lambda$ for its stable category, we have a triangle equivalence*

$$Z^0 \colon \mathbf{K}_{\mathrm{ac}}(\operatorname{proj} \Lambda) \xrightarrow{\sim} \underline{\operatorname{Gproj}} \Lambda.$$

Also, the composite

$$F \colon \operatorname{Gproj} \Lambda \rightarrowtail \mathbf{D}^b(\operatorname{mod} \Lambda) \twoheadrightarrow \mathbf{D}_{\mathrm{sg}}(\Lambda)$$

induces a triangle equivalence

$$\underline{\operatorname{Gproj}} \Lambda \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(\Lambda).$$

Proof It follows from Lemma 6.2.2 that $\operatorname{Gproj} \Lambda$ is a Frobenius category. The projective Λ-modules form the subcategory of objects that are projective and injective. Thus the first triangle equivalence follows from Proposition 4.4.18.

The functor F is exact: it takes an exact sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{Gproj} \Lambda$ to an exact triangle $F(X) \to F(Y) \to F(Z) \to F(X)[1]$. Also,

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F annihilates all projective Λ-modules and yields therefore an exact functor $\bar{F}\colon \underline{\mathrm{Gproj}}\,\Lambda \to \mathbf{D}_{\mathrm{sg}}(\Lambda)$. The suspension in $\underline{\mathrm{Gproj}}\,\Lambda$ takes X to $\Omega^{-1}X$, and

$$F(\Omega^{-1}X) \cong F(X)[1].$$

We construct a quasi-inverse for $\bar{F}$ as follows.

Consider the category of complexes $\mathbf{K}(\mathrm{proj}\,\Lambda)$ of finitely generated projective Λ-modules up to homotopy. We identify the subcategories

$$\mathbf{K}^b(\mathrm{proj}\,\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{proj}\,\Lambda) \quad \text{and} \quad \mathbf{K}^{-,b}(\mathrm{proj}\,\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{mod}\,\Lambda).$$

For a complex X and $n \in \mathbb{Z}$ we use the following truncation:

$$\begin{array}{ccccccccccc}
X & & \cdots \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow \cdots \\
\downarrow & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & \\
\sigma_{\le n}X & & \cdots \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

Now fix a complex X in $\mathbf{K}^{-,b}(\mathrm{proj}\,\Lambda)$ and choose $n \in \mathbb{Z}$ such that $H^i(X) = 0$ for all $i \le n + d$. Then $\mathrm{Coker}(X^{n-1} \to X^n)$ is Gorenstein projective, by Lemma 6.2.2. Note that the cone of $X \to \sigma_{\le n}X$ belongs to $\mathbf{K}^b(\mathrm{proj}\,\Lambda)$. Thus $X \cong \sigma_{\le n}X$ in $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ and the assignment

$$X \longmapsto \Omega^n\,\mathrm{Coker}(X^{n-1} \to X^n)$$

yields a functor $G\colon \mathbf{D}_{\mathrm{sg}}(\Lambda) \to \underline{\mathrm{Gproj}}\,\Lambda$ which does not depend on n. It is not difficult to check that $G \circ \bar{F} \cong \mathrm{id}$ and $\bar{F} \circ G \cong \mathrm{id}$. □

We observe that the stable category $\underline{\mathrm{Gproj}}\,\Lambda$ and the equivalent singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ are idempotent complete when the Gorenstein projective modules form a Krull–Schmidt category. For instance, this holds when Λ is an Artin algebra. For an example when $\underline{\mathrm{Gproj}}\,\Lambda$ is not idempotent complete, see Lemma 6.2.12.

Examples of Gorenstein Rings

Gorenstein rings are ubiquitous and we provide several examples.

Commutative Rings

Let Λ be a commutative noetherian ring. In that context one uses a local property and calls Λ *Gorenstein* if for each prime ideal $\mathfrak{p}$ the localisation $\Lambda_{\mathfrak{p}}$ has finite injective dimension as a module over $\Lambda_{\mathfrak{p}}$. Clearly, this definition coincides with our original definition when Λ is local. Important examples are *hypersurface rings* and more generally *complete intersection rings*.

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Non-commutative Rings

Let Λ be a (not necessarily commutative) two-sided noetherian ring. We begin with two extreme cases, where Gproj Λ is either all or nothing.

The ring Λ is Gorenstein of dimension zero if and only if it is a *quasi-Frobenius ring*. Then Λ is a two-sided artinian ring and projective and injective Λ-modules coincide. Clearly, in that case all Λ-modules are Gorenstein projective.

If Λ has finite global dimension, say d, then Λ is Gorenstein of dimension d. In that case only the projective Λ-modules are Gorenstein projective.

Let G be a finite group. Then the *integral group algebra* $\mathbb{Z}G$ is Gorenstein of dimension one. A $\mathbb{Z}G$-module is Gorenstein projective if and only if the underlying $\mathbb{Z}$-module is Gorenstein projective if and only if the underlying $\mathbb{Z}$-module is projective.

Further interesting examples of Gorenstein rings arise from the study of graded rings.

Artin Algebras

We fix a field k and discuss two specific constructions of Artin k-algebras that are Gorenstein.

For a Gorenstein ring Λ let $\operatorname{Gor.dim}\Lambda$ denote its dimension. Note that $\operatorname{Gor.dim}\Lambda$ equals the global dimension of Λ when both dimensions are finite.

Proposition 6.2.6. *Let Γ and Λ be finite dimensional k-algebras. If Γ and Λ are Gorenstein, then the tensor product $\Gamma \otimes_k \Lambda$ is Gorenstein and*

$$\operatorname{Gor.dim}\Gamma \otimes_k \Lambda = \operatorname{Gor.dim}\Gamma + \operatorname{Gor.dim}\Lambda.$$

We need some preparation. Given chain complexes of k-modules X and Y, we consider the tensor product $X \otimes_k Y$ given by $(X \otimes_k Y)_n = \bigoplus_{i+j=n} X_i \otimes_k Y_j$ with differential $\partial(x \otimes y) = \partial x \otimes y + (-1)^{|x|} x \otimes \partial y$, where x and y are any homogeneous elements in X and Y respectively, and $|x|$ denotes the degree of x.

Lemma 6.2.7. *Let Γ and Λ be finite dimensional k-algebras. If P and Q are minimal projective resolutions of modules ${}_\Gamma M$ and N_Λ, then the tensor product $P \otimes_k Q$ is a minimal projective resolution of the $\Gamma \otimes_k \Lambda$-module $M \otimes_k N$.*

Proof The Künneth formula implies that $P \otimes_k Q$ is a projective resolution of $M \otimes_k N$; see [46, Theorem VI.3.1]. Next observe that $J(\Gamma) \otimes_k \Lambda + \Gamma \otimes_k J(\Lambda)$ is a nilpotent two-sided ideal in $\Gamma \otimes_k \Lambda$, and therefore it is contained in $J(\Gamma \otimes_k \Lambda)$.

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For any modules ${}_\Gamma X$ and Y_Λ, we have $\operatorname{rad}(X) = J(\Gamma)X$ and $\operatorname{rad}(Y) = YJ(\Lambda)$. Thus

$$\operatorname{rad}(X) \otimes_k Y + X \otimes_k \operatorname{rad}(Y) \subseteq \operatorname{rad}(X \otimes_k Y)$$

as $\Gamma \otimes_k \Lambda$-module. The lemma now follows from the fact that a projective resolution is minimal if and only if the image of each differential lands in the radical of the next module (Lemma 2.1.21). □

Proof of Proposition 6.2.6 Set $c = \operatorname{Gor.dim}\Gamma$ and $d = \operatorname{Gor.dim}\Lambda$. We have

$$\operatorname{proj.dim}({}_\Lambda D(\Lambda)) = \operatorname{inj.dim}(\Lambda_\Lambda) \quad \text{and} \quad \operatorname{proj.dim}(D(\Gamma)_\Gamma) = \operatorname{inj.dim}({}_\Gamma\Gamma).$$

Thus the assumptions yield minimal projective resolutions

$$\cdots \longrightarrow 0 \longrightarrow P_d \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_\Lambda D(\Lambda) \longrightarrow 0$$

and

$$\cdots \longrightarrow 0 \longrightarrow Q_c \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow D(\Gamma)_\Gamma \longrightarrow 0.$$

The tensor product $P \otimes_k Q$ yields a minimal projective resolution of $D(\Gamma \otimes_k \Lambda) \cong D(\Lambda) \otimes_k D(\Gamma)$ over $\Lambda \otimes_k \Gamma$ by the above lemma. This gives

$$\operatorname{inj.dim}({}_\Gamma(\Gamma \otimes_k \Lambda)_\Lambda) = \operatorname{proj.dim}({}_\Lambda D(\Gamma \otimes_k \Lambda)_\Gamma) = c + d.$$

The same computation for $(\Gamma \otimes_k \Lambda)^{\mathrm{op}}$ then shows that $\Gamma \otimes_k \Lambda$ has dimension $c + d$. □

Of particular interest is the case when Γ is the algebra $k[\varepsilon]$ of dual numbers. We have $k[\varepsilon] \otimes_k \Lambda \cong \Lambda[\varepsilon]$, and the $\Lambda[\varepsilon]$-modules identify with *differential modules* over Λ, that is, pairs (X, d) consisting of a Λ-module X with an endomorphism $d \colon X \to X$ satisfying $d^2 = 0$.

Gentle Algebras

Let k be a field. A k-algebra is called *gentle* if it is Morita equivalent to an algebra of the form kQ/I where $Q = (Q_0, Q_1, s, t)$ is a finite quiver and I is an ideal generated by paths of length two, subject to the following conditions.

(Ge1) For each $x \in Q_0$, there are at most two arrows starting at x.
(Ge2) For each $x \in Q_0$, there are at most two arrows ending at x.
(Ge3) For each $\alpha \in Q_1$, there is at most one arrow β such that $s(\beta) = t(\alpha)$ and $\beta\alpha \in I$, and there is at most one arrow β' such that $s(\beta') = t(\alpha)$ and $\beta'\alpha \notin I$.
(Ge4) For each $\beta \in Q_1$, there is at most one arrow α such that $t(\alpha) = s(\beta)$ and $\beta\alpha \in I$, and there is at most one arrow α' such that $t(\alpha') = s(\beta)$ and $\beta\alpha' \notin I$.

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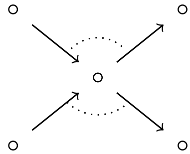
The following diagram shows the local shape of (Q, I) when kQ/I is gentle:



Now fix a pair (Q, I) such that the algebra $\Lambda = kQ/I$ satisfies the above conditions (Ge1)–(Ge4). A non-trivial path α in Q is a *primitive cycle* if $s(\alpha) = t(\alpha)$, $\alpha^r \notin I$ for all $r > 0$, and α is not a power of a cycle of smaller length. For $x \in Q_0$ let $c_x \in \Lambda$ denote the sum of all primitive cycles α with $t(\alpha) = x$. Note that there are at most two primitive cycles ending at x. If $\alpha \neq \beta$ are such cycles, then $\alpha\beta = 0 = \beta\alpha$ in Λ. Moreover, for any arrow $\alpha \colon x \to y$ we have $c_y\alpha = \alpha c_x$.

Let $k[c]$ denote the polynomial ring in one indeterminate c. Then the assignment $c \mapsto \sum_{x\in Q_0} c_x$ yields a $k[c]$-algebra structure for Λ.

Lemma 6.2.8. *The gentle algebra $\Lambda = kQ/I$ is a noetherian $k[c]$-algebra.*

Proof For each pair $x, y \in Q_0$ we consider the non-trivial paths in Q that generate $e_x\Lambda e_y$, where e_x and e_y denote the idempotents corresponding to x and y respectively. The conditions for a gentle algebra imply that all are of the form $\alpha^r\beta$ for some non-trivial paths α, β and some $r \geq 0$. If there are infinitely many such paths, then α is a primitive cycle, so $\alpha^r\beta = c_x^r\beta$ in Λ. Thus the $k[c]$-module $e_x\Lambda e_y$ is finitely generated. □

Proposition 6.2.9. *A gentle algebra is Gorenstein.*

We fix a gentle algebra $\Lambda = kQ/I$. A possibly infinite path $\alpha_1\alpha_2\alpha_3\cdots$ in Q is called *differential* if $\alpha_i\alpha_{i+1} \in I$ for all $i \geq 1$. Such a path is *maximal* if it has finite length, say n, and cannot be extended to a differential path of length $n+1$. Note that there are only finitely many maximal differential paths in Q.

The proof of the proposition uses the following reduction argument.

Lemma 6.2.10. *Let Λ be a noetherian $k[c]$-algebra and Y a finitely generated Λ-module. Then* $\operatorname{inj.dim} Y \leq d$ *if* $\operatorname{Ext}^n_\Lambda(S, Y) = 0$ *for every simple Λ-module S and $n > d$.*

Proof Baer's criterion implies that it suffices to show $\operatorname{Ext}^n_\Lambda(X, Y) = 0$ for every finitely generated Λ-module X and $n > d$. Let X be finitely generated and set $t(X) = \{x \in X \mid xc^p = 0 \text{ for } p \gg 0\}$. Then $t(X)$ has finite length, and therefore it suffices to show that $\operatorname{Ext}^n_\Lambda(\bar{X}, Y) = 0$ for $\bar{X} = X/t(X)$ and every $n > d$. The exact sequence $0 \to \bar{X} \xrightarrow{c} \bar{X} \to \bar{X}/\bar{X}c \to 0$ induces a bijection

$\mathrm{Ext}^n_\Lambda(\bar{X}, Y) \xrightarrow{c} \mathrm{Ext}^n_\Lambda(\bar{X}, Y)$ for $n > d$. It follows that $\mathrm{Ext}^n_\Lambda(\bar{X}, Y) = 0$ since Y is finitely generated. □

Proof of Proposition 6.2.9 We wish to apply Lemma 6.2.10 and consider a simple Λ-module S. Then $\mathrm{Hom}_\Lambda(P_x, S) \neq 0$ for some $x \in Q_0$, where P_x denotes the indecomposable projective Λ-module corresponding to $x \in Q_0$, with k-basis given by all paths in Q ending at x and not contained in I. There are two possible cases. Either $S\alpha = 0$ for every non-trivial path α, and then we write $S = S_x$. Otherwise, there is a primitive cycle α ending at x and an irreducible polynomial $f \in k[t, t^{-1}]$ such that S fits into an exact sequence

$$0 \longrightarrow P_x \xrightarrow{f(\alpha)} P_x \longrightarrow S \longrightarrow 0;$$

cf. [60, Theorem 1.2]. In this case we have $\mathrm{proj.dim}\, S = 1$.

For $S = S_x$ we show that $\mathrm{Ext}^n_\Lambda(S_x, \Lambda) \neq 0$ and $n > 0$ imply the existence of a maximal differential path of length n ending at x. There are at most two arrows ending at x

$$u_1 \xrightarrow{\alpha_1} x \xleftarrow{\beta_1} v_1$$

and then a projective resolution of S_x has the following form

$$\cdots \longrightarrow P_{u_2} \oplus P_{v_2} \longrightarrow P_{u_1} \oplus P_{v_1} \longrightarrow P_x \longrightarrow S_x \longrightarrow 0.$$

The differentials are given by differential paths

$$\cdots \longrightarrow u_3 \xrightarrow{\alpha_3} u_2 \xrightarrow{\alpha_2} u_1 \xrightarrow{\alpha_1} x \quad \text{and} \quad \cdots \longrightarrow v_3 \xrightarrow{\beta_3} v_2 \xrightarrow{\beta_2} v_1 \xrightarrow{\beta_1} x$$

which may be infinite. Let $y \in Q_0$ and $\mathrm{Ext}^n_\Lambda(S_x, P_y) \neq 0$ with $n > 0$. A non-zero cocycle is given by a morphism $P_{u_n} \oplus P_{v_n} \to P_y$, and we may assume that the first component is non-zero. If $P_{u_n} \to P_y$ is invertible, then $\alpha_1 \cdots \alpha_n$ is maximal differential, because a composite $P_{u_{n+1}} \to P_{u_n} \to P_y$ would be zero. A radical morphism yields a path $u_n \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_r} y$ with $\gamma_1 \neq \alpha_n$. This implies that $\alpha_1 \cdots \alpha_n$ is maximal differential, since $\alpha_n \alpha_{n+1} \in I$ and $\gamma_1 \alpha_{n+1} \in I$ is impossible. Here we use that the ideal I of kQ is generated by paths of length two, because a cocycle means that the composite $P_{u_{n+1}} \to P_{u_n} \to P_y$ is zero, and therefore necessarily $\gamma_1 \alpha_{n+1} \in I$ since $\gamma_r \cdots \gamma_1 \notin I$.

Now we apply Lemma 6.2.10 and conclude that $\mathrm{inj.dim}\, \Lambda$ equals the maximal length of a maximal differential path in Q. An exception is the case that this equals zero and there exists a primitive cycle; then $\mathrm{inj.dim}\, \Lambda = 1$. □

Corollary 6.2.11. *Let Λ be a gentle algebra. Then its dimension as a Gorenstein algebra equals the maximal length of a maximal differential path in its quiver.*

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An exception is the case that this equals zero and there is a primitive cycle; then the dimension equals one. □

A Complete Intersection

Let us consider the complete intersection ring

$$\Lambda = R_{(x,y)} \qquad \text{where} \qquad R = \mathbb{C}[x,y]/(x^2 - y^2(y+1)).$$

The ring Λ is Gorenstein. Moreover, Λ is an integral domain with non-local integral closure. We use these facts to exhibit some phenomena of the stable category $\underline{\mathrm{mod}}\,\Lambda$.

Let Γ denote the integral closure of Λ. Since Γ is in the field of fractions of Λ, we know that Γ contains no proper direct summands. On the other hand the completion $\widehat{\Gamma}$ is the direct sum of two local rings $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$, one for each maximal ideal. Now observe that for each $X \in \mathrm{mod}\,\Lambda$ the module $\mathrm{Ext}^1_\Lambda(\Gamma, X)$ is of finite length. This yields a decomposition of the functor $\mathrm{Ext}^1_\Lambda(\Gamma, -)$, since

$$\mathrm{Ext}^1_\Lambda(\Gamma, X) \cong \mathrm{Ext}^1_{\widehat{\Lambda}}(\widehat{\Gamma}, \widehat{X}) = \mathrm{Ext}^1_{\widehat{\Lambda}}(\widehat{\Gamma}_1, \widehat{X}) \oplus \mathrm{Ext}^1_{\widehat{\Lambda}}(\widehat{\Gamma}_2, \widehat{X}).$$

Next observe that $X \mapsto \mathrm{Ext}^1_\Lambda(X, -)$ provides a fully faithful functor

$$\underline{\mathrm{mod}}\,\Lambda \longrightarrow \mathrm{Fp}(\mathrm{mod}\,\Lambda, \mathrm{Ab})$$

into the category of finitely presented functors $\mathrm{mod}\,\Lambda \to \mathrm{Ab}$ by Lemma 2.1.26.

Lemma 6.2.12. *There is a proper idempotent in* $\underline{\mathrm{End}}_\Lambda(\Gamma)$ *reflecting the decomposition of* $\mathrm{Ext}^1_\Lambda(\Gamma, -)$. *This idempotent has no kernel in* $\underline{\mathrm{mod}}\,\Lambda$. *In particular, there is a direct summand of* $\mathrm{Ext}^1_\Lambda(\Gamma, -)$ *which is not isomorphic to* $\mathrm{Ext}^1_\Lambda(X, -)$ *for some* Λ*-module* X.

Proof Suppose there is a decomposition $\Gamma = U \oplus V$ in $\underline{\mathrm{mod}}\,\Lambda$, and therefore $\Gamma \oplus P \cong U \oplus V \oplus Q$ for some projective Λ-modules P, Q by Lemma 2.1.27. The ring Λ is local, so all projective modules are free. Thus we may remove the indecomposable summands corresponding to P from the right-hand side, since these summands have local endomorphism rings. This yields a decomposition $\Gamma \cong U' \oplus V'$ in $\mathrm{mod}\,\Lambda$. Thus $U' = 0$ or $V' = 0$, and therefore $U = 0$ or $V = 0$ in $\underline{\mathrm{mod}}\,\Lambda$. The assertion about $\mathrm{Ext}^1_\Lambda(X, -)$ now follows, since any decomposition

$$\mathrm{Ext}^1_\Lambda(\Gamma, -) = \mathrm{Ext}^1_\Lambda(X, -) \oplus \mathrm{Ext}^1_\Lambda(Y, -)$$

is equivalent to a decomposition $\Gamma = X \oplus Y$ in $\underline{\mathrm{mod}}\,\Lambda$. □

Finally, observe that the Λ-module Γ is Gorenstein projective. It follows that the stable category $\underline{\mathrm{Gproj}}\,\Lambda$ and the equivalent singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ are not idempotent complete.

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6.3 Serre Duality

A special feature of Gorenstein algebras is Serre duality. In fact, there are several results and we formulate them in the context of Artin algebras.

Let k be a commutative artinian ring and Λ an Artin k-algebra. We write $D = \operatorname{Hom}_k(-, E)$ for the Matlis duality over k, which is given by a minimal injective cogenerator E. Thus $X \xrightarrow{\sim} D^2X$ for every k-module X of finite length.

The derived category $\mathbf{D}^b(\operatorname{mod}\Lambda)$ of a Gorenstein algebra Λ 'decomposes' into the category of *perfect complexes*

$$\mathbf{D}^{\mathrm{perf}}(\Lambda) = \mathbf{D}^b(\operatorname{proj}\Lambda)$$

and the singularity category

$$\underline{\operatorname{Gproj}}\,\Lambda \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(\Lambda).$$

This reflects the decomposition of the module category $\operatorname{mod}\Lambda$ from Theorem 6.2.4. We establish Serre duality for both categories. In fact, we derive Serre duality for the stable category of Gorenstein projective modules from Auslander–Reiten duality. The following section then discusses Serre duality for perfect complexes in terms of the derived Nakayama functor.

Serre Functors

Let $\mathcal{C}$ be a k-linear and Hom-finite additive category. Thus $\operatorname{Hom}(X, Y)$ is a k-module of finite length for all $X, Y \in \mathcal{C}$. A *Serre functor* is an equivalence $F: \mathcal{C} \to \mathcal{C}$ together with natural isomorphisms

$$\eta_{X,Y}: \operatorname{Hom}(X, Y) \xrightarrow{\sim} D\operatorname{Hom}(Y, FX)$$

for all objects X, Y in $\mathcal{C}$. Note that a Serre functor is determined by the natural isomorphisms $\eta_{X,Y}$ since FX represents the functor $D\operatorname{Hom}(X, -)$.

A Serre functor yields for each object X a morphism

$$\eta_X := \eta_{X,X}(\mathrm{id}_X): \operatorname{Hom}(X, FX) \longrightarrow E.$$

Lemma 6.3.1. *The morphisms $(\eta_X)_{X\in\mathcal{C}}$ have the following properties for all objects X, Y in $\mathcal{C}$.*

(1) *For all $\phi: X \to Y$ and $\psi: Y \to FX$ we have*

$$\eta_X(\psi\phi) = \eta_{X,Y}(\phi)(\psi) = \eta_Y(F(\phi)\psi).$$

(2) *The map $F_{X,Y}$ equals the composite*

$$\operatorname{Hom}(X, Y) \xrightarrow{\eta_{X,Y}} D\operatorname{Hom}(Y, FX) \xrightarrow{(D\eta_{Y,FX})^{-1}} \operatorname{Hom}(FX, FY).$$

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(3) *The composite*

$$\operatorname{Hom}(Y, FX) \times \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(X, FX) \xrightarrow{\eta_X} E$$

is a non-degenerate pairing.

Moreover, any bijection $X \mapsto FX$ on the isomorphism classes of objects in $\mathcal{C}$ together with a choice of k-linear maps $(\eta_X)_{X\in\mathcal{C}}$ satisfying (3) *yield a Serre functor.*

Proof The calculations are straightforward. (1) uses the naturality of the $\eta_{X,Y}$. (2) follows from the identity in (1). The non-degeneracy in (3) follows from the fact that the $\eta_{X,Y}$ are isomorphisms. For the last assertion, observe that the $\eta_{X,Y}$ and $F_{X,Y}$ are obtained via the identities in (1) and (2). Also, (3) implies that the $\eta_{X,Y}$ are isomorphisms. In particular, F is fully faithful and therefore an equivalence. □

The following remark collects some useful properties of Serre functors.

Remark 6.3.2. Let $F\colon \mathcal{C} \to \mathcal{C}$ be a Serre functor with maps $(\eta_{X,Y})_{X,Y\in\mathcal{C}}$.

(1) For any Serre functor $F'\colon \mathcal{C} \to \mathcal{C}$, there is a canonical isomorphism $F \xrightarrow{\sim} F'$ which is compatible with the $\eta_{X,Y}$. This follows from Yoneda's lemma since for any object X we have the isomorphism

$$\operatorname{Hom}(-, FX) \xrightarrow{D\eta_{X,-}} D\operatorname{Hom}(X, -) \xrightarrow{(D\eta'_{X,-})^{-1}} \operatorname{Hom}(-, F'X).$$

(2) Let $\Sigma\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ be an autoequivalence. Then $\Sigma^{-}F\Sigma$ is a Serre functor, so $\Sigma^{-}F\Sigma \cong F$, and therefore $F\Sigma \cong \Sigma F$.

(3) When $\mathcal{C}$ is a triangulated category with suspension Σ, then F is exact. Thus there is a canonical isomorphism $F\Sigma \cong \Sigma F$ and F maps exact triangles to exact triangles.

(4) Given an abelian category $\mathcal{A}$ and a Serre functor $F\colon \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})$, we may choose $(\eta_X)_{X\in\mathcal{C}}$ such that $\eta_{\Sigma X}(\Sigma\phi) = \eta_X(\phi)$ for all $\phi\colon X \to FX$.

Auslander–Reiten Duality

Given Λ-modules X and Y, we set

$$\underline{\operatorname{Hom}}_\Lambda(X, Y) = \operatorname{Hom}_\Lambda(X, Y)/\{\phi \mid \phi \text{ factors through a projective module}\}$$

and

$$\overline{\operatorname{Hom}}_\Lambda(X, Y) = \operatorname{Hom}_\Lambda(X, Y)/\{\phi \mid \phi \text{ factors through an injective module}\}.$$

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In this way we obtain the *projectively stable category* $\underline{\mathrm{mod}}\,\Lambda$ as additive quotient $(\mathrm{mod}\,\Lambda)/(\mathrm{proj}\,\Lambda)$. Analogously, the *injectively stable category* $\overline{\mathrm{mod}}\,\Lambda$ is defined.

For a finitely presented Λ-module X choose a projective presentation

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

such that the P_i are finitely generated. The *transpose* $\mathrm{Tr}\,X$ is defined by the exactness of the following sequence of Λ^{op}-modules

$$P_0^* \longrightarrow P_1^* \longrightarrow \mathrm{Tr}\,X \longrightarrow 0$$

where $P^* = \mathrm{Hom}_\Lambda(P, \Lambda)$.

Lemma 6.3.3. *The transpose induces mutually inverse equivalences*

$$(\underline{\mathrm{mod}}\,\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \underline{\mathrm{mod}}(\Lambda^{\mathrm{op}}) \qquad \textit{and} \qquad \underline{\mathrm{mod}}(\Lambda^{\mathrm{op}}) \xrightarrow{\sim} (\underline{\mathrm{mod}}\,\Lambda)^{\mathrm{op}}.$$

Proof The transpose depends on the choice of a projective presentation and is therefore unique up to morphisms that factor through a projective module. For a finitely generated projective Λ-module P, we have a natural isomorphism $P \xrightarrow{\sim} P^{**}$. Thus $\mathrm{Tr}\,\mathrm{Tr}\,X \cong X$ in $\underline{\mathrm{mod}}\,\Lambda$. □

From this it follows that the functors $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$ induce mutually inverse equivalences:

$$\underline{\mathrm{mod}}\,\Lambda \underset{\mathrm{Tr}}{\overset{\mathrm{Tr}}{\rightleftarrows}} \underline{\mathrm{mod}}(\Lambda^{\mathrm{op}}) \underset{D}{\overset{D}{\rightleftarrows}} \overline{\mathrm{mod}}\,\Lambda.$$

Lemma 6.3.4. *We have a natural isomorphism*

$$\underline{\mathrm{Hom}}_\Lambda(X, -) \cong \mathrm{Tor}_1^\Lambda(-, \mathrm{Tr}\,X).$$

Proof A projective presentation $P_1 \to P_0 \to X \to 0$ induces for any Λ-module A the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & & & A \otimes_\Lambda P_0^* & \longrightarrow & A \otimes_\Lambda P_1^* & \longrightarrow & A \otimes_\Lambda \mathrm{Tr}\,X \to 0 \\
 & & & & \downarrow\wr & & \downarrow\wr & & \\
0 \to & \mathrm{Hom}_\Lambda(X, A) & \to & & \mathrm{Hom}_\Lambda(P_0, A) & \to & \mathrm{Hom}_\Lambda(P_1, A) & &
\end{array}$$

Therefore an exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

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induces the following commutative diagram with exact rows and colums.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & \operatorname{Hom}_\Lambda(X,A) & \to & \operatorname{Hom}_\Lambda(P_0,A) & \to & \operatorname{Hom}_\Lambda(P_1,A) & \to & A\otimes_\Lambda \operatorname{Tr}X & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \operatorname{Hom}_\Lambda(X,B) & \to & \operatorname{Hom}_\Lambda(P_0,B) & \to & \operatorname{Hom}_\Lambda(P_1,B) & \to & B\otimes_\Lambda \operatorname{Tr}X & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \operatorname{Hom}_\Lambda(X,C) & \to & \operatorname{Hom}_\Lambda(P_0,C) & \to & \operatorname{Hom}_\Lambda(P_1,C) & \to & C\otimes_\Lambda \operatorname{Tr}X & \to 0\\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 & & 0 &
\end{array}$$

Now suppose that B is projective. Using the snake lemma we get

$$\begin{aligned}
\underline{\operatorname{Hom}}_\Lambda(X,C) &\cong \operatorname{Coker}\operatorname{Hom}_\Lambda(X,\psi)\\
&\cong \operatorname{Ker}(\phi\otimes_\Lambda \operatorname{Tr}X)\\
&\cong \operatorname{Tor}_1^\Lambda(C,\operatorname{Tr}X).
\end{aligned}$$

□

We obtain the following *Auslander–Reiten formulas*.

Proposition 6.3.5 (Auslander–Reiten)**.** *For all $X \in \operatorname{mod}\Lambda$ there are natural isomorphisms*

$$D\,\underline{\operatorname{Hom}}_\Lambda(X,-) \cong \operatorname{Ext}^1_\Lambda(-, D\operatorname{Tr}X)$$

and

$$D\operatorname{Ext}^1_\Lambda(X,-) \cong \overline{\operatorname{Hom}}_\Lambda(-, D\operatorname{Tr}X).$$

Proof The first isomorphism follows from the above lemma together with Lemma 4.3.17. The second isomorphism follows from the first via Matlis duality. □

Gorenstein Projective and Injective Modules

Let Λ be Gorenstein. We recall that a Λ-module X is *Gorenstein projective* if $\operatorname{Ext}^i_\Lambda(X,\Lambda) = 0$ for all $i \neq 0$. Dually, the module X is *Gorenstein injective* if $\operatorname{Ext}^i_\Lambda(D(\Lambda),X) = 0$ for all $i \neq 0$. We set

$$\operatorname{Ginj}\Lambda = \{X \in \operatorname{mod}\Lambda \mid X \text{ is Gorenstein injective}\}.$$

The duality D induces an equivalence

$$(\operatorname{Gproj}\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Ginj}(\Lambda^{\mathrm{op}}).$$

Observe that a Λ-module has finite projective dimension if and only if it has finite injective dimension, because Λ is Gorenstein. Then Theorem 6.2.4 yields two cotorsion pairs

$$(\operatorname{Gproj}\Lambda, \mathcal{Y}) \qquad \text{and} \qquad (\mathcal{Y}, \operatorname{Ginj}\Lambda)$$

for $\operatorname{mod}\Lambda$, where $\mathcal{Y}$ denotes the subcategory of modules having finite projective and finite injective dimension.

We consider the full subcategories

$$\underline{\operatorname{Gproj}}\,\Lambda \hookrightarrow \underline{\operatorname{mod}}\,\Lambda \qquad \text{and} \qquad \overline{\operatorname{Ginj}}\,\Lambda \hookrightarrow \overline{\operatorname{mod}}\,\Lambda.$$

The following lemma yields adjoints

$$\operatorname{GP}\colon \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{Gproj}}\,\Lambda \qquad \text{and} \qquad \operatorname{GI}\colon \overline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{Ginj}}\,\Lambda.$$

Lemma 6.3.6. *The inclusion* $\underline{\operatorname{Gproj}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda$ *admits a right adjoint and the inclusion* $\overline{\operatorname{Ginj}}\,\Lambda \to \overline{\operatorname{mod}}\,\Lambda$ *admits a left adjoint.*

Proof The existence of the right adjoint follows from Theorem 6.2.4, using also Remark 6.1.2. Applying Matlis duality, we obtain the left adjoint. With the notation of the approximation sequence (6.1.1), we get $\operatorname{GP}(A) = X_A$ and $\operatorname{GI}(A) = Y^A$ for a Λ-module A. □

We collect further properties of Gorenstein projective and injective modules.

Lemma 6.3.7. *Let* $X \in \operatorname{mod}\Lambda$ *be Gorenstein projective and* $Y \in \operatorname{mod}\Lambda$ *be Gorenstein injective. Then the following holds.*

(1) $D\operatorname{Tr}X$ *is Gorenstein injective and* $\operatorname{Tr}DY$ *is Gorenstein projective.*
(2) $\operatorname{GP}(\operatorname{GI}(X)) \cong X$ *in* $\underline{\operatorname{mod}}\,\Lambda$ *and* $\operatorname{GI}(\operatorname{GP}(Y)) \cong Y$ *in* $\overline{\operatorname{mod}}\,\Lambda$.
(3) $\underline{\operatorname{Hom}}_\Lambda(X,Y) = \overline{\operatorname{Hom}}_\Lambda(X,Y)$.

Proof (1) If X is Gorenstein projective, then $\operatorname{Tr}X$ is a Gorenstein projective Λ^{op}-module by Lemma 6.2.2. Thus $D\operatorname{Tr}X$ is Gorenstein injective. The argument for Y is dual.

(2) Consider the approximation sequence $0 \to X \to \operatorname{GI}(X) \to Y \to 0$, where Y is of finite injective dimension and therefore of finite projective dimension. Let $P \to Y$ be a projective cover and form the following pullback.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & X \oplus P & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & \operatorname{GI}(X) & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

The morphism $X \oplus P \to \operatorname{GI}(X)$ is a Gorenstein projective approximation,

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since its kernel has finite projective dimension. Thus we have $\mathrm{GP}(\mathrm{GI}(X)) \cong X$ stably. The argument for Y is dual.

(3) Fix a morphism $\phi\colon X \to Y$ that factors through an injective module. Let $X \to P$ denote the approximation with P of finite projective dimension, which exists by Theorem 6.2.4. Then P is projective, by Remark 6.1.2, and ϕ factors through P, since any injective module has finite projective dimension. The dual argument shows that ϕ factors through an injective module when one assumes that it factors through a projective module. □

Serre Duality for the Stable Category

Auslander–Reiten duality translates into Serre duality for the stable category of Gorenstein projective Λ-modules. Recall that $\operatorname{Gproj}\Lambda$ is a Frobenius category, and we denote by $\Omega^{-1}\colon \underline{\operatorname{Gproj}}\,\Lambda \xrightarrow{\sim} \underline{\operatorname{Gproj}}\,\Lambda$ the suspension of the stable category, which takes a module X to the cokernel of a monomorphism $X \to P$ into a projective module P.

Proposition 6.3.8. *Let Λ be Gorenstein. For Gorenstein projective Λ-modules X, Y there are natural isomorphisms*

$$\underline{\operatorname{Hom}}_\Lambda(\operatorname{Tr} D(\operatorname{GI}\Omega Y), X) \cong D\,\underline{\operatorname{Hom}}_\Lambda(X, Y) \cong \underline{\operatorname{Hom}}_\Lambda(Y, \Omega^{-1}\operatorname{GP}(D\operatorname{Tr} X)).$$

Proof We have

$$\begin{aligned} D\,\underline{\operatorname{Hom}}_\Lambda(X, Y) &\cong D\operatorname{Ext}^1_\Lambda(X, \Omega Y) \\ &\cong \overline{\operatorname{Hom}}_\Lambda(\Omega Y, D\operatorname{Tr} X) \\ &\cong \overline{\operatorname{Hom}}_\Lambda(\operatorname{GI}\Omega Y, D\operatorname{Tr} X) \\ &\cong \underline{\operatorname{Hom}}_\Lambda(\operatorname{Tr} D(\operatorname{GI}\Omega Y), X). \end{aligned}$$

The first isomorphism is obtained by applying $\operatorname{Hom}_\Lambda(X, -)$ to an exact sequence

$$0 \longrightarrow \Omega Y \longrightarrow P \longrightarrow Y \longrightarrow 0$$

with P projective. The second isomorphism is Auslander–Reiten duality; see Proposition 6.3.5. The third isomorphism is induced by $\Omega Y \to \operatorname{GI}(\Omega Y)$; see Lemma 6.3.7. The last isomorphism is obtained by applying $\operatorname{Tr} D$.

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A similar sequence of arguments yields

$$\begin{aligned} D\,\underline{\operatorname{Hom}}_\Lambda(X,Y) &\cong D\operatorname{Ext}^1_\Lambda(X,\Omega Y) \\ &\cong \overline{\operatorname{Hom}}_\Lambda(\Omega Y, D\operatorname{Tr} X) \\ &= \underline{\operatorname{Hom}}_\Lambda(\Omega Y, D\operatorname{Tr} X) \\ &\cong \underline{\operatorname{Hom}}_\Lambda(\Omega Y, \operatorname{GP}(D\operatorname{Tr} X)) \\ &\cong \underline{\operatorname{Hom}}_\Lambda(Y, \Omega^{-1}\operatorname{GP}(D\operatorname{Tr} X)). \end{aligned}$$ □

Corollary 6.3.9. *Let* Λ *be Gorenstein. The assignments*

$$X \mapsto \Omega^{-1}\operatorname{GP}(D\operatorname{Tr} X) \quad \textit{and} \quad Y \mapsto \operatorname{Tr} D(\operatorname{GI}\Omega Y)$$

yield mutually inverse equivalences $\underline{\operatorname{Gproj}}\,\Lambda \xrightarrow{\sim} \underline{\operatorname{Gproj}}\,\Lambda$. *In particular, the composite* $\Omega^{-1}\circ\operatorname{GP}\circ D\operatorname{Tr}$ *is a Serre functor for* $\underline{\operatorname{Gproj}}\,\Lambda$.

Proof We have $\Omega^{-1}\circ\Omega \cong \mathrm{id} \cong \Omega\circ\Omega^{-1}$ since $\operatorname{Gproj}\Lambda$ is a Frobenius category; see Theorem 6.2.5. Also, $D\operatorname{Tr}\circ\operatorname{Tr} D \cong \mathrm{id}$ and $\operatorname{Tr} D\circ D\operatorname{Tr} \cong \mathrm{id}$. Finally, $\operatorname{GP}\circ\operatorname{GI} \cong \mathrm{id}$ and $\operatorname{GI}\circ\operatorname{GP} \cong \mathrm{id}$ by Lemma 6.3.7. The isomorphism in Proposition 6.3.8 then shows that $\Omega^{-1}\circ\operatorname{GP}\circ D\operatorname{Tr}$ is a Serre functor. □

6.4 The Derived Nakayama Functor

In this section we introduce the derived Nakayama functor and use this to establish Serre duality for the category of perfect complexes. We keep the setting from the previous section and consider an Artin algebra Λ over a commutative ring k. Also, we show that $\mathbf{D}^b(\operatorname{mod}\Lambda)$ admits a Serre functor if and only if Λ has finite global dimension.

The Nakayama Functor

The *Nakayama functor* $\operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ is given by

$$\nu X := X \otimes_\Lambda D(\Lambda) \cong D\operatorname{Hom}_\Lambda(X,\Lambda).$$

It restricts to an equivalence $\operatorname{proj}\Lambda \xrightarrow{\sim} \operatorname{inj}\Lambda$, where $\operatorname{inj}\Lambda$ denotes the category of finitely generated injective Λ-modules.

A projective presentation $P_1 \to P_0 \to X \to 0$ induces an exact sequence

$$0 \longrightarrow D\operatorname{Tr} X \longrightarrow \nu P_1 \longrightarrow \nu P_0 \longrightarrow \nu X \longrightarrow 0 \tag{6.4.1}$$

and therefore

$$\Omega^{-2}(D\operatorname{Tr} X) \cong \nu X.$$

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Of particular interest is the following case.

Proposition 6.4.2. *Let* Λ *be self-injective. Then the functor*

$$\underline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda, \quad X \mapsto \Omega\nu X \cong \nu\Omega X,$$

is a Serre functor, and therefore

$$D\,\underline{\mathrm{Hom}}_\Lambda(X,-) \cong \underline{\mathrm{Hom}}_\Lambda(-,\Omega\nu X).$$

Proof This follows from Corollary 6.3.9 since $\underline{\mathrm{mod}}\,\Lambda = \underline{\mathrm{Gproj}}\,\Lambda$. □

Suppose the algebra Λ is *symmetric* so that $D\Lambda \cong \Lambda$ as Λ-Λ-bimodules. Then we have $\nu = \mathrm{id}$. An example is the group algebra of a finite group, and in that case the above Serre duality is known as Tate duality.

Let us identify $\mathbf{K}^-(\mathrm{proj}\,\Lambda) \xrightarrow{\sim} \mathbf{D}^-(\mathrm{mod}\,\Lambda)$. Then the *derived Nakayama functor*

$$X \longmapsto X \otimes^L_\Lambda D(\Lambda)$$

is by definition the composite

$$\mathbf{D}^-(\mathrm{mod}\,\Lambda) \xrightarrow{\sim} \mathbf{K}^-(\mathrm{proj}\,\Lambda) \xrightarrow{-\otimes_\Lambda D(\Lambda)} \mathbf{K}^-(\mathrm{mod}\,\Lambda) \xrightarrow{\mathrm{can}} \mathbf{D}^-(\mathrm{mod}\,\Lambda).$$

Serre Duality for Perfect Complexes

We show that the category of perfect complexes

$$\mathbf{D}^{\mathrm{perf}}(\Lambda) = \mathbf{D}^b(\mathrm{proj}\,\Lambda)$$

admits a Serre functor if and only if the algebra Λ is Gorenstein. In fact, the derived Nakayama functor restricts to a Serre functor for $\mathbf{D}^{\mathrm{perf}}(\Lambda)$ when Λ is Gorenstein.

We recall the following standard isomorphisms and extend them to isomorphisms of complexes.

Lemma 6.4.3. *Let* $(A_\Lambda, {}_\Gamma B_\Lambda, {}_\Gamma C)$ *be modules and suppose that* A_Λ *is finitely generated projective. Then there are natural isomorphisms*

$$B \otimes_\Lambda \mathrm{Hom}_\Lambda(A,\Lambda) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(A,B)$$

and

$$A \otimes_\Lambda \mathrm{Hom}_\Gamma(B,C) \xrightarrow{\sim} \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(A,B),C).$$ □

A complex X is called *bounded* if $X^n = 0$ for almost all $n \in \mathbb{Z}$. A pair of complexes (X,Y) is *bounded* if for each $n \in \mathbb{Z}$ we have for almost all pairs of integers (p,q) with $-p+q = n$ that $X^p = 0$ or $Y^q = 0$.

Lemma 6.4.4. *Let* $(X_\Lambda, {}_\Gamma Y_\Lambda, {}_\Gamma Z)$ *be complexes of modules with each* X^n *finitely generated projective. Suppose that* (X, Y) *and* Z *are bounded. Then there are natural isomorphisms*

$$Y \otimes_\Lambda \operatorname{Hom}_\Lambda(X, \Lambda) \xrightarrow{\sim} \operatorname{Hom}_\Lambda(X, Y)$$

and

$$X \otimes_\Lambda \operatorname{Hom}_\Gamma(Y, Z) \xrightarrow{\sim} \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(X, Y), Z).$$

Proof We may apply Lemma 6.4.3 degreewise, thanks to the boundedness assumptions. □

It is convenient to set $\mathbf{K}(\Lambda) = \mathbf{K}(\operatorname{Mod}\Lambda)$ and $\mathbf{D}(\Lambda) = \mathbf{D}(\operatorname{Mod}\Lambda)$.

Lemma 6.4.5. *Let* X, Y *be complexes of* Λ*-modules and suppose that* X *is perfect. Then we have natural isomorphisms*

$$D \operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, Y) \cong \operatorname{Hom}_{\mathbf{K}(\Lambda)}(Y, X \otimes_\Lambda D(\Lambda))$$

and

$$D \operatorname{Hom}_{\mathbf{D}(\Lambda)}(X, Y) \cong \operatorname{Hom}_{\mathbf{D}(\Lambda)}(Y, X \otimes_\Lambda D(\Lambda)).$$

Proof Let X be a bounded complex of finitely generated projective modules. Then we have the following sequence of isomorphisms:

$$\begin{aligned}
D \operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, Y) &\cong \operatorname{Hom}_k(H^0 \operatorname{Hom}_\Lambda(X, Y), E) && 4.3.14\\
&\cong H^0 \operatorname{Hom}_k(\operatorname{Hom}_\Lambda(X, Y), E) &&\\
&\cong H^0 \operatorname{Hom}_k(Y \otimes_\Lambda \operatorname{Hom}_\Lambda(X, \Lambda), E) && 6.4.4\\
&\cong H^0 \operatorname{Hom}_\Lambda(Y, \operatorname{Hom}_k(\operatorname{Hom}_\Lambda(X, \Lambda), E)) && 4.3.13\\
&\cong H^0 \operatorname{Hom}_\Lambda(Y, X \otimes_\Lambda \operatorname{Hom}_k(\Lambda, E)) && 6.4.4\\
&\cong \operatorname{Hom}_{\mathbf{K}(\Lambda)}(Y, X \otimes_\Lambda D(\Lambda)). && 4.3.14
\end{aligned}$$

The above isomorphism for a perfect complex X carries over to $\mathbf{D}(\Lambda)$ since

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, -) \cong \operatorname{Hom}_{\mathbf{D}(\Lambda)}(X, -)$$

and

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(-, X \otimes_\Lambda D(\Lambda)) \cong \operatorname{Hom}_{\mathbf{D}(\Lambda)}(-, X \otimes_\Lambda D(\Lambda)).$$ □

Theorem 6.4.6. *An Artin algebra* Λ *is Gorenstein if and only if the category of perfect complexes* $\mathbf{D}^{\mathrm{perf}}(\Lambda)$ *admits a Serre functor. In this case, the Serre functor is given by the derived Nakayama functor.*

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Proof Set $\mathcal{P} = \operatorname{Thick}(\operatorname{proj}\Lambda)$ and $\mathcal{I} = \operatorname{Thick}(\operatorname{inj}\Lambda)$. We have equivalences

$$\mathbf{K}^b(\operatorname{proj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{proj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{P})$$

and analogously

$$\mathbf{K}^b(\operatorname{inj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{inj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{I}).$$

The Nakayama functor $-\otimes_\Lambda D(\Lambda)$ and its adjoint $\operatorname{Hom}_\Lambda(D(\Lambda),-)$ induce mutually inverse equivalences:

$$\mathbf{D}^b(\mathcal{P}) \xleftarrow{\sim} \mathbf{K}^b(\operatorname{proj}\Lambda) \underset{\operatorname{Hom}_\Lambda(D(\Lambda),-)}{\overset{-\otimes_\Lambda D(\Lambda)}{\rightleftarrows}} \mathbf{K}^b(\operatorname{inj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{I}).$$

If Λ is Gorenstein, then we have $\mathcal{P} = \mathcal{I}$. Thus the Nakayama functor gives an equivalence $\mathbf{D}^{\text{perf}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^{\text{perf}}(\Lambda)$, and this is a Serre functor by the isomorphism from Lemma 6.4.5.

Now suppose that $F\colon \mathbf{D}^{\text{perf}}(\Lambda) \to \mathbf{D}^{\text{perf}}(\Lambda)$ is a Serre functor. Then we have isomorphisms

$$\operatorname{Hom}_{\mathbf{D}(\Lambda)}(-,F\Lambda) \cong D\operatorname{Hom}_{\mathbf{D}(\Lambda)}(\Lambda,-) \cong \operatorname{Hom}_{\mathbf{D}(\Lambda)}(-,D(\Lambda))$$

of functors on $\mathbf{D}^{\text{perf}}(\Lambda)$. The first isomorphism is clear from the definition of a Serre functor, and the second by Lemma 6.4.5. The isomorphism of functors is induced by a morphism $F\Lambda \to D(\Lambda)$ and this is a quasi-isomorphism since

$$H^n(F\Lambda) \cong \operatorname{Hom}_{\mathbf{D}(\Lambda)}(\Sigma^{-n}\Lambda, F\Lambda) \cong \operatorname{Hom}_{\mathbf{D}(\Lambda)}(\Sigma^{-n}\Lambda, D(\Lambda)) \cong H^n(D(\Lambda)).$$

It follows that $D(\Lambda)_\Lambda$ has finite projective dimension. The functor $\operatorname{Hom}_\Lambda(-,\Lambda)$ induces a triangle equivalence

$$\mathbf{D}^{\text{perf}}(\Lambda)^{\text{op}} \xrightarrow{\sim} \mathbf{D}^{\text{perf}}(\Lambda^{\text{op}}),$$

and it follows that $\mathbf{D}^{\text{perf}}(\Lambda^{\text{op}})$ admits a Serre functor. Thus ${}_\Lambda D(\Lambda)$ has finite projective dimension. Using Matlis duality, it follows that Λ_Λ and ${}_\Lambda\Lambda$ have finite injective dimension. We conclude that Λ is Gorenstein. □

Serre Duality for the Singularity Category

We get back to the stable category of a Gorenstein algebra and provide another description of the Serre functor which uses the derived Nakayama functor.

Proposition 6.4.7. *Let Λ be Gorenstein. Then the derived Nakayama functor yields an equivalence*

$$\hat{\nu}\colon \mathbf{D}^b(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$$

satisfying $\hat{\nu}(\mathbf{D}^{\text{perf}}(\Lambda)) = \mathbf{D}^{\text{perf}}(\Lambda)$.

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Proof The category $\mathbf{D}^b(\operatorname{mod}\Lambda)$ identifies with the full subcategory of objects $X \in \mathbf{K}^-(\operatorname{proj}\Lambda)$ such that the cohomology of X is concentrated in finitely many degrees. Suppose first that $H^n X = 0$ for all $n \neq 0$. Then we have

$$H^i(X \otimes_\Lambda D(\Lambda)) \cong \operatorname{Tor}^\Lambda_{-i}(H^0 X, D(\Lambda)).$$

Thus $X \otimes^L_\Lambda D(\Lambda)$ belongs to $\mathbf{D}^b(\operatorname{mod}\Lambda)$ since $D(\Lambda)$ has finite projective dimension. From the exactness of the derived Nakayama functor it follows that we get a functor $\mathbf{D}^b(\operatorname{mod}\Lambda) \to \mathbf{D}^b(\operatorname{mod}\Lambda)$, because the objects with cohomology concentrated in degree zero generate $\mathbf{D}^b(\operatorname{mod}\Lambda)$. A quasi-inverse is given by $\operatorname{RHom}_\Lambda(D(\Lambda), -)$, and that $\hat{\nu}$ identifies perfect complexes with perfect complexes follows from Theorem 6.4.6. □

There is a converse of the above proposition for which we refer to Proposition 9.2.17.

Example 6.4.8. Let Λ be hereditary and X a Λ-module, viewed as a complex concentrated in degree zero. Then we have

$$X \otimes^L_\Lambda D(\Lambda) = \nu X \oplus (D \operatorname{Tr} X)[1].$$

This follows from the sequence (6.4.1).

The above proposition implies that the derived Nakayama functor induces an equivalence $\bar{\nu}\colon \mathbf{D}_{\mathrm{sg}}(\Lambda) \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(\Lambda)$ making the following diagram commutative.

$$\begin{array}{ccccc}
\mathbf{D}^b(\operatorname{proj}\Lambda) & \rightarrowtail & \mathbf{D}^b(\operatorname{mod}\Lambda) & \twoheadrightarrow & \mathbf{D}_{\mathrm{sg}}(\Lambda) \\
\downarrow & & \downarrow{\scriptstyle\hat{\nu}} & & \downarrow{\scriptstyle\bar{\nu}} \\
\mathbf{D}^b(\operatorname{proj}\Lambda) & \rightarrowtail & \mathbf{D}^b(\operatorname{mod}\Lambda) & \twoheadrightarrow & \mathbf{D}_{\mathrm{sg}}(\Lambda)
\end{array}$$

Moreover, the equivalence $\bar{\nu}$ makes the following square commutative

$$\begin{array}{ccc}
\underline{\operatorname{Gproj}}\,\Lambda & \xrightarrow[p]{\sim} & \mathbf{D}_{\mathrm{sg}}(\Lambda) \\
{\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\bar{\nu}} \\
\overline{\operatorname{Ginj}}\,\Lambda & \xrightarrow[q]{\sim} & \mathbf{D}_{\mathrm{sg}}(\Lambda)
\end{array}$$

where the horizontal equivalence p is from Theorem 6.2.5 and q is its analogue for Gorenstein injectives.

Lemma 6.4.9. *The Gorenstein projective approximation* GP *induces a triangle equivalence* $\overline{\operatorname{Ginj}}\,\Lambda \xrightarrow{\sim} \underline{\operatorname{Gproj}}\,\Lambda$. *Moreover, we have*

$$q \cong p \circ \mathrm{GP} \qquad \textit{and} \qquad \bar{\nu} \circ p \cong q \circ \nu \cong q \circ \Omega^{-2} \circ D\operatorname{Tr}.$$

Proof For any Λ-module X there is an exact sequence $0 \to X' \to \mathrm{GP}(X) \to X \to 0$ such that X' has finite projective dimension, by Theorem 6.2.4. Thus $q \cong p \circ \mathrm{GP}$. It follows that GP induces a triangle equivalence $\overline{\mathrm{Ginj}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{Gproj}}\,\Lambda$ since p and q are triangle equivalences. The isomorphism $\bar{\nu} \circ p \cong q \circ \nu$ is clear and the last one follows from (6.4.1). □

In Corollary 6.3.9 we have already seen that the stable category of Gorenstein projectives admits a Serre functor and now we have an alternative description.

Corollary 6.4.10. *Let Λ be Gorenstein. Then*

$$\Sigma^{-1} \circ \bar{\nu} \colon \mathbf{D}_{\mathrm{sg}}(\Lambda) \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(\Lambda) \quad \text{and} \quad \Omega \circ \mathrm{GP} \circ \nu \colon \underline{\mathrm{Gproj}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{Gproj}}\,\Lambda$$

are Serre functors.

Proof We apply Proposition 6.3.8 and Lemma 6.4.9. Thus for Gorenstein projective Λ-modules X, Y we compute in $\mathbf{D}_{\mathrm{sg}}(\Lambda)$

$$\begin{aligned} D\operatorname{Hom}(pX, pY) &\cong \operatorname{Hom}(pY, p\Omega^{-1}\,\mathrm{GP}(D\operatorname{Tr} X)) \\ &\cong \operatorname{Hom}(pY, \Sigma^{-1} p\,\mathrm{GP}\,\Omega^{-2}(D\operatorname{Tr} X)) \\ &\cong \operatorname{Hom}(pY, \Sigma^{-1} q\Omega^{-2}(D\operatorname{Tr} X)) \\ &\cong \operatorname{Hom}(pY, \Sigma^{-1}\bar{\nu} pX). \end{aligned}$$

Thus $\Sigma^{-1} \circ \bar{\nu}$ is a Serre functor for $\mathbf{D}_{\mathrm{sg}}(\Lambda)$. We have $\bar{\nu} \circ p \cong p \circ \mathrm{GP} \circ \nu$, and it follows that $\Omega \circ \mathrm{GP} \circ \nu$ is a Serre functor for $\underline{\mathrm{Gproj}}\,\Lambda$. □

Finite Global Dimension

We show that an Artin algebra Λ has finite global dimension if and only if $\mathbf{D}^b(\operatorname{mod}\Lambda)$ admits a Serre functor. This requires various computations in $\mathbf{K}(\Lambda) := \mathbf{K}(\operatorname{Mod}\Lambda)$. We begin with a statement about complexes of injective modules.

Lemma 6.4.11. *Let Λ be a right noetherian ring and X an object in $\mathbf{K}(\operatorname{Inj}\Lambda)$. For a Λ-module A an injective resolution $A \to iA$ induces an isomorphism*

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(iA, X) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{K}(\Lambda)}(A, X).$$

Moreover, if $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(A, \Sigma^n X) = 0$ for all $A \in \operatorname{mod}\Lambda$ and $n \in \mathbb{Z}$, then $X = 0$.

Proof For the first assertion see Lemma 4.2.6.

Now suppose $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(A, \Sigma^n X) = 0$ for all $A \in \operatorname{mod}\Lambda$ and $n \in \mathbb{Z}$. Suppose first $H^n X \neq 0$ for some n. Choose $A \in \operatorname{mod}\Lambda$ and a morphism $A \to Z^n X$ inducing a non-zero morphism $A \to H^n X$. We obtain a morphism $A \to \Sigma^n X$ which induces a non-zero element in $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(A, \Sigma^n X)$.

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Now suppose $H^n X = 0$ for all n. We can choose n such that $Z^n X$ is non-injective. Applying Baer's criterion, there exists $A \in \operatorname{mod}\Lambda$ such that $\operatorname{Ext}^1_\Lambda(A, Z^n X)$ is non-zero. Observe that

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(A, \Sigma^{n+p} X) \cong \operatorname{Ext}^p_\Lambda(A, Z^n X)$$

for all $p \geq 1$. Thus $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(A, \Sigma^{n+1} X) \neq 0$. □

We consider the triangle equivalence

$$\mathbf{K}^{+,b}(\operatorname{Inj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{Mod}\Lambda) \xrightarrow{\sim} \mathbf{K}^{-,b}(\operatorname{Proj}\Lambda), \qquad X \longmapsto \mathbf{p}X$$

from Corollary 4.2.9, which takes a complex to a homotopy projective resolution. This restricts to an equivalence

$$\mathcal{C} \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathbf{K}^{-,b}(\operatorname{proj}\Lambda)$$

where $\mathcal{C} \subseteq \mathbf{K}^{+,b}(\operatorname{Inj}\Lambda)$ denotes the thick subcategory generated by all injective resolutions of finitely generated Λ-modules.

Lemma 6.4.12. *For $X, Y \in \mathbf{K}(\operatorname{Inj}\Lambda)$ with $X \in \mathcal{C}$ we have a natural isomorphism*

$$D\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, Y) \cong \operatorname{Hom}_{\mathbf{K}(\Lambda)}(Y, \mathbf{p}X \otimes_\Lambda D(\Lambda)).$$

Proof We may assume that $Y^n = 0$ for $n \ll 0$ since Y can be written as a homotopy colimit of objects in $\mathbf{K}^+(\operatorname{Inj}\Lambda)$ (using (4.2.3)). In fact, one checks that $D\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, -)$ and $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(-, \mathbf{p}X \otimes_\Lambda D(\Lambda))$ preserve homotopy colimits. This is clear in the second case. In the first case we may reduce to the case that X is the injective resolution of a finitely generated module A, so there is a quasi-isomorphism $A \to X$ in $\mathbf{K}(\Lambda)$. Then

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, -) \cong \operatorname{Hom}_{\mathbf{K}(\Lambda)}(A, -)$$

by Lemma 6.4.11, and this preserves coproducts since A is finitely generated.

Keeping in mind that Y is in $\mathbf{K}^+(\operatorname{Inj}\Lambda)$, we have the following sequence of isomorphisms:

$$\begin{aligned}
D\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, Y) &\cong \operatorname{Hom}_k(\operatorname{Hom}_{\mathbf{K}(\Lambda)}(\mathbf{p}X, Y), E) && 4.2.5\\
&\cong \operatorname{Hom}_k(H^0\operatorname{Hom}_\Lambda(\mathbf{p}X, Y), E) && 4.3.14\\
&\cong H^0\operatorname{Hom}_k(\operatorname{Hom}_\Lambda(\mathbf{p}X, Y), E) && \\
&\cong H^0\operatorname{Hom}_k(Y \otimes_\Lambda \operatorname{Hom}_\Lambda(\mathbf{p}X, \Lambda), E) && 6.4.4\\
&\cong H^0\operatorname{Hom}_\Lambda(Y, \operatorname{Hom}_k(\operatorname{Hom}_\Lambda(\mathbf{p}X, \Lambda), E)) && 4.3.13\\
&\cong H^0\operatorname{Hom}_\Lambda(Y, \mathbf{p}X \otimes_\Lambda \operatorname{Hom}_k(\Lambda, E)) && 6.4.4\\
&\cong \operatorname{Hom}_{\mathbf{K}(\Lambda)}(Y, \mathbf{p}X \otimes_\Lambda D(\Lambda)). && 4.3.14
\end{aligned}$$

With our first observation the isomorphism follows for an arbitrary complex Y in $\mathbf{K}(\operatorname{Inj}\Lambda)$. □

Theorem 6.4.13. *For an Artin algebra Λ the following are equivalent.*

(1) *The algebra Λ has finite global dimension.*
(2) *The canonical functor $\mathbf{D}^b(\operatorname{proj}\Lambda) \to \mathbf{D}^b(\operatorname{mod}\Lambda)$ is an equivalence.*
(3) *The category $\mathbf{D}^b(\operatorname{mod}\Lambda)$ admits a Serre functor.*
(4) *The algebra Λ is Gorenstein and each acyclic complex of finitely generated injective Λ-modules is contractible.*
(5) *The algebra Λ is Gorenstein and each acyclic complex of finitely generated projective Λ-modules is contractible.*

Proof (1) ⇔ (2): The canonical functor $\mathbf{D}^b(\operatorname{proj}\Lambda) \to \mathbf{D}^b(\operatorname{mod}\Lambda)$ is fully faithful, and it is an equivalence if and only if every object in $\operatorname{mod}\Lambda$ has finite projective dimension. It remains to observe that the global dimension of Λ equals the maximum of the projective dimensions of the simple Λ-modules.

(1) ⇒ (3): When Λ has finite global dimension, then we have $\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$, and it follows from Theorem 6.4.6 that $\mathbf{D}^b(\operatorname{mod}\Lambda)$ admits a Serre functor.

(3) ⇒ (4): Suppose that $F\colon \mathbf{D}^b(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$ is a Serre functor and set $S := \Lambda/\operatorname{rad}\Lambda$. Then

$$D\operatorname{Ext}^n_\Lambda(S,\Lambda) \cong D\operatorname{Hom}_{\mathbf{D}(\Lambda)}(\Sigma^{-n}S,\Lambda) \cong \operatorname{Hom}_{\mathbf{D}(\Lambda)}(\Lambda,\Sigma^{-n}F(S)) = H^{-n}F(S)$$

and therefore

$$\begin{aligned}\operatorname{inj.dim}_\Lambda \Lambda &= \sup\{n \geq 0 \mid \operatorname{Ext}^n_\Lambda(S,\Lambda) \neq 0\}\\ &= \sup\{n \geq 0 \mid H^{-n}(FS) \neq 0\}\end{aligned}$$

which is finite. It follows that Λ is Gorenstein, keeping in mind that $\mathbf{D}^b(\operatorname{mod}\Lambda^{\mathrm{op}})$ admits a Serre functor as well.

Let us identify $\mathcal{T} := \mathbf{K}^{+,b}(\operatorname{inj}\Lambda) = \mathbf{D}^b(\operatorname{mod}\Lambda)$. Then Lemma 6.4.12 implies for each $X \in \mathcal{T}$ a natural isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(-,F(X)) \cong D\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X,-) \cong \operatorname{Hom}_{\mathbf{K}(\Lambda)}(-,\mathbf{p}X \otimes_\Lambda D(\Lambda))$$

of functors on $\mathcal{T}$. This is induced by a morphism $\phi\colon F(X) \to \mathbf{p}X \otimes_\Lambda D(\Lambda)$, which is an isomorphism by Lemma 6.4.11, since $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(-,\operatorname{Cone}\phi)|_{\mathcal{T}} = 0$. Thus $\mathbf{p}X \otimes_\Lambda D(\Lambda)$ belongs to $\mathcal{T}$. Now choose a complex V in $\mathbf{K}(\operatorname{inj}\Lambda)$ which is acyclic. Then

$$D\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X,V) \cong \operatorname{Hom}_{\mathbf{K}(\Lambda)}(V,\mathbf{p}X \otimes_\Lambda D(\Lambda)) = 0.$$

The first isomorphism is by Lemma 6.4.12 and the second by Lemma 4.2.5. Thus $V = 0$ by Lemma 6.4.11.

(4) ⇔ (5): The Nakayama functor yields an equivalence $\mathbf{K}(\operatorname{proj}\Lambda) \xrightarrow{\sim} \mathbf{K}(\operatorname{inj}\Lambda)$ that identifies the acyclic complexes in both categories. This follows from Lemma 6.2.2.

(5) ⇒ (1): Each Gorenstein projective module admits a complete resolution, so is of the form Z^0X for some acyclic complex X of projective Λ-modules, by Theorem 6.2.5. If X is contractible, then Z^0X is projective. Thus every Λ-module has finite projective dimension by Theorem 6.2.4. □

6.5 Examples

We discuss two examples. The first one gives an application of Serre duality for hereditary abelian categories. The second example provides explicit computations for a Gorenstein algebra of dimension one.

Hereditary Categories

We give an application of Serre duality for hereditary abelian categories. Fix a commutative ring k and a k-linear hereditary abelian category $\mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(X,Y)$ and $\operatorname{Ext}^1_{\mathcal{A}}(X,Y)$ are finite length k-modules for all objects X, Y. Let $D = \operatorname{Hom}_k(-E)$ denote Matlis duality given by an injective k-module E.

Proposition 6.5.1. *Suppose that $\mathcal{A}$ admits a tilting object and that $\mathcal{A}$ has no non-zero projective or injective objects. Then there is an equivalence $F\colon \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ together with natural isomorphisms*

$$D\operatorname{Ext}^1_{\mathcal{A}}(X,Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(Y,FX) \qquad (X,Y \in \mathcal{A}). \tag{6.5.2}$$

Proof Let $T \in \mathcal{A}$ be a tilting object and set $\Lambda = \operatorname{End}_{\mathcal{A}}(T)$. Then it follows from Theorem 5.1.2 that $\operatorname{RHom}_{\mathcal{A}}(T,-)$ induces a triangle equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$. Moreover, the algebra Λ has finite global dimension. Next we apply Theorem 6.4.13. Thus there is a triangle equivalence $F\colon \mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A})$ together with natural isomorphisms

$$D\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(Y,FX) \qquad (X,Y \in \mathbf{D}^b(\mathcal{A})).$$

We identify $\mathcal{A}$ with the full subcategory of complexes in $\mathbf{D}^b(\mathcal{A})$ that are concentrated in degree zero and claim that $F(\mathcal{A}) \subseteq \Sigma(\mathcal{A})$. Fix an indecomposable object $X \in \mathcal{A}$. Then we find $Y \in \mathcal{A}$ such that $FX = \Sigma^n Y$ for some $n \in \mathbb{Z}$ since

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$\mathcal{A}$ is hereditary (Proposition 4.4.15). We compute

$$D\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, \Sigma^n Y) \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(\Sigma^n Y, \Sigma^n Y) \neq 0,$$

and therefore $n \in \{0, 1\}$. Suppose $n = 0$ and let $Z \in \mathcal{A}$. Then we have

$$D\operatorname{Ext}^1_{\mathcal{A}}(X, Z) = D\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, \Sigma Z) \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(\Sigma Y, Y) = 0$$

which means that X is projective. This is a contradiction, and therefore $n = 1$. The dual result shows for a quasi-inverse F^- of F that $F^-(\mathcal{A}) \subseteq \Sigma^{-1}(\mathcal{A})$. Thus $F' = \Sigma^{-1} \circ F$ yields an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ together with natural isomorphisms

$$D\operatorname{Ext}^1_{\mathcal{A}}(X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(Y, F'X) \qquad (X, Y \in \mathcal{A}). \qquad \square$$

Remark 6.5.3. Let $\mathcal{A}$ be an abelian category with an equivalence $F\colon \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ and a natural isomorphism (6.5.2). Then $\mathcal{A}$ is hereditary and $\mathcal{A}$ has no non-zero projective or injective objects.

We obtain a version of Serre duality for the projective line over a field.

Example 6.5.4. The category $\operatorname{coh}\mathbb{P}^1_k$ of coherent sheaves on the projective line over a field k admits a tilting object (Lemma 5.1.16) and therefore an equivalence $\operatorname{coh}\mathbb{P}^1_k \xrightarrow{\sim} \operatorname{coh}\mathbb{P}^1_k$ providing a natural isomorphism

$$D\operatorname{Ext}^1(X, Y) \xrightarrow{\sim} \operatorname{Hom}(Y, X \otimes \Omega_{\mathbb{P}^1_k}) \qquad (X, Y \in \operatorname{coh}\mathbb{P}^1_k)$$

where $\Omega_{\mathbb{P}^1_k} \cong \mathscr{O}(-2)$ denotes the *sheaf of differential forms*.

A Gorenstein Algebra of Dimension One

We discuss a small example of an Artin algebra that is Gorenstein, but neither of finite global dimension nor self-injective.

Fix a field k and consider the finite dimensional algebra $\Lambda = k[\varepsilon] \otimes_k \left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right]$, which is Gorenstein of dimension one and isomorphic to the path algebra of the quiver

$$\varepsilon_1 \circlearrowright 1 \xrightarrow{\ \alpha\ } 2 \circlearrowleft \varepsilon_2$$

modulo the relations

$$\varepsilon_1^2 = 0 = \varepsilon_2^2 \qquad \text{and} \qquad \alpha\varepsilon_1 = \varepsilon_2\alpha.$$

There are two simple modules corresponding to the vertices 1 and 2. The indecomposable projective modules are given by idempotents e_1 and e_2 as follows:

$$e_1\Lambda = ke_1 \oplus k\varepsilon_1 \quad \text{and} \quad e_2\Lambda = ke_2 \oplus k\varepsilon_2 \oplus k\alpha \oplus k\varepsilon_2\alpha.$$

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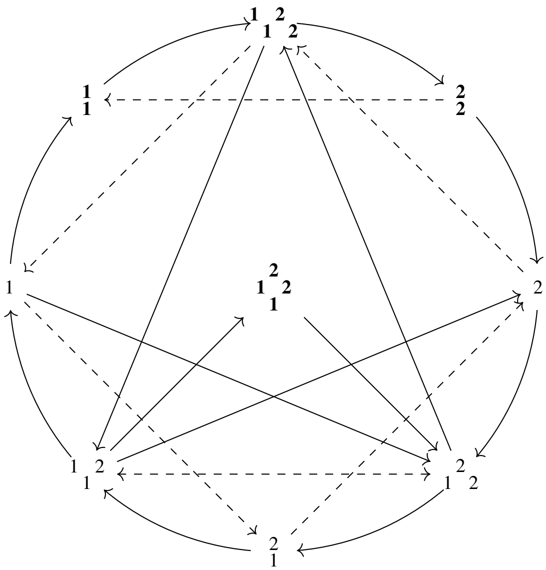
Figure 6.1 Auslander–Reiten quiver of Λ



The algebra Λ is representation finite and has precisely nine indecomposable modules. The Auslander–Reiten quiver of Λ is shown in Figure 6.1. The vertices represent the indecomposables via their composition series. There is a solid arrow $X \longrightarrow Y$ if there is an irreducible morphism, and a dashed arrow $X \dashrightarrow Y$ when $Y = D\,\mathrm{Tr}\,X$. The modules of finite projective and injective dimension have bold dimension vectors. The indecomposable Gorenstein projectives and Gorenstein injectives are given by

$$\operatorname{Gproj}\Lambda = \left\{ 1,\ \substack{1\\1},\ \substack{2\\1},\ \substack{1\ \ 2\\1},\ \substack{2\\1\ \ 2\\1} \right\} \quad \text{and} \quad \operatorname{Ginj}\Lambda = \left\{ 2,\ \substack{2\\2},\ \substack{2\\1},\ \substack{2\\1\ \ 2},\ \substack{2\\1\ \ 2\\1} \right\}.$$

A module belongs to all three classes if and only if it is projective and injective; there is a unique indecomposable with this property.

Notes

The ubiquity of Gorenstein rings was pointed out in a seminal article by Bass [22]. The terminology goes back to Grothendieck who studied duality phenomena for Gorenstein schemes, as explained in notes by Hartshorne [106].

For non-commutative rings it was Iwanaga who proposed to call a ring Gorenstein when it is of finite self-injective dimension on both sides [116]. The lemma that justifies the dimension of a Gorenstein ring is due to Zaks [202].

The decomposition theorem and the existence of approximations for modules over Gorenstein rings were established by Auslander and Buchweitz [13, 44]. The notion of a cotorsion pair provides a convenient language; it was introduced by Salce [179] in the context of abelian groups and we refer to work of Beligiannis and Reiten [28] for a comprehensive study in more general contexts.

For a Gorenstein ring the stable category of Gorenstein projective modules is discussed extensively in notes by Buchweitz [44]. In this work he introduced the singularity category (under the name 'stabilised derived category') that was later rediscovered by Orlov in the geometric context [152]. The example of a module M such that $\mathrm{Ext}^1(M,-)$ has a direct summand not given by a direct summand of M is due to Auslander [8]. This example shows that the singularity category need not be idempotent complete; see [153] for a geometric analysis of this phenomenon.

Serre duality for the derived category of a finite dimensional algebra of finite global dimension appears implicitly in Happel's work [101], while the notion of a Serre functor was introduced by Bondal and Kapranov [38] formalising Serre's duality for algebraic varieties [189]. In fact, Happel showed that the derived category of a finite dimensional algebra has Auslander–Reiten triangles if and only if the algebra has finite global dimension [103], while Reiten and Van den Bergh showed for any triangulated category that the existence of Auslander–Reiten triangles is equivalent to the existence of a Serre functor [168]. The Nakayama functor was introduced by Gabriel [80]; it is the categorical analogue of the Nakayama automorphism that permutes the isomorphism classes of simple modules over a self-injective algebra.

Auslander–Reiten duality based on the dual of transpose $D\,\mathrm{Tr}$ was initiated for Artin algebras in [15] by Auslander and Reiten and then extended to more general settings in [11]. For Gorenstein projective modules the Auslander–Reiten theory was developed in [17]. In the context of modular representations of finite groups a classical version of Serre duality is due to Tate [46].

Gentle algebras were introduced by Assem and Skowroński [6]. The fact that gentle algebras are Gorenstein is due to Geiß and Reiten [90]; the proof given here was suggested by Plamondon, with a modification by Briggs and Bennett-Tennenhaus. For the noetherianness of gentle algebras, see [60].