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Artinianness of Composed Graded Local Cohomology Modules

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Abstract. Let $R = \bigoplus_{n\geq 0} R_n$ be a graded Noetherian ring with local base ring $(R_0, \mathfrak{m}_0)$ and let $R_+ = \bigoplus_{n>0} R_n$. Let M and N be finitely generated graded R-modules and let $\mathfrak{a} = \mathfrak{a}_0 + R_+$ an ideal of R. We show that $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ and $H^i_{\mathfrak{a}}(M,N)/\mathfrak{b}_0 H^i_{\mathfrak{a}}(M,N)$ are Artinian for some i s and j s with a specified property, where $\mathfrak{b}_o$ is an ideal of R_0 such that $\mathfrak{a}_0 + \mathfrak{b}_0$ is an $\mathfrak{m}_0$-primary ideal.

1 Introduction

Throughout this paper, we assume that $R = \bigoplus_{n\geq 0} R_n$ is a graded Noetherian ring with local base ring $(R_0, \mathfrak{m}_0)$. In addition, we use $\mathfrak{a}_0$ and $\mathfrak{b}_0$ to denote two proper ideals of R_0 such that $\mathfrak{a}_0 + \mathfrak{b}_0$ is an $\mathfrak{m}_0$-primary ideal. We set $R_+ = \bigoplus_{n>0} R_n$, the irrelevant ideal of R, $\mathfrak{a} = \mathfrak{a}_0 + R_+$, and $\mathfrak{m} = \mathfrak{m}_0 + R_+$. Also, we use M, N to denote non-zero, finitely generated, graded R-modules. It is well known that, for each $i \in \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of all non-negative integers), the i-th generalized local cohomology module $H^i_{\mathfrak{a}}(M,N)$ of M and N with respect to $\mathfrak{a}$ inherits natural grading. For each $n \in \mathbb{Z}$ (where $\mathbb{Z}$ denotes the set of integers), we use the notation $H^i_{\mathfrak{a}}(M,N)_n$ to denote the n-th graded component of $H^i_{\mathfrak{a}}(M,N)$. Then, according to [7], for each $i \geq 0$, the R_0-module $H^i_{\mathfrak{a}}(M,N)_n$ is finitely generated in certain cases and vanishes for all $n \gg 0$. Therefore, the asymptotic behavior of $H^i_{\mathfrak{a}}(M,N)_n$ when $n \to -\infty$ holds a lot of interest.

The concept of tameness is the most fundamental concept related to the asymptotic behavior of cohomology modules. A graded R-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is said to be *tame* or *asymptotic gap-free* ([2, Definition 4.1]) if either $T_n = 0$ for all $n \ll 0$ else $T_n \neq 0$ for all $n \ll 0$. It is well known that any graded Artinian R-module is tame [1, Remark 4.2]. In this paper, we study the Artinianness of modules $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ and $H^i_{\mathfrak{a}}(M,N)/\mathfrak{b}_0 H^i_{\mathfrak{a}}(M,N)$ for some i s and j s with a specified property. At first we show that if t is smallest positive integer such that $H^t_{\mathfrak{a}}(M,N)$ is not Artinian, then $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite and Artinian for all $i < t$ (see Theorem 2.2). We also prove that if $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all $i < r$, then $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ is Artinian $\mathfrak{a}$-cofinite for all $i < r$ and all $j \geq 0$. Moreover, $\Gamma_{\mathfrak{b}_0 R}(H^r_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{a}$-cofinite and $H^i_{\mathfrak{a}}(M,N)/\mathfrak{m} H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \leq r$ (see Corollaries 2.4 and 2.6). The generalized homological finite length dimension and cohomological dimension of M and N with respect to $\mathfrak{a}$ is denoted by $g_{\mathfrak{a}}(M,N)$ and $cd_{\mathfrak{a}}(M,N)$), respectively.

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Also, $q_{\mathfrak{a}}(M,N)$ is the largest non-negative integer i such that $H^i_{\mathfrak{a}}(M,N)$ is not Artinian. We show that $H^i_{\mathfrak{a}}(M,N)/\mathfrak{b}_0 H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \geq q_{\mathfrak{a}}(M,N)$ and $\Gamma_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ is tame for all $i \leq g_{\mathfrak{a}}(M,N)$ (see Theorems 2.8 and 2.9). Furthermore, we prove that if $cd_{\mathfrak{a}}(M,N) = 2$, then $H^i_{\mathfrak{b}_0}(H^2_{\mathfrak{a}}(M,N))$ is Artinian if and only if $H^{i+2}_{\mathfrak{b}_0}(H^1_{\mathfrak{a}}(M,N))$ is Artinian (see Theorem 2.13).

For notation and terminology not given in this paper, the reader is referred to [3,4], if necessary.

2 Main Results

We keep the notation and hypotheses given in the introduction and continue with the following definition.

Definition 2.1 (i) An R-module T is said to be $\mathfrak{a}$-cofinite if Supp $T \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, T)$ is finitely generated R-module for all $i \geq 0$.

(ii) For a graded ideal $\mathfrak{a}$ in R, the generalized homological finite length dimension of N and M with respect to $\mathfrak{a}$ is defined as

$$g_{\mathfrak{a}}(M,N) = \inf\left\{ i \in \mathbb{N}_0 \mid \ell_{R_0} H^i_{\mathfrak{a}}(M,N)_n = \infty \text{ for finitely many } n \in \mathbb{Z} \right\},$$

where we denote by $\ell_{R_0} T$ the length over R_0 of T for an R_0-module T. Also, the notation $q_{\mathfrak{a}}(M,N)$ is the largest non-negative integer i such that $H^i_{\mathfrak{a}}(M,N)$ is not Artinian R-module.

In addition, for an ideal $\mathfrak{a}$ in R, the cohomological dimension of M and N with respect to $\mathfrak{a}$ is denoted by $cd_{\mathfrak{a}}(M,N)$. Thus, $cd_{\mathfrak{a}}(M,N)$ is the largest non-negative integer i such that $H^i_{\mathfrak{a}}(M,N)$ is non-zero and finiteness dimension of M and N with respect to $\mathfrak{a}$, denoted $f_{\mathfrak{a}}(M,N)$, is defined by

$$f_{\mathfrak{a}}(M,N) = \inf\left\{ i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M,N) \text{ is not finitely generated} \right\}.$$

Theorem 2.2 *Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i < t$. Then $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all $i < t$.*

Proof We prove this by induction on $t \geq 0$. If $t = 0$, then the result is clear. Assume that $t > 0$, and the result holds for $t-1$. In view of [5, Corollary 2.3] and our hypotheses, in conjunction with the fact that $H^i_{\mathfrak{a}+\operatorname{Ann}(M)}(M,N) \cong H^i_{\mathfrak{a}}(M,N)$, we see that $\Gamma_{\mathfrak{a}}(N)$ is Artinian. Therefore,

$$\operatorname{Ext}^i_R\big(M, \Gamma_{\mathfrak{a}}(N)\big) \cong H^i_{\mathfrak{a}}(M,N)$$

is Artinian for all $i \geq 0$. From the exact sequence $0 \to \Gamma_{\mathfrak{a}}(N) \to N \to N/\Gamma_{\mathfrak{a}}(N) \to 0$, we get the long exact sequence

$$H^i_{\mathfrak{a}}\big(M, \Gamma_{\mathfrak{a}}(N)\big) \xrightarrow{\phi_i} H^i_{\mathfrak{a}}(M,N) \xrightarrow{\psi_i} H^i_{\mathfrak{a}}\big(M, N/\Gamma_{\mathfrak{a}}(N)\big) \xrightarrow{\lambda_i} H^{i+1}_{\mathfrak{a}}\big(M, \Gamma_{\mathfrak{a}}(N)\big)$$

for all $i \geq 0$. We split the above exact sequence into the following two exact sequences:

$$0 \longrightarrow \operatorname{im}\phi_i \longrightarrow H^i_{\mathfrak{a}}(M,N) \longrightarrow \operatorname{im}\psi_i \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{im}\psi_i \longrightarrow H^i_{\mathfrak{a}}\big(M, N/\Gamma_{\mathfrak{a}}(N)\big) \longrightarrow \operatorname{im}\lambda_i \longrightarrow 0.$$

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Note that $\operatorname{im}\phi_i$ and $\operatorname{im}\lambda_i$ are Artinian and finitely generated R-module. It follows that for all $i \geq 0$, $H^i_{\mathfrak{a}}(M,N)$ is Artinian and $\mathfrak{a}$-cofinite if and only if the same is true for $H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N))$. Hence, we assume that $\Gamma_{\mathfrak{a}}(N)=0$. Then the ideal $\mathfrak{a}$ contains an element x that avoids all members of $\operatorname{Ass} N$. Therefore, the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

induces a long exact sequence

$$H^{i-1}_{\mathfrak{a}}(M,N/xN) \longrightarrow H^i_{\mathfrak{a}}(M,N) \xrightarrow{x} H^i_{\mathfrak{a}}(M,N) \longrightarrow H^i_{\mathfrak{a}}(M,N/xN).$$

By using the above exact sequence in conjunction with the inductive hypothesis we see that the R-module $(0:_{H^i_{\mathfrak{a}}(M,N)}x)$ is Artinian and $\mathfrak{a}$-cofinite. Therefore, in view of [9, Theorem 4.1], $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite and Artinian. ∎

Theorem 2.3 *Let r be a non-negative integer and let X be an arbitrary R-module such that for all $n \in \mathbb{N}$, $\operatorname{Ext}^i_R(R/\mathfrak{a}^n,X)$ is finitely generated for any $i \leq r$. Let $H^i_{\mathfrak{a}}(M,X)$ be $\mathfrak{a}$-cofinite for all $i<r$. Then $H^i_{\mathfrak{a}}(M,X)/\mathfrak{a}H^i_{\mathfrak{a}}(M,X)$ and $\operatorname{Hom}(R/\mathfrak{a},H^i_{\mathfrak{a}}(M,X))$ are finitely generated for all $i \leq r$.*

Proof If $i<r$, then the conclusion is clear by [10, Corollary 1.2]. Thus, we consider the case where $i=r$. We argue by induction on r. If $r=0$, then $H^0_{\mathfrak{a}}(M,X) \cong \operatorname{Hom}(M,\Gamma_{\mathfrak{a}}(X))$. Hence, the result is true by the assumption as well as [3, Theorem 1.2.11] and [9, Theorem 2.1]. Now, inductively assume that $r>0$ and that the assertion has been proved for $r-1$. Since $H^i_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(X)) \cong \operatorname{Ext}^i_R(M,\Gamma_{\mathfrak{a}}(X))$, by using the exact sequence $0 \to \Gamma_{\mathfrak{a}}(X) \to X \to X/\Gamma_{\mathfrak{a}}(X) \to 0$ and our hypotheses, we have that $H^i_{\mathfrak{a}}(M,X)$ is $\mathfrak{a}$-cofinite for all $i<r$ if and only if $H^i_{\mathfrak{a}}(M,X/\Gamma_{\mathfrak{a}}(X))$ is $\mathfrak{a}$-cofinite for all $i<r$. On the other hand, $\operatorname{Hom}(R/\mathfrak{a},H^i_{\mathfrak{a}}(M,X/\Gamma_{\mathfrak{a}}(X))$ is finitely generated for all $i \leq r$ if and only if $\operatorname{Hom}(R/\mathfrak{a},H^i_{\mathfrak{a}}(M,X))$ is finitely generated for all $i \leq r$. Thus, we may assume that $\Gamma_{\mathfrak{a}}(X)=0$. Let E be an injective hull of X and put $L=E/X$. Then $\Gamma_{\mathfrak{a}}(E)=0$. Consequently, $\operatorname{Ext}^i_R(R/\mathfrak{a}^n,L) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}^n,X)$ and $H^i_{\mathfrak{a}}(M,L) \cong H^{i+1}_{\mathfrak{a}}(M,X)$ for all $i \geq 0$. Now the induction hypothesis yields that $\operatorname{Hom}(R/\mathfrak{a},H^{r-1}_{\mathfrak{a}}(M,L))$ and $H^{r-1}_{\mathfrak{a}}(M,L)/\mathfrak{a}H^{r-1}_{\mathfrak{a}}(M,L)$ are finitely generated, and hence

$$\operatorname{Hom}\big(R/\mathfrak{a},H^r_{\mathfrak{a}}(M,X)\big) \quad \text{and} \quad H^r_{\mathfrak{a}}(M,X)/\mathfrak{a}H^r_{\mathfrak{a}}(M,X)$$

are finitely generated. ∎

Corollary 2.4 *Let r be a non-negative integer. Let $H^i_{\mathfrak{a}}(M,N)$ be $\mathfrak{a}$-cofinite for all $i<r$. Then $H^i_{\mathfrak{a}}(M,N)/\mathfrak{m}H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \leq r$.*

Proof Using Theorem 2.3, $H^i_{\mathfrak{a}}(M,N)/\mathfrak{a}H^i_{\mathfrak{a}}(M,N)$ is finitely generated for all $i \leq r$. So, $R_0/\mathfrak{b}_0 \otimes H^i_{\mathfrak{a}}(M,N)/\mathfrak{a}H^i_{\mathfrak{a}}(M,N)$ is finitely generated for all $i \leq r$. Therefore, since the radical of annihilator of $H^i_{\mathfrak{a}}(M,N)/(\mathfrak{b}_0+\mathfrak{a})H^i_{\mathfrak{a}}(M,N)$ equals $\mathfrak{m}=\mathfrak{m}_0+R_+$, the R-module $H^i_{\mathfrak{a}}(M,N)/\mathfrak{m}H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \leq r$. This proves the claim. ∎

Theorem 2.5 *Let T be an $\mathfrak{a}$-torsion and $\mathfrak{a}$-cofinite module. Then $H^i_{\mathfrak{b}_0}(T)$ is Artinian and $\mathfrak{a}$-cofinite for all $i \geq 0$.*

Proof It is enough, in view of $H^i_{\mathfrak{b}_0}(T) \cong H^i_{\mathfrak{b}_0}(\Gamma_{\mathfrak{a}}(T)) \cong H^i_{\mathfrak{m}}(T)$, to show that the R-module $H^i_{\mathfrak{m}}(T)$ is Artinian and $\mathfrak{a}$-cofinite. We use induction on i. Since T is $\mathfrak{a}$-cofinite, $\Gamma_{\mathfrak{m}}(T)$ is Artinian and $\mathfrak{a}$-cofinite by [10, Corollary 1.8]. Thus, $T/\Gamma_{\mathfrak{m}}(T)$ is $\mathfrak{a}$-cofinite. Now suppose, inductively, that $i > 0$ and we have shown that $H^{i-1}_{\mathfrak{m}}(T')$ is Artinian and $\mathfrak{a}$-cofinite for any $\mathfrak{a}$-cofinite R-module T'. Now $H^i_{\mathfrak{m}}(T) \cong H^i_{\mathfrak{m}}(T/\Gamma_{\mathfrak{m}}(T))$ for $i > 0$. We can assume that $\Gamma_{\mathfrak{m}}(T) = 0$. Then $\mathfrak{m} \notin \operatorname{Ass}(T)$, and since the set $\operatorname{Ass}(T)$ is finite (see [10, Corollary 1.4]), we can, by prime avoidance, take an element $x \in \mathfrak{m} - \bigcup_{\mathfrak{p} \in \operatorname{Ass} T} \mathfrak{p}$. From the exact sequence

$$0 \longrightarrow T \xrightarrow{x} T \longrightarrow T/xT \longrightarrow 0,$$

we get that T/xT is $\mathfrak{a}$-cofinite. This yields the exact sequence

$$H^{i-1}_{\mathfrak{m}}(T/xT) \longrightarrow H^i_{\mathfrak{m}}(T) \xrightarrow{x} H^i_{\mathfrak{m}}(T) \longrightarrow H^i_{\mathfrak{m}}(T/xT).$$

One can deduce from the above exact sequence, by using the inductive hypothesis, that the R-module $(0 :_{H^i_{\mathfrak{m}}(T)} x)$ is Artinian and $\mathfrak{a}$-cofinite. It follows $H^i_{\mathfrak{m}}(T)$ is Artinian and $\mathfrak{a}$-cofinite by [9, Proposition 4.1]. ∎

Corollary 2.6 *Let r be a non-negative integer. Let $H^i_{\mathfrak{a}}(M,N)$ be $\mathfrak{a}$-cofinite for all $i < r$. Then $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{a}$-cofinite for all $i < r$ and $j \geq 0$. In addition, $\Gamma_{\mathfrak{b}_0 R}(H^r_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{a}$-cofinite.*

Proof If $i < r$, then, in view of Theorem 2.5, $H^j_{\mathfrak{b}_0 R}(H^i_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{a}$-cofinite for all $j \geq 0$. On the other hand, using Theorem 2.3, $\operatorname{Hom}(R/\mathfrak{a}, H^r_{\mathfrak{a}}(M,N))$ is finitely generated. This fact implies that

$$\begin{aligned}\Gamma_{\mathfrak{m}_0 R}(\operatorname{Hom}(R/\mathfrak{a}, H^r_{\mathfrak{a}}(M,N))) &\cong \Gamma_{\mathfrak{m}_0 R}(0 :_{H^r_{\mathfrak{a}}(M,N)} \mathfrak{a}) \\ &\cong (0 :_{\Gamma_{\mathfrak{m}_0 R}(H^r_{\mathfrak{a}}(M,N))} \mathfrak{a}) \cong (0 :_{\Gamma_{\mathfrak{b}_0 R}(H^r_{\mathfrak{a}}(M,N))} \mathfrak{a})\end{aligned}$$

has finite length, by [3, Theorem 7.1.3]. Now, it follows from [9, Proposition 4.1] that $\Gamma_{\mathfrak{b}_0 R}(H^r_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{a}$-cofinite. ∎

Proposition 2.7 *Let $i \geq 0$. Then the R-modules $H^i_{\mathfrak{a}}(\Gamma_{\mathfrak{b}_0}(N))$ and $H^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{b}_0}(N))$ are Artinian and tame.*

Proof Using [3, Theorem 7.1.3], $H^i_{\mathfrak{a}}(\Gamma_{\mathfrak{b}_0}(N)) \cong H^i_{\mathfrak{a}+\mathfrak{b}_0}(\Gamma_{\mathfrak{b}_0}(N)) \cong H^i_{\mathfrak{m}}(\Gamma_{\mathfrak{b}_0}(N))$ is Artinian. In view of [6, Theorem 2.1], $H^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{b}_0}(N))$ is Artinian and tame. ∎

Theorem 2.8 *Let $i \geq q_{\mathfrak{a}}(M,N) = q$. Then $H^i_{\mathfrak{a}}(M,N)/\mathfrak{b}_0 H^i_{\mathfrak{a}}(M,N)$ is Artinian and tame.*

Proof When $i > q_{\mathfrak{a}}(M,N)$, the result is obvious by the definition of $q_{\mathfrak{a}}(M,N)$. So, it only remains to show that $H^q_{\mathfrak{a}}(M,N)/\mathfrak{b}_0 H^q_{\mathfrak{a}}(M,N)$ is an Artinian R-module. We prove the result by induction on $d = \dim N$. If $d = 0$, then $H^i_{\mathfrak{a}}(N) = H^i_{\mathfrak{a}}(\Gamma_{\mathfrak{m}}(N)) = H^i_{\mathfrak{m}}(N)$ is Artinian for all $i \geq 0$. As a result of [6, Theorem 2.1], $H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \geq 0$, and there is nothing to prove. So, suppose that $d > 0$ and that the result has been proved for $d - 1$. In view of the long exact sequence of generalized local cohomology modules that is induced by the exact sequence $0 \to \Gamma_{\mathfrak{b}_0}(N) \to N \to$

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$N/\Gamma_{\mathfrak{b}_0}(N) \to 0$ and Proposition 2.7, we have $q_{\mathfrak{a}}(M,N) = q_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{b}_0}(N))$. Now, consider the exact sequence

$$H^i_{\mathfrak{a}}\big(M,\Gamma_{\mathfrak{b}_0}(N)\big) \xrightarrow{\psi} H^i_{\mathfrak{a}}(M,N) \longrightarrow H^i_{\mathfrak{a}}\big(M,N/\Gamma_{\mathfrak{b}_0}(N)\big) \xrightarrow{\phi} H^{i+1}_{\mathfrak{a}}\big(M,\Gamma_{\mathfrak{b}_0}(N)\big),$$

which induces the following two exact sequences

$$0 \longrightarrow \operatorname{im}\psi \longrightarrow H^i_{\mathfrak{a}}(M,N) \longrightarrow \ker\phi \longrightarrow 0,$$
$$0 \longrightarrow \ker\phi \longrightarrow H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{b}_0}(N)) \longrightarrow \operatorname{im}\phi \longrightarrow 0.$$

Therefore, we can obtain the following two exact sequences:

$$(2.1)\qquad \longrightarrow R_0/\mathfrak{b}_0 \otimes \operatorname{im}\psi \longrightarrow R_0/\mathfrak{b}_0 \otimes H^i_{\mathfrak{a}}(M,N) \longrightarrow R_0/\mathfrak{b}_0 \otimes \ker\phi \longrightarrow 0,$$
$$(2.2)\quad R_0/\mathfrak{b}_0 \otimes \ker\phi \longrightarrow R_0/\mathfrak{b}_0 \otimes H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{b}_0}(N)) \longrightarrow R_0/\mathfrak{b}_0 \otimes \operatorname{im}\phi \longrightarrow 0.$$

In view of Proposition 2.7, $\operatorname{im}\psi$ and $\operatorname{im}\phi$ are Artinian, and hence so are $\operatorname{im}\psi/\mathfrak{b}_0 \operatorname{im}\psi$ and $\operatorname{im}\phi/\mathfrak{b}_0 \operatorname{im}\phi$. According to the exact sequences (2.1) and (2.2), we can easily conclude that $R_0/\mathfrak{b}_0 \otimes H^i_{\mathfrak{a}}(M,N)$ is Artinian if and only if $R_0/\mathfrak{b}_0 \otimes H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{b}_0}(N))$ is Artinian. We can assume that $\Gamma_{\mathfrak{b}_0}(N) = 0$. The last fact implies that there is an element $x \in \mathfrak{b}_0$ that is an N-sequence, and hence there is the following exact sequence of R-modules

$$(2.3)\qquad H^{i-1}_{\mathfrak{a}}(M,N/xN) \longrightarrow H^i_{\mathfrak{a}}(M,N) \xrightarrow{x} H^i_{\mathfrak{a}}(M,N) \longrightarrow H^i_{\mathfrak{a}}(M,N/xN).$$

Therefore, the exact sequence (2.3) yields $q_{\mathfrak{a}}(M,N/xN) \le q_{\mathfrak{a}}(M,N)$ and induces an exact sequence of R-modules and R-homomorphisms

$$(2.4)\qquad 0 \longrightarrow H^q_{\mathfrak{a}}(M,N)/xH^q_{\mathfrak{a}}(M,N) \longrightarrow H^q_{\mathfrak{a}}(M,N/xN) \xrightarrow{\lambda} H^{q+1}_{\mathfrak{a}}(M,N).$$

If we apply the functor $Tor^{R_0}_i(R_0/\mathfrak{b}_0,)$ to the exact sequence (2.4), we have the following exact sequence

$$Tor^{R_0}_1(R_0/\mathfrak{b}_0, \operatorname{im}\lambda) \longrightarrow R_0/\mathfrak{b}_0 \otimes H^q_{\mathfrak{a}}(M,N)/xH^q_{\mathfrak{a}}(M,N)$$
$$\longrightarrow R_0/\mathfrak{b}_0 \otimes H^q_{\mathfrak{a}}(M,N/xN) \longrightarrow \operatorname{im}\lambda \otimes R_0/\mathfrak{b}_0 \longrightarrow 0.$$

Since $\operatorname{im}\lambda$ is Artinian, it is seen that $\operatorname{Tor}^{R_0}_1(R_0/\mathfrak{b}_0, \operatorname{im}\lambda)$ is Artinian. If

$$q_{\mathfrak{a}}(M,N/xN) = q_{\mathfrak{a}}(M,N),$$

by using last exact sequence in conjunction with the inductive hypothesis and $x \in \mathfrak{b}_0$, we see that the R-module

$$H^q_{\mathfrak{a}}(M,N)/\mathfrak{b}_0 H^q_{\mathfrak{a}}(M,N) = R_0/\mathfrak{b}_0 \otimes H^q_{\mathfrak{a}}(M,N)/xH^q_{\mathfrak{a}}(M,N)$$

is Artinian and tame. If $q_{\mathfrak{a}}(M,N/xN) < q_{\mathfrak{a}}(M,N)$, then $H^q_{\mathfrak{a}}(M,N/xN)$ is Artinian. Again we can use the above exact sequence to obtain the result. ∎

Theorem 2.9 *Let $i \le g_{\mathfrak{a}}(M,N)$. Then $\Gamma_{\mathfrak{b}_0}(H^g_{\mathfrak{a}}(M,N))$ is tame. Furthermore, if $H^i_{\mathfrak{a}}(M,N)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$, then $\Gamma_{\mathfrak{b}_0}(H^g_{\mathfrak{a}}(M,N))$ is an Artinian R-module.*

Proof If $i < g_{\mathfrak{a}}(M,N)$, then in view of the definition of $g_{\mathfrak{a}}(M,N)$, $\ell_{R_0} H^i_{\mathfrak{a}}(M,N)_n$ is finite for all $n \ll 0$ and the result is clear. Consider the Grothendieck spectral sequence [10, Theorem 11.38]

$$(E_2^{p,i})_n = H^p_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N)_n) \overset{p}{\Longrightarrow} H^{p+i}_{\mathfrak{m}}(M,N)_n.$$

It is easy to see that there exists $n_0 \in \mathbb{Z}$ such that, for all $n < n_0$, $(E_2^{p,i})_n = 0$ for all $i < g_{\mathfrak{a}}(M,N)$ and $p \in \mathbb{N}$. Now, the convergence of the above spectral sequence implies that $H^0_{\mathfrak{b}_0}(H^g_{\mathfrak{a}}(M,N)_n) \cong H^g_{\mathfrak{m}}(M,N)_n$ for all $n < n_0$. Since all graded Artinian R-modules are tame, it is seen that $H^0_{\mathfrak{b}_0 R}(H^g_{\mathfrak{a}}(M,N))$ is tame. The last part of the theorem follows from Kirby's Artinian criterion [8, Theorem 1]. ∎

Theorem 2.10 *Let t be a non-negative integer and let $H^i_{\mathfrak{b}_0 R}(H^j_{\mathfrak{a}}(M,N))$ be Artinian for all $j \neq t$ and for all i. Then $H^i_{\mathfrak{b}_0 R}(H^t_{\mathfrak{a}}(M,N))$ is Artinian for all i.*

Proof Using [10, Theorem 11.38] there exists a Grothendieck spectral sequence

$$(E_2^{p,q})_n = H^p_{\mathfrak{b}_0}\big(H^q_{\mathfrak{a}}(M,N)_n\big) \overset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N)_n.$$

Also, there is a bounded filtration

$$0 = \phi^{n+1}H^n \subseteq \phi^n H^n \subseteq \cdots \subseteq \phi^1 H^n \subseteq \phi^0 H^n = H^n_{\mathfrak{m}}(M,N)$$

such that $E_\infty^{i,n-i} = \phi^i H^n / \phi^{i+1} H^n$ for all $0 \le i \le n$, and hence $E_\infty^{p,q}$ is Artinian. Note that $E_\infty^{p,q} = E_r^{p,q}$ for large r and each p and q. It follows that there is an integer $\ell \ge 2$ such that $E_r^{p,q}$ is Artinian for all $r \ge \ell$. We now argue by descending induction on ℓ. Assume that $2 < \ell < r$ and that the claim holds for ℓ. Since $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $p, q \in \mathbb{N}_0$, the hypotheses give that $E_r^{p+r,t-r+1}$ is Artinian for all $r \ge 2$. In addition,

$$E_\ell^{p,t} = \ker d_{\ell-1}^{p,t} / \operatorname{im} d_{\ell-1}^{p-\ell+1,t+\ell-2}$$

and $\operatorname{im} d_{\ell-1}^{p-\ell+1,t+\ell-2}$ are Artinian for all $p \ge 0$. It follows that $\ker d_{\ell-1}^{p,t}$ is Artinian for all $\ell > 2$ and $p \ge 0$. Let $r \ge 2$ and $p \ge 0$. We consider the sequence

$$0 \longrightarrow \ker d_r^{p,t} \longrightarrow E_r^{p,t} \longrightarrow E_r^{p+r,t-r+1}.$$

Since both $\ker d_{\ell-1}^{p,t}$ and $E_{\ell-1}^{p+\ell-1,t-\ell+2}$ are Artinian, it follows that $E_{\ell-1}^{p,t}$ is Artinian for $p \ge 0$. This completes the inductive step. ∎

Proposition 2.11 *Let $f = f_{\mathfrak{a}}(M,N) = cd_{\mathfrak{a}}(M,N)$. Then $H^j_{\mathfrak{b}_0 R}(H^i_{\mathfrak{a}}(M,N))$ is Artinian for all i and j.*

Proof If $i < f$, then, in view of the definition of $f_{\mathfrak{a}}(M,N)$, $H^i_{\mathfrak{a}}(M,N)$ is an $\mathfrak{a}$-cofinite R-module. It follows from Theorem 2.5 that $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{a}$-cofinite. On the other hand $H^j_{\mathfrak{b}_0}(H^i_{\mathfrak{a}}(M,N)) = 0$ for all $i > f$. Therefore, in view of the spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{b}_0}(H^q_{\mathfrak{a}}(M,N)) \overset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N),$$

the result follows by similar argument as used in Theorem 2.10. ∎

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As an application of Proposition 2.11, we have the following corollary.

Corollary 2.12 *Let $cd_{\mathfrak{a}}(M,N) = 1$. Then $H^{j}_{\mathfrak{b}_0 R}(H^{i}_{\mathfrak{a}}(M,N))$ is Artinian for all i and j.*

Theorem 2.13 *Let $i \in \mathbb{N}_0$ and $cd_{\mathfrak{a}}(M,N) = 2$. Then $H^{i}_{\mathfrak{b}_0}(H^{2}_{\mathfrak{a}}(M,N))$ is an Artinian R-module if and only $H^{i+2}_{\mathfrak{b}_0}(H^{1}_{\mathfrak{a}}(M,N))$ is an Artinian R-module.*

Proof Using [10, Theorem 11.38] there exists a Grothendieck spectral sequence

$$E_2^{p,q} = H^{p}_{\mathfrak{b}_0}\big(H^{q}_{\mathfrak{a}}(M,N)\big) \overset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N).$$

Also, there is a bounded filtration

$$0 = \phi^{n+1}H^n \subseteq \phi^{n}H^n \subseteq \cdots \subseteq \phi^{1}H^n \subseteq \phi^{0}H^n = H^{p+q}_{\mathfrak{m}}(M,N)$$

such that $E_\infty^{i,n-i} \cong \phi^i H^n/\phi^{i+1}H^n$ for all i, and hence $E_\infty^{p,q}$ is Artinian for all p,q. Note that $E_\infty^{p,q} = E_r^{p,q}$ for large r and each p and q. For all $r \geq 2$ and $p,q \geq 0$, we consider the exact sequence

$$0 \longrightarrow \ker d_r^{p,q} \longrightarrow E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r,q-r+1} \xrightarrow{d_r^{p+r,q-r+1}} . \tag{2.5}$$

In view of the definition of $cd_{\mathfrak{a}}(M,N)$, $(E_2^{p,q})_n = 0$ for all $n \in \mathbb{Z}$ and $q > 2$. On the other hand, $E_{r+1}^{p,q} \cong \ker d_r^{p,q}/\operatorname{im} d_r^{p-r,q+r-1}$ imply that $E_{r+1}^{i,r} \cong \ker d_r^{i,2}$ and

$$E_2^{i+2,1}/\operatorname{im} d_2^{i,2} \cong \ker d_2^{i+2,1}/\operatorname{im} d_2^{i,2} \cong E_3^{i+2,1}.$$

Now, we use the exact sequence (2.5) to obtain exact sequences

$$0 \longrightarrow (E_\infty^{i,2})_n \longrightarrow H^{i}_{\mathfrak{b}_0 R}\big(H^{2}_{\mathfrak{a}}(M,N)\big)_n \longrightarrow \operatorname{im}(d_2^{i,2})_n \longrightarrow 0, \tag{2.6}$$

$$0 \longrightarrow \operatorname{im}(d_2^{i,2})_n \longrightarrow H^{i+2}_{\mathfrak{b}_0 R}\big(H^{1}_{\mathfrak{a}}(M,N)\big)_n \longrightarrow (E_\infty^{i+2,1})_n \longrightarrow 0, \tag{2.7}$$

which in turn yield the exact sequences

$$0 \longrightarrow (0:_{(E_\infty^{i,2})_n} R_1) \longrightarrow (0:_{H^{i}_{\mathfrak{b}_0 R}(H^{2}_{\mathfrak{a}}(M,N))_n} R_1) \longrightarrow (0:_{\operatorname{im}(d_2^{i,2})_n} R_1),$$

$$0 \longrightarrow (0:_{\operatorname{im}(d_2^{i,2})_n} R_1) \longrightarrow (0:_{H^{i+2}_{\mathfrak{b}_0 R}(H^{1}_{\mathfrak{a}}(M,N))_n} R_1) \longrightarrow (0:_{(E_\infty^{i+2,1})_n} R_1).$$

Note that for each $i, j \in \mathbb{N}_0$, $E_\infty^{i,j}$ is an Artinian graded R-module. Therefore, using Kirby's Artinian criterion ([8, Theorem 1]), we deduce that

$$\big(0:_{(E_\infty^{i+2,1})_n} R_1\big) = 0 = \big(0:_{(E_\infty^{i,2})_n} R_1\big)$$

for $n \ll 0$. Now, we can use the last two displayed exact sequences to see that $(0:_{H^{i}_{\mathfrak{b}_0}(H^{2}_{\mathfrak{a}}(M,N))_n} R_1) = 0$ for all $n \ll 0$ if and only if $(0:_{H^{i+2}_{\mathfrak{b}_0}(H^{1}_{\mathfrak{a}}(M,N))_n} R_1) = 0$ for all $n \ll 0$. In addition, since

$$H^{i}_{\mathfrak{b}_0 R}\big(H^{j}_{\mathfrak{a}}(M,N)\big)_n \cong H^{i}_{\mathfrak{b}_0}\big(H^{j}_{\mathfrak{a}}(M,N)_n\big)$$

for all $i \geq 0$ and all $n \in \mathbb{Z}$, then $H^{i}_{\mathfrak{b}_0 R}(H^{j}_{\mathfrak{a}}(M,N))_n = 0$ for all $n \gg 0$. Again, using the fact that $E_\infty^{i,j}$ is an Artinian graded R-module, together with exact sequences (2.6) and (2.7), we see that $H^{i+2}_{\mathfrak{b}_0 R}(H^{1}_{\mathfrak{a}}(M,N))_n$ is an Artinian R_0-module for all $n \in \mathbb{Z}$ if and only if $H^{i}_{\mathfrak{b}_0 R}(H^{2}_{\mathfrak{a}}(M,N))_n$ is an Artinian R_0-module for all $n \in \mathbb{Z}$. Therefore, in view of [8, Theorem 1], the result follows. ∎

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References

[1] M. P. Brodmann, *Asymptotic behaviour of cohomology: tameness, supports and associated primes.* In: Commutative algebra and algebraic geometry, Contemp. Math., 390, American Mathematical Society, Providence, RI, 2005, pp. 31–61. http://dx.doi.org/10.1090/conm/390/07292

[2] M. P. Brodmann and M. Hellus, *Cohomological pattern of coherent sheaves over projective schemes.* J. Pure Appl. Algebra **172**(2002), no. 2–3, 165–182. http://dx.doi.org/10.1016/S0022-4049(01)00144-X

[3] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*. Cambridge Studies in Mathematics, 60, Cambridge University Press, Cambridge, 1998. http://dx.doi.org/10.1017/CBO9780511629204

[4] W. Bruns and J. Herzog, *Cohen-Macaulay rings.* Cambridge Studies in Mathematics, 39, Cambridge University Press, Cambridge, 1993.

[5] L.Chu and Z. Tang, *On the Artinianness of generalized local cohomology.* Comm. Algebra **35**(2007), no 12, 3821–3827. http://dx.doi.org/10.1080/00927870701511517

[6] F. Dehghani-Zadeh, *Finiteness properties generalized local cohomology with respect to an ideal containing the irrelevant ideal.* J. Korean Math. Soc. **49**(2012), no. 6, 1215–1227. http://dx.doi.org/10.4134/JKMS.2012.49.6.1215

[7] M. Jahangiri, N. Shirmohammadi, and Sh. Tahamtan, *Tameness and Artinianness of graded generalized local cohomology modules.* Algebra Colloq. **22**(2015), no. 1, 131–146. http://dx.doi.org/10.1142/S1005386715000127

[8] D. Kirby, *Artinian modules and Hilbert polynomials.* Quart. J. Math. Oxford Ser. (2) **24**(1973), 47–57.

[9] L. Melkerson, *Modules cofinite with respect to an ideal.* J. Algebra, **285**(2005), no. 2, 649–668. http://dx.doi.org/10.1016/j.jalgebra.2004.08.037

[10] ———, *Properties of cofinite modules and applications to local cohomology.* Math. Proc. Cambridge Philos. Soc. **125**(1999), no. 3, 417–423. http://dx.doi.org/10.1017/S0305004198003041

[11] J. J. Rotman, *An introduction to homological algebra.* Pure and Applied Mathematics, 85, Academic Press, New York-London, 1979.

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