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Functional Equations and Fourier Analysis

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Abstract. By exploring the relations among functional equations, harmonic analysis and representation theory, we give a unified and very accessible approach to solve three important functional equations — the d'Alembert equation, the Wilson equation, and the d'Alembert long equation — on compact groups.

Recently, three important equations — the d'Alembert equation, the Wilson equation, and the d'Alembert long equation — have been attracting a great deal of attention. (See [1–6, 8–10] and the references therein.) It turns out that their solutions have very nice structures. In particular, their solutions on compact groups were obtained as consequences of the main results of [2], where a much more general class of functional equations was studied.

Because of the increasing importance of these three equations, it is worthwhile giving a much more transparent approach. This is the first main purpose of this short note: We shall give a unified, transparent, and very accessible approach to solving these equations on compact groups.

The second main purpose is, as in [1, 2, 9, 10], to explore the relations among different areas whenever possible, such as the areas of functional equations, harmonic analysis, and representation theory considered here.

1 Preliminaries

In this section, we set up some notation and conventions, briefly review some fundamental facts in Fourier analysis which will be used later, and introduce the functional equations we shall be concerned with.

1.1 Fourier Analysis

Let G be a compact group with the normalized Haar measure dx. Let $\widehat{G}$ stand for the set of equivalence classes of irreducible unitary representations of G. For $[\pi] \in \widehat{G}$, the notation d_π denotes the dimension of the representation space of π. For $f \in L^2(G)$, the *Fourier transform* of f is defined by

$$\hat{f}(\pi) = d_\pi \int_G f(x)\pi(x)^{-1}\, dx \in \mathbb{M}_{d_\pi}(\mathbb{C}) \quad \text{for all } [\pi] \in \widehat{G},$$

where $\mathbb{M}_n(\mathbb{C})$ is the space of all $n \times n$ complex matrices.

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As usual, the left and right regular representations of G in $L^2(G)$ are defined by $(L_y f)(x) = f(y^{-1}x)$ and $(R_y f)(x) = f(xy)$, respectively, where $f \in L^2(G)$ and $x, y \in G$. A crucial property of the Fourier transform is that it converts the regular representations of G into matrix multiplications.

The following facts will be useful later.

(i) The *Fourier inversion formula* is given by

$$f(x) = \sum_{[\pi]\in\widehat{G}} \operatorname{tr}(\hat{f}(\pi)\pi(x)) \quad \text{for all} x \in G.$$

(ii) The following identities hold:

$$(L_y f)\hat{}(\pi) = \hat{f}(\pi)\pi(y)^{-1}, \quad (R_y f)\hat{}(\pi) = \pi(y)\hat{f}(\pi)$$

for all $y \in G$ and $\pi \in \hat{G}$.

For more information about the topics of this subsection, see [7, Chapter 5].

1.2 Functional Equations

Let G be a compact group, and f, g be complex functions on G. In this note, we study the following functional equations:

$$f(xy) + f(xy^{-1}) = 2f(x)f(y), \tag{1.1}$$

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \tag{1.2}$$

$$f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y). \tag{1.3}$$

These are called the *d'Alembert equation*, the *Wilson equation*, and the *d'Alembert long equation*, respectively. It is known that those equations are closely related to each other. For example, (1.1) is a special case of (1.2); (1.3) becomes (1.1) if f in (1.3) is central, *i.e.*, $f(xy) = f(yx)$ for all $x, y \in G$; and if (f, g) with $f \neq 0$ satisfies (1.2), then g also satisfies (1.3).

Recently much effort has been put into solving those equations. It turns out that their solutions have very nice and interesting structures. See [1,5,6] for more details.

Remark 1.1 Throughout the rest of this note, the group G is always assumed to be compact. By solutions (resp. representations), we always mean continuous solutions (resp. continuous representations).

2 Small Dimension Lemma

We will prove a very useful lemma in this section, which may be of independent interest. The lemma, roughly speaking, says that for an *irreducible* representation π of a compact group G, if the operators $\pi(x) + \pi(x)^{-1}$ for all $x \in G$ have a common nonzero eigenvector, then the dimension d_π of π has to be rather small: $d_\pi \leq 2$. For obvious reasons, we call this the *Small Dimension Lemma*. In the next section, we will apply it to give a unified approach to solving (1.1), (1.2), and (1.3).

Small Dimension Lemma *Let G be a compact group, and $\pi\colon G \to U(n)$ an irreducible representation of G. Suppose that there exists a nonzero vector $v \in \mathbb{C}^n$ such that*

$$(\pi(x) + \pi(x)^{-1})v \in \mathbb{C}v \quad \textit{for all } x \in G. \tag{2.1}$$

Then either $n = 1$ and π is a unitary group character, or $n = 2$ and $\pi(G) \subseteq SU(2)$.

Proof For $x \in G$, let $\pi_{ij}(x) = \langle \pi(x)e_j, e_i\rangle$ denote the (i, j)-th entry of $\pi(x)$. Consider the subspaces $\mathcal{E}_i$ of $L^2(G)$:

$$\mathcal{E}_i = \operatorname{span}\{\pi_{ij} \mid j = 1, \dots, n\}, \quad i = 1, \dots, n.$$

Let $R|_{\mathcal{E}_i}$ be the restriction of the right regular representation R of G to $\mathcal{E}_i$. Then $R|_{\mathcal{E}_i}$ is equivalent to π. Indeed, let $U\colon \mathcal{E}_i \to \mathbb{C}^n$ be the unitary operator defined via $U(\pi_{ij}) = e_j$; then one can easily check that $U^*\pi(\,\cdot\,)U = R|_{\mathcal{E}_i}(\,\cdot\,)$. In particular, $R|_{\mathcal{E}_i}$ is irreducible as π is irreducible. Thus,

$$\mathcal{E}_i = \operatorname{span}\{R_x\pi_{ij} \mid x \in G\} \quad \text{for any } j = 1, \dots, n.$$

After a similarity, we may assume that the nonzero vector v in (2.1) is given by $v = (1, 0, \dots, 0)^t$, where t denotes transpose. From condition (2.1), it follows that

$$\pi_{i1} = -\overline{\pi_{1i}} \quad \text{for all } i = 2, \dots, n. \tag{2.2}$$

Here $\overline{\pi_{1i}}$ denotes the complex conjugate of π_{1i}. Hence we have

$$\begin{aligned}\mathcal{E}_i &= \operatorname{span}\{R_x\pi_{i1} \mid x \in G\} = \operatorname{span}\{R_x\overline{\pi_{1i}} \mid x \in G\}\\ &= \operatorname{span}\{\overline{R_x\pi_{1i}} \mid x \in G\} = \overline{\mathcal{E}}_1\end{aligned}$$

for each $i = 2, \dots, n$. If $n > 2$, then $\mathcal{E}_2 = \mathcal{E}_3 = \cdots = \mathcal{E}_n$. This is impossible by Schur's orthogonality relations (see [7]). Thus $n \leq 2$.

In the case $n = 1$, π is, of course, a unitary group character. In the case $n = 2$, it follows from $\pi_{21} = -\overline{\pi_{12}}$ in (2.2) that $\pi(x) = \left(\begin{smallmatrix} a & b \\ -\bar{b} & c\end{smallmatrix}\right)$. As $\pi(x)$ is a unitary operator, some simple calculations give either $\pi(x) = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right) \in SU(2)$ with $b \neq 0$, or $\pi(x) = \left(\begin{smallmatrix} a & 0 \\ 0 & c\end{smallmatrix}\right)$. But the product of any such two elements should also be one of the forms. This forces that either $\pi(x) \in SU(2)$ for all $x \in G$, or $\pi(x) \in \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & c\end{smallmatrix}\right) \in U(2)\right\}$ for all $x \in G$. Since π is irreducible, we necessarily have $\pi(G) \subseteq SU(2)$. ∎

3 Solving Functional Equations on Compact Groups

In this section, we shall apply the Small Dimension Lemma from Section 2 to solve the d'Alembert equation, the Wilson equation, and the d'Alembert long equation on compact groups.

The idea behind the method here is the following. We first convert the functional equations in hand into matrix equations by taking Fourier transforms; then we invoke the Small Dimension Lemma, so that those matrix equations become very easy

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to handle. Finally, if necessary, we apply the Fourier inversion formula to obtain the solutions of the functional equations.

Before solving our functional equations, we first give a simple identity which will be frequently used in the sequel. If $\pi\colon G \to SU(2)$ is a representation, then

$$\pi(x) + \pi(x)^{-1} = \chi_\pi(x) \quad \text{for all } x \in G, \tag{3.1}$$

where χ_π denotes the character of π: $\chi_\pi(x) = \operatorname{tr}(\pi(x))$ for all $x \in G$.

Theorem 3.1 *Suppose f is a nonzero solution of the d'Alembert equation* (1.1). *Then there is a representation $\varphi\colon G \to SU(2)$ such that*

$$f(x) = \frac{\chi_\varphi(x)}{2} \quad \textit{for all } x \in G.$$

Proof Suppose that f satisfies (1.1). Rewrite (1.1) as

$$R_y f + R_{y^{-1}} f = 2f(y)f \quad \text{for all } y \in G.$$

Taking the Fourier transform to the above equation and using the identities given in Subsection 1.1, we have

$$(\pi(y) + \pi(y)^{-1})\hat{f}(\pi) = 2f(y)\hat{f}(\pi). \tag{3.2}$$

Since $f \not\equiv 0$, there exists $[\pi] \in \widehat{G}$ with $\hat{f}(\pi) \neq 0$. Applying the Small Dimension Lemma to a nonzero column of $\hat{f}(\pi)$, we conclude from (3.2) that either $d_\pi = 1$, or $d_\pi = 2$ and $\pi(G) \subseteq SU(2)$.

If $d_\pi = 1$, then (3.2) implies $f = \frac{1}{2}(\pi + \bar{\pi})$. Let φ be the direct sum of π and $\bar{\pi}$: $\varphi = \pi \oplus \bar{\pi}$. Then $f = \frac{\chi_\varphi}{2}$.

If $d_\pi = 2$ and $\pi(G) \subseteq SU(2)$, then substituting (3.1) into (3.2) gives $f = \frac{\chi_\pi}{2}$. Letting $\varphi := \pi$ ends our proof. ■

Remark 3.2 In complete contrast to the standard methods in the theory of functional equations, in the proof of Theorem 3.1 we did not use *any* property of the solution f, not even including the (probably) most important and crucial property of f — the centralness, that is, $f(xy) = f(yx)$ for all $x, y \in G$.

Theorem 3.3 *Suppose a* 2*-tuple* (f, g) *satisfies the Wilson equation* (1.2). *Then* (f, g) *is one of the following forms:*

(i) $f \equiv 0$ *and g arbitrary;*
(ii) *there is a representation $\varphi\colon G \to SU(2)$ and $A \in \mathbb{M}_2(\mathbb{C})$ so that*

$$f(x) = \operatorname{tr}(A\varphi(x)) \quad \textit{and} \quad g(x) = \frac{\chi_\varphi(x)}{2} \quad \textit{for all } x \in G.$$

Proof Suppose $f \not\equiv 0$. In what follows, we wish to show that the 2-tuple (f, g) is of the form given in (ii).

Since (1.2) is equivalent to $R_y f + R_{y^{-1}} f = 2g(y) f$ for all $y \in G$, we have

$$(\pi(y) + \pi(y)^{-1})\hat{f}(\pi) = 2g(y)\hat{f}(\pi) \tag{3.3}$$

by taking the Fourier transform.

If $[\pi] \in \operatorname{supp} \hat{f}$, as before, invoking the Small Dimension Lemma, we obtain from (3.3) that either $d_\pi = 1$, or $d_\pi = 2$ and $\pi(G) \subseteq SU(2)$. In the former case, π is a unitary group character, say $\pi = \chi^\pi$, and we deduce from (3.3) that $2g = \chi^\pi + \overline{\chi^\pi}$. For the latter case, it follows from (3.1) and (3.3) that $2g = \chi_\pi 4$.

Since $f \not\equiv 0$, there is $[\pi_0] \in \operatorname{supp} \hat{f}$. From the above analysis, there is either

- a unitary group character χ_0 such that $2g = \chi_0 + \overline{\chi_0}$, or
- a 2-dimensional irreducible representation π_0 with $\pi_0(G) \subseteq SU(2)$ such that $2g = \chi_{\pi_0}$.

Therefore, for a fixed g, if $[\pi] \in \operatorname{supp} \hat{f}$, we have simultaneously

$$2g = \begin{cases} \chi^\pi + \overline{\chi^\pi} & \text{or} \\ \chi_\pi \end{cases} \qquad \text{and} \quad 2g = \begin{cases} \chi_0 + \overline{\chi_0} & \text{or} \\ \chi_{\pi_0}. \end{cases}$$

By the linear independence of characters, there are only two possibilities: $[\pi] \in \{\chi_0, \overline{\chi_0}\}$, and $[\pi] = [\pi_0]$.

Now a simple application of the Fourier inversion formula ends the proof. ■

Some remarks are in order.

- Of course, letting $f = g$ in Theorem 3.3, we recover Theorem 3.1.
- As mentioned before, if (f, g) satisfies the Wilson equation, then in general g is a solution of the d'Alembert *long* equation. Theorem 3.3 implies that, on compact groups, g is actually a solution of the d'Alembert (short) equation.
- Once again, no properties of the solution are needed in the proof of Theorem 3.3.

We are now in position to solve the d'Alembert long equation on compact groups.

Theorem 3.4 *Suppose a nonzero function f satisfies the d'Alembert long equation* (1.3). *Then there is a representation $\varphi\colon G \to SU(2)$ such that*

$$f(x) = \frac{\chi_\varphi(x)}{2} \quad \textit{for all } x \in G.$$

As an immediate consequence of Theorem 3.4, we have the following result.

Corollary 3.5 *The d'Alembert (short) equation* (1.1) *and the d'Alembert long equation* (1.3) *are equivalent on compact groups.*

Proof of Theorem 3.4 Clearly, (1.3) is equivalent to

$$R_y f + R_{y^{-1}} f + L_y f + L_{y^{-1}} f = 4f(y) f \quad \text{for all } y \in G.$$

Let $[\pi] \in \hat{G}$. As before, taking the Fourier transform gives

$$(\pi(y) + \pi(y)^{-1})\hat{f}(\pi) + \hat{f}(\pi)(\pi(y) + \pi(y)^{-1}) = 4f(y)\hat{f}(\pi). \tag{3.4}$$

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For $[\pi] \in \widehat{G}$, we define an operator-valued function δ_π on G by

$$\delta_\pi(y) = \pi(y) + \pi(y)^{-1} - 2f(y)I_{d_\pi} \quad \text{for all } y \in G. \tag{3.5}$$

Then (3.4) is now equivalent to

$$\delta_\pi(y)\hat{f}(\pi) + \hat{f}(\pi)\delta_\pi(y) = 0 \quad \text{for all } y \in G. \tag{3.6}$$

In the sequel, we claim that

$$\delta_\pi(y)^2 = \delta_\pi(y^2) - 4f(y)\delta_\pi(y). \tag{3.7}$$

To this end, recall that

$$2f(y)^2 = f(y^2) + 1 \quad \text{for all } y \in G \tag{3.8}$$

(see [8]). It now follows that

$$\begin{aligned}
\delta_\pi(y)^2 &= \pi(y)^2 + \pi(y)^{-2} + 2I_{d_\pi} - 4f(y)(\pi(y) + \pi(y)^{-1}) \\
&\qquad + 4f(y)^2 I_{d_\pi} \text{ (by (3.5))} \\
&= (\pi(y^2) + \pi(y^2)^{-1} - 2f(y^2)I_{d_\pi}) - 4f(y)(\pi(y) + \pi(y)^{-1} - 2f(y)I_{d_\pi}) \\
&\qquad + 2(f(y^2) + 1 - 2f(y)^2) \\
&= \delta_\pi(y^2) - 4f(y)\delta_\pi(y)I_{d_\pi} \text{ (by (3.5) and (3.8))}.
\end{aligned}$$

This proves (3.7).

From (3.6) and (3.7), we arrive at $\delta_\pi(y)^2\hat{f}(\pi) + \hat{f}(\pi)\delta_\pi(y)^2 = 0$. On the other hand, using (3.6) twice gives $\delta_\pi(y)^2\hat{f}(\pi) = \hat{f}(\pi)\delta_\pi(y)^2$. Clearly, combining the above two identities implies

$$\delta_\pi(y)^2\hat{f}(\pi) = 0. \tag{3.9}$$

As $\pi(y)$ is a unitary operator and $f(y) \in \mathbb{R}$ (see [8, Proposition 2.10]), one has from (3.5) that $\delta_\pi(y)$ is a self-adjoint operator, *i.e.*, $\delta_\pi(y)^* = \delta_\pi(y)$. We now have from (3.9)

$$(\delta_\pi(y)\hat{f}(\pi))^*\delta_\pi(y)\hat{f}(\pi) = \hat{f}(\pi)^*\delta_\pi(y)^2\hat{f}(\pi) = 0,$$

namely, $\delta_\pi(y)\hat{f}(\pi) = 0$. Therefore, $(\pi(y) + \pi(y)^{-1})\hat{f}(\pi) = 2f(y)\hat{f}(\pi)$. Notice that this relation is the same as (3.2). Now, following the same line as the proof of Theorem 3.1, we finish the proof. ■

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