A NON-TRIVIAL RING WITH NON-RATIONAL INJECTIVE HULL

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1. Introduction. Several authors have investigated "rings of quotients" of a given ring R. (See, for example, Johnson [7], Johnson and Wong [8], Utumi [11], Findlay and Lambek [5], Lambek [9], and Bourbaki [2].) These are rings Q containing R such that Q_R is an essential extension of R_R. If the injective hull $E(R)$ of R is a rational extension of R, that is, if the only map from $E(R)$ to $E(R)$ whose kernel contains R is the zero map, then $E(R)$ can be made into a maximal quotient ring of R containing a copy of every quotient ring of R. In [10], I construct a ring R whose injective hull is a quotient ring of R but not a rational extension, and a second ring S whose injective hull cannot be made into a ring extending module multiplication by S. These examples are rather trivial, and the question arises whether all "nice" rings have their injective hulls a rational extension of the ring. If R is primitive with a minimal right ideal, or if R is self-injective, this must be the case. However, in this note I construct a semi-simple prime ring whose injective hull is not a rational extension of the ring.

Let $J(R)$ denote the Jacobson radical of R, and $Z(R)$ the singular ideal $= \{x \in R \mid R$ is an essential extension of the right annihilator of $x\}$.

Let F be the free algebra over Z_2 generated by

$$\{X, Y_i \mid i = 0, 1, 2, \ldots\} .$$

A word W is a finite product of generators

$$W = X^{i_1} Y_{j_2} X^{i_2} \ldots Y_{j_n} X^{i_n} \quad i_k, j_k \geq 0, \; n \geq 1$$

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where $X^0 = 1$. The length of W is defined by

$$\ell(W) = \sum_{k=1}^{n} i_k$$

and the maximum subscript $m(W)$ by

$$m(W) = \begin{cases} \text{largest subscript of } Y \text{ in } W & \text{if } n \geq 2 \\ 0 & \text{if } n = 1 . \end{cases}$$

Let I be the ideal of F generated by all words W such that

$$m(W) > 0$$

and

$$\ell(W) > m(W) \text{ times the number of times } Y_{m(W)} \text{ appears in } W .$$

Let $R = F / I$. We will show that this ring has $J(R) = 0$, but $E(R)$ is not a rational extension of R. We will apply the notions of word, length, and maximum subscript to R as well as to F.

LEMMA 1. Let $\{W_i \mid i = 1, \ldots, n\}$ be distinct words of F. Then $\Sigma_{i=1}^{n} W_i \in I$ if and only if $W_i \in I$ for $i = 1, \ldots, n$. Distinct appearing words of R are either distinct elements or both equal to 0.

<u>Proof.</u> $x \in I$ if and only if

$$x = \sum_{j=1}^{m} \Big(\sum_{i=1}^{n_j} W_{ij} \Big) W_j' \Big(\sum_{k=1}^{p_j} W_{kj}'' \Big) ,$$

where for each j, $1 \leq j \leq m$, $\{W_{ij} \mid 1 \leq i \leq n_j\}$ and $\{W_{kj}'' \mid 1 \leq k \leq p_j\}$ are sets of distinct words of F and W_j' represents 0 or a generator of I. Multiplying out, we get $x \in I$ if and only if

$$x = \sum_{j=1}^{m} \sum_{k=1}^{p_j} \sum_{i=1}^{n_j} W_{ij} W_j' W_{kj}'' ,$$

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where each word of the sum is a member of I. Since distinct appearing sums of words in F represent distinct elements, the first part of the statement of the lemma follows. If two words in F represent the same element in R, then their difference lies in I, so they are either identical or both lie in I, giving the second part.

Now let

$$0 \neq p = \sum_{i=1}^{n} W_i ,$$

where the W_i are distinct words of R. Define $j(p)$ by

$$j(p) = \sum_{i=1}^{n} [2\ell(W_i) + m(W_i)] + 1 .$$

LEMMA 2. $J(R) = 0$.

Proof. Let $0 \neq p = \Sigma_{i=1}^{n} W_i \varepsilon R$. Let

$$p' = pY_{j(p)} .$$

If p' is quasi-regular, there is an $r \varepsilon R$ with

$$p' + r + p'r = 0 .$$

Let $r = \Sigma_{j=1}^{m} V_j$, where the V_j are distinct words of R. Then

$$\sum_{i=1}^{n} W_i Y_{j(p)} + \sum_{j=1}^{m} V_j + \sum_{i=1}^{n} \sum_{j=1}^{m} W_i Y_{j(p)} V_j = 0 .$$

Since every term in p' and $p'r$ has a factor $Y_{j(p)}$, the term 1 cannot appear in r.

$W_1 Y_{j(p)} \neq 0$ since $W_1 \notin I$ and

$$j(p) = m(W' Y_{j(p)}) > \ell(W')$$

for any subword W' of W_1, so $W' Y_{j(p)} \notin I$.

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$W_1 Y_{j(p)}$ appears in p', but not in $p'r$ since each word in $p'r$ has a subword $Y_{j(p)} V_j$ where $V_j \neq 1$. Hence $W_1 Y_{j(p)}$ must be one of the V_j appearing in r.

Assume $(W_1 Y_{j(p)})^n \neq 0$ is one of the V_j, for $n \geq 1$. Then $(W_1 Y_{j(p)})^{n+1}$ appears in the above sum for $p'r$. As $(W_1 Y_{j(p)})^{n+1}$ involves at least two $Y_{j(p)}$, it cannot appear in p', and so is either 0 or one of the V_j appearing in r.

Let W' be a subword of $(W_1 Y_{j(p)})^{n+1}$. If W' does not involve $Y_{j(p)}$, then $W' \notin I$ since $W_1 \notin I$. If W' contains $Y_{j(p)}$ k times,

$$\ell(W') \leq (k+1)\ell(W_1) \leq 2k\,\ell(W_1) < k\, j(p)$$

so W' cannot be a generator of I. Then $(W_1 Y_{j(p)})^{n+1}$ is not zero.

Thus r contains the infinite sum of distinct non-zero terms $(W_1 Y_{j(p)})^n$ for all $n \geq 1$, a contradiction. We conclude that p' is not quasi-regular, so $p \notin J(R)$ and $J(R) = 0$.

LEMMA 3. R is prime.

Proof. Let $a \neq 0$, $b \neq 0 \in R$. Let $j = j(a) + j(b)$. If $a = \Sigma_{i=1}^{n} W_i$ and $b = \Sigma_{k=1}^{m} V_k$, where the terms of each sum are distinct non-zero words of R, then no $W_i Y_j V_k$ can belong to I, so $aY_j b \neq 0$ and R is prime.

LEMMA 4. $Z(R)$ = the ideal generated by X.

Proof. Let $p \neq 0 \in Z(R)$. If $\ell(p) = 0$, then p is a polynomial in the Y_j and $|p) \cap (0:p) = 0$, a contradiction. Hence $Z(R) \subseteq$ the ideal generated by X.

Let $p \neq 0 \in R$. Then if $p' = pY_{j(p)}$,

$$p'X^o \neq 0 , \quad p'X^{j(p)+1} = 0 .$$

Let $n > 0$ be the smallest integer such that $p'X^n = 0$. Then $0 \neq p'' = p'X^{n-1} = \Sigma_{i=1}^{m} W_i$, where each $W_i \notin I$, but $W_i X \varepsilon I$. $W_i = V_i Y_{j(p)} X^{n-1}$ must have length $\geq j(p)$ since any subword U of $W_i X$ such that $U \varepsilon I$ must include the last X and hence $Y_{j(p)}$, and so $j(p) < \ell(U) \leq \ell(W_i) + 1$. Also, $\ell(W_i) \leq j(p)$ since $W_i \notin I$. Then $\ell(W_i) = j(p)$.

Then $Xp'' = \Sigma_{i=1}^{m} XW_i$ is a sum of generators of I, and hence is 0 in R. Thus $X \varepsilon Z(R)$.

Since $Z(R)$ is a two-sided ideal of R (see Johnson [6]) the lemma follows.

We say that the word W' is an initial subword of the word W if

$$W = X^{i_1} Y_{j_2} \dots Y_{j_n} X^{i_n}$$

and for some $k \leq n$ and $m \leq i_k$,

$$W' = X^{i_1} Y_{j_2} \dots Y_{j_k} X^m .$$

Let W be a word in $(0:X)$. W is called primitive if no proper initial subword of W lies in $(0:X)$. Every word in $(0:X)$ contains a primitive initial subword.

By Lemma 1, we have:

(1) If $Xp = X \Sigma_{i=1}^{n} W_i = 0$, where the W_i are distinct words, then $XW_i = 0$ for $1 \leq i \leq n$.

(2) If $\{W_i \mid i \varepsilon \mathcal{J}\}$ are distinct words of R, none of which is an initial subword of any of the others, then the sum $\Sigma_{i \varepsilon \mathcal{J}} W_i R$ is a direct sum.

By (1) and (2), if $\mathcal{J}$ is the set of primitive words of $(0:X)$,

$$(0:X) = \sum_{W \in \mathscr{I}} \oplus WR .$$

Thus any map from $\mathscr{I}$ into R extends to an R-homomorphism of $(0:X)$ into R.

LEMMA 5. $E(R)$ is not a rational extension of R.

Proof. Let $W \in \mathscr{I}$, the set of primitive words of $(0:X)$. Define $f(W) = WY_o$. f extends to an R-homomorphism of $(0:X)$ into R.

Since $m(W) > 0$ for any generator of I, and since $\ell(WV) = \ell(WY_oV)$ and $m(WV) = m(WY_oV)$ for any words W and V in F, $WY_oV \in I \Leftrightarrow WV \in I$.

Then f maps WR one-to-one into itself, so f is one-to-one.

Since $E(R)$ is injective, there is an $m \in E(R)$ such that $mr = f(r)$ for all $r \in (0:X)$. We show that $(R:m) = (0:X)$.

Clearly $(R:m) \supseteq (0:X)$.

Let $p = \Sigma_{i=1}^{h} W_i \in (R:m)$, and let $mp = \Sigma_{j=1}^{n} U_j$.

We use induction on n, the number of distinct words appearing in mp, to prove $p \in (0:X)$.

If $n = 0$, $mp = 0$. Then $mp((0:X):p) = 0$. But $mp((0:X):p) = f(p((0:X):p))$, and f is a monomorphism. Thus $p((0:X):p) = |p) \cap (0:X) = 0$, so by Lemma 4, $p = 0 \in (0:X)$.

Now assume $n > 0$. Let $q = Y_{j(p)}X^k$ be such that $0 \neq pq \in (0:X)$. q was shown to exist in the proof of Lemma 4. Then

$$(mp)q = \Sigma_{j=1}^{n} U_jq ,$$

$$m(pq) = f(pq) = \Sigma_{i=1}^{h} f(W_iq) \neq 0 .$$

Now each $f(W_iq)$ contains an initial subword of the form

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$W'Y_o$, where $W' \varepsilon \mathcal{J}$. Hence for each non-zero $U_j q$, U_j' contains an initial subword of the form $W_j'Y_o$, with $W_j' \varepsilon \mathcal{J}$. Say $U_j' = W_j'Y_oV_j$. Then, for $U_j q \neq 0$, $m(p - W_j'V_j)$ is a proper subsum of the U_i', so by the induction hypothesis, $p - W_j'V_j \varepsilon (0:X)$. Since $W_j'V_j$ is also in $(0:X)$, $p \varepsilon (0:X)$.

We may then define a homomorphism on $mR + R$ to $E(R)$ such that $m \rightarrow X$ and $R \rightarrow 0$. This map shows that $E(R)$ is not a rational extension of R.

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