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The Ideal Structures of Crossed Products of Cuntz Algebras by Quasi-Free Actions of Abelian Groups

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Abstract. We completely determine the ideal structures of the crossed products of Cuntz algebras by quasi-free actions of abelian groups and give another proof of A. Kishimoto's result on the simplicity of such crossed products. We also give a necessary and sufficient condition that our algebras become primitive, and compute the Connes spectra and K-groups of our algebras.

1 Introduction

Recently the classification theory of simple C^*-algebras has developed rapidly. One of the most important questions in the classification theory of C^*-algebras is to determine whether a given C^*-algebra is simple or not. It is also important to examine the ideal structure of a given C^*-algebra if it turns out to be non-simple. There have been many works examining the ideal structures of some classes of C^*-algebras. J. Cuntz examined the ideal structures of Cuntz-Krieger algebras under a certain condition in [C2]. In [aHR], A. an Huef and I. Raeburn determined the ideal structures of arbitrary Cuntz-Krieger algebras. There have been many extensions of Cuntz-Krieger algebras, for example, Cuntz-Pimsner algebras [Pi], graph algebras and Exel-Laca algebras [EL], and there have also been many results about the ideal structures of such algebras (for example, [KPW], [KPRR], [BPRS] and [EL]).

The crossed products of C^*-algebras give us plenty of interesting examples and the structures of them have been examined by several authors. In [Ki], A. Kishimoto gave a necessary and sufficient condition that the crossed products by abelian groups become simple in terms of the strong Connes spectrum. For the case of the crossed products of the Cuntz algebras by so-called quasi-free actions of abelian groups, he gave a condition for simplicity which is easy to check and computed the strong Connes spectra of some of such actions. It is hard to compute the strong Connes spectrum by its definition and there have been few examples of actions whose strong Connes spectra have been computed.

In this paper, we deal with crossed products of Cuntz algebras $\mathcal{O}_n$ by quasi-free actions of arbitrary locally compact, second countable, abelian groups G. For similar results on crossed products of Cuntz algebras $\mathcal{O}_\infty$, see [Ka2]. The class of our algebras has many examples of simple stably projectionless C^*-algebras as well as AF-algebras and purely infinite C^*-algebras (see [KK1], [KK2] or [Ka1]). Our algebras may be considered as continuous counterparts of Cuntz-Krieger algebras or graph algebras

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(see [Ka3, Ka4]). The main purpose of this paper is to determine the ideal structures of our algebras in terms of the spectrum of the action, which is a finite subset of the dual group Γ of G. This paper is organized as follows. After some preliminaries, we prove that the set of all ideals that are invariant under the gauge action is in a one-to-one correspondence to the set of closed subsets of the dual group Γ of G satisfying certain conditions (Theorem 3.14). Next we give a necessary and sufficient condition that our algebras become simple (Theorem 4.8), which gives another proof of A. Kishimoto's result. We also give a necessary and sufficient condition that our algebras become primitive (Theorem 4.12). In Section 5, we completely determine the ideal structures of our algebras. If actions satisfy a certain condition which is an analogue of Condition (II) in the case of Cuntz-Krieger algebras [C2] or Condition (K) in the case of graph algebras [KPRR], then one can show that all ideals are invariant under the gauge action and so one can describe all ideals in terms of closed sets of the group Γ (Theorem 5.2). It is rather difficult to describe the ideal structures when actions do not satisfy the condition. We have to determine all primitive ideals and investigate the topology of the primitive ideal spaces of our algebras. After that, we show that when actions do not satisfy the condition, the set of all ideals corresponds bijectively to the set of closed subsets of a certain topological space satisfying a certain condition (Theorem 5.49). As a consequence of knowing the ideal structures completely, we can compute the strong Connes spectra of quasi-free actions on Cuntz algebras. Our algebras can be considered as Cuntz-Pimsner algebras and by using this fact we compute the K-groups of our algebras. Finally we conclude this paper by giving some examples and remarks.

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2 Preliminaries

In this section, we review some basic objects and fix the notation.

For $n = 2, 3, \ldots$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by n isometries $S_1, S_2, \ldots, S_n$, satisfying $\sum_{i=1}^n S_i S_i^* = 1$ [C1]. For $k \in \mathbb{N} = \{0, 1, \ldots\}$, we define the set $\mathcal{W}_n^{(k)}$ of k-tuples by $\mathcal{W}_n^{(0)} = \{\varnothing\}$ and

$$\mathcal{W}_n^{(k)} = \big\{ (i_1, i_2, \ldots, i_k) \mid i_j \in \{1, 2, \ldots, n\} \big\}.$$

We set $\mathcal{W}_n = \bigcup_{k=0}^\infty \mathcal{W}_n^{(k)}$. For $\mu = (i_1, i_2, \ldots, i_k) \in \mathcal{W}_n$, we denote its length k by $|\mu|$, and set $S_\mu = S_{i_1} S_{i_2} \cdots S_{i_k} \in \mathcal{O}_n$. Note that $|\varnothing| = 0$, $S_\varnothing = 1$. For $\mu = (i_1, i_2, \ldots, i_k)$, $\nu = (j_1, j_2, \ldots, j_l) \in \mathcal{W}_n$, we define their product $\mu\nu \in \mathcal{W}_n$ by $\mu\nu = (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l)$.

We fix a locally compact abelian group G which satisfies the second axiom of countability. The dual group of G is denoted by Γ which is also a locally compact abelian group satisfying the second axiom of countability. We always use + for multiplicative operations of abelian groups except for $\mathbb{T}$, which is the group of the unit circle in the complex plane $\mathbb{C}$. The pairing of $t \in G$ and $\gamma \in \Gamma$ is denoted by $\langle t \mid \gamma \rangle \in \mathbb{T}$.

Let us take $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Gamma^n$ and fix it. Since the n isometries $\langle t \mid \omega_1 \rangle S_1, \langle t \mid \omega_2 \rangle S_2, \ldots, \langle t \mid \omega_n \rangle S_n$ also satisfy the relation above for any $t \in G$, there is a $*$-automorphism $\alpha_t^\omega \colon \mathcal{O}_n \to \mathcal{O}_n$ such that $\alpha_t^\omega(S_i) = \langle t \mid \omega_i \rangle S_i$ for $i = 1, 2, \ldots, n$. One can see that $\alpha^\omega \colon G \ni t \mapsto \alpha_t^\omega \in \operatorname{Aut}(\mathcal{O}_n)$ is a strongly continuous group homomorphism.

Definition 2.1 Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Gamma^n$ be given. We define the action $\alpha^\omega \colon G \curvearrowright \mathcal{O}_n$ by

$$\alpha_t^\omega(S_i) = \langle t \mid \omega_i \rangle S_i \quad (i = 1, 2, \ldots, n, t \in G).$$

The action $\alpha^\omega \colon G \curvearrowright \mathcal{O}_n$ becomes quasi-free (for a definition of quasi-free actions on the Cuntz algebras, see [E]). Conversely, any quasi-free action of abelian group G on $\mathcal{O}_n$ is conjugate to α^ω for some $\omega \in \Gamma^n$.

By definition, the (full) crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is the universal C^*-algebra generated by the $*$-algebra $L^1(G, \mathcal{O}_n)$ whose multiplication and involution are defined as the following:

$$fg(t) = \int_G f(s)\alpha_s^\omega\big(g(t-s)\big)\, ds, \qquad f^*(t) = \alpha_t^\omega\big(f(-t)^*\big),$$

for $f, g \in L^1(G, \mathcal{O}_n)$ (*cf.* [Pe]). The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ has a C^*-subalgebra $\mathbb{C}1 \rtimes_{\alpha^\omega} G$, which is isomorphic to $C_0(\Gamma)$ via the map $\mathbb{C}1 \rtimes_{\alpha^\omega} G \supset L^1(G) \ni f \mapsto \hat{f} \in C_0(\Gamma)$, where

$$\hat{f}(\gamma) = \int_G \langle t \mid \gamma \rangle f(t)\, dt.$$

Throughout this paper, we always consider $C_0(\Gamma)$ as a C^*-subalgebra of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, and use $f, g, \ldots$ for denoting elements of $C_0(\Gamma) \subset \mathcal{O}_n \rtimes_{\alpha^\omega} G$. The Cuntz algebra $\mathcal{O}_n$ is naturally embedded into the multiplier algebra $M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. For each $\mu = (i_1, i_2, \ldots, i_k)$ in $\mathcal{W}_n$, we define an element ω_μ of Γ by $\omega_\mu = \sum_{j=1}^k \omega_{i_j}$. For $\gamma_0 \in \Gamma$, we define a (reverse) shift automorphism $\sigma_{\gamma_0} \colon C_0(\Gamma) \to C_0(\Gamma)$ by $(\sigma_{\gamma_0} f)(\gamma) = f(\gamma + \gamma_0)$ for $f \in C_0(\Gamma)$.

Once noting that $\alpha_t^\omega(S_\mu) = \langle t \mid \omega_\mu \rangle S_\mu$ for $\mu \in \mathcal{W}_n$, one can easily verify the following.

Proposition 2.2 *For any $f \in C_0(\Gamma) \subset \mathcal{O}_n \rtimes_{\alpha^\omega} G$ and any $\mu \in \mathcal{W}_n$, we have $fS_\mu = S_\mu \sigma_{\omega_\mu} f$.*

For a subset X of a C^*-algebra, the linear span of X is denoted by $\operatorname{span} X$, and the closure of $\operatorname{span} X$ is denoted by $\overline{\operatorname{span}}\, X$.

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Proposition 2.3 *We have* $\mathcal{O}_n \rtimes_{\alpha^\omega} G = \overline{\operatorname{span}}\,\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ f \in C_0(\Gamma)\}$.

Proof By Proposition 2.2,

$$\overline{\operatorname{span}}\,\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ f \in C_0(\Gamma)\} = \overline{\operatorname{span}}\,\{S_\mu S_\nu^* f \mid \mu, \nu \in \mathcal{W}_n,\ f \in C_0(\Gamma)\}.$$

Obviously $\overline{\operatorname{span}}\,\{S_\mu S_\nu^* f \mid \mu, \nu \in \mathcal{W}_n,\ f \in C_0(\Gamma)\}$ contains all elements of $L^1(G, \mathcal{O}_n)$, which is dense in $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. The proof is complete. ■

We denote by $\mathbb{M}_k$ the C^*-algebra of $k \times k$ matrices for $k = 1, 2, \dots$, and by $\mathbb{K}$ the C^*-algebra of compact operators of the infinite dimensional separable Hilbert space.

3 Gauge Invariant Ideals

There is an action β of $\mathbb{T}$ on $\mathcal{O}_n$ called the *gauge action* which is defined by $\beta_t(S_i) = tS_i$ for $t \in \mathbb{T}$, $i = 1, 2, \dots, n$. We can extend this action to $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ which is also called the *gauge action* and denoted by β. Explicitly, $\beta_t(S_\mu f S_\nu^*) = t^{|\mu|-|\nu|} S_\mu f S_\nu^*$ for $\mu, \nu \in \mathcal{W}_n$, $f \in C_0(\Gamma)$ and $t \in \mathbb{T}$.

By an ideal we mean a closed two-sided ideal. In this section, we determine all the ideals which are globally invariant under the gauge action.

Definition 3.1 For an ideal I of the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, we define the closed subset X_I of Γ by

$$X_I = \bigcap_{f \in I \cap C_0(\Gamma)} \{\gamma \in \Gamma \mid f(\gamma) = 0\}.$$

In other words, X_I is determined by $C_0(\Gamma \setminus X_I) = I \cap C_0(\Gamma)$ where for a closed subset X of Γ, $C_0(\Gamma \setminus X)$ means the set of functions in $C_0(\Gamma)$ which vanish on X. In particular, $C_0(\Gamma \setminus \Gamma) = \{0\}$. One can easily see that $I_1 \subset I_2$ implies $X_{I_1} \supset X_{I_2}$ and $X_{I_1 \cap I_2} = X_{I_1} \cup X_{I_2}$ for ideals I_1, I_2 of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$.

Definition 3.2 A subset X of Γ is called ω*-invariant* if X is a closed set satisfying the following two conditions:

(i) For any $\gamma \in X$ and any $i \in \{1, 2, \dots, n\}$, we have $\gamma + \omega_i \in X$.
(ii) For any $\gamma \in X$, there exists $i \in \{1, 2, \dots, n\}$ such that $\gamma - \omega_i \in X$.

For an element γ of an ω-invariant set X, one can easily show that $\gamma + \omega_\mu \in X$ for any $\mu \in \mathcal{W}_n$ and that there exists $i \in \{1, 2, \dots, n\}$ such that $\gamma - m\omega_i \in X$ for any $m \in \mathbb{N}$. For a subset X of Γ and an element γ_0 of Γ, we define the subset $X + \gamma_0$ of Γ by $X + \gamma_0 = \{\gamma + \gamma_0 \mid \gamma \in X\}$. Similarly, we define $X_1 + X_2 = \{\gamma_1 + \gamma_2 \mid \gamma_1 \in X_1, \gamma_2 \in X_2\}$ for $X_1, X_2 \subset \Gamma$. A closed set X is ω-invariant if and only if $X = \bigcup_{i=1}^n (X + \omega_i)$.

Proposition 3.3 *For any ideal I of the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, the closed set X_I is ω-invariant.*

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Proof Take $\gamma \in X_I$ and $i \in \{1, 2, \ldots, n\}$ arbitrarily. Let f be an element of $I \cap C_0(\Gamma)$. By Proposition 2.2, $S_i^* f S_i = S_i^* S_i \sigma_{\omega_i} f = \sigma_{\omega_i} f$. Hence $\sigma_{\omega_i} f \in I \cap C_0(\Gamma)$, so we have $\sigma_{\omega_i} f(\gamma) = 0$. Thus, $f(\gamma + \omega_i) = 0$ for any $f \in I \cap C_0(\Gamma)$. It implies $\gamma + \omega_i \in X_I$.

Let γ_0 be a point of Γ such that $\gamma_0 - \omega_i \notin X_I$ for any $i = 1, 2, \ldots, n$, and we will show that $\gamma_0 \notin X_I$. Since $\Gamma \setminus X_I$ is open, there is a neighborhood U of $\gamma_0 \in \Gamma$ such that $U - \omega_i \subset \Gamma \setminus X_I$ for any $i = 1, 2, \ldots, n$. There exists $f \in C_0(\Gamma)$ such that $f(\gamma_0) \neq 0$ and $f(\gamma) = 0$ for any $\gamma \notin U$. Then $U - \omega_i \subset \Gamma \setminus X_I$ implies $\sigma_{\omega_i} f \in C_0(\Gamma \setminus X_I) \subset I$ for $i = 1, 2, \ldots, n$. Since

$$f = \sum_{i=1}^{n} S_i S_i^* f = \sum_{i=1}^{n} S_i \sigma_{\omega_i} f S_i^*,$$

we have $f \in I$. It implies $\gamma_0 \notin X_I$. Thus X_I is ω-invariant. ∎

We will show that for any ω-invariant subset X, there exists a gauge invariant ideal I such that $X = X_I$ (Proposition 3.6).

Definition 3.4 Let X be an ω-invariant subset of Γ. We define $I_X \subset \mathcal{O}_n \rtimes_{\alpha^\omega} G$ by

$$I_X = \overline{\operatorname{span}} \{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n, f \in C_0(\Gamma \setminus X)\}.$$

Proposition 3.5 *For an ω-invariant subset X of Γ, the set I_X becomes a gauge invariant ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$.*

Proof Clearly I_X is a $*$-invariant closed linear space. Since $\beta_t(S_\mu f S_\nu^*) = t^{|\mu|-|\nu|} S_\mu f S_\nu^*$ for $t \in \mathbb{T}$, I_X is invariant under the gauge action β. By Proposition 2.3, it suffices to show that for any $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{W}_n$ and any $f \in C_0(\Gamma \setminus X)$, $g \in C_0(\Gamma)$, the product xy of $x = S_{\mu_1} f S_{\nu_1}^* \in I_X$ and $y = S_{\mu_2} g S_{\nu_2}^* \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$ is an element of I_X.

If $S_{\nu_1}^* S_{\mu_2} = 0$, then $xy = 0 \in I_X$. Otherwise, $S_{\nu_1}^* S_{\mu_2} = S_\mu$ or $S_{\nu_1}^* S_{\mu_2} = S_\mu^*$ for some $\mu \in \mathcal{W}_n$. For the case $S_{\nu_1}^* S_{\mu_2} = S_\mu$,

$$xy = S_{\mu_1} f S_{\nu_1}^* S_{\mu_2} g S_{\nu_2}^* = S_{\mu_1} f S_\mu g S_{\nu_2}^* = S_{\mu_1 \mu} (\sigma_{\omega_\mu} f) g S_{\nu_2}^*.$$

Since $f \in C_0(\Gamma \setminus X)$ and X is ω-invariant, we have $\sigma_{\omega_\mu} f \in C_0(\Gamma \setminus X)$. This implies $(\sigma_{\omega_\mu} f) g \in C_0(\Gamma \setminus X)$ and so $xy \in I_X$. For the case $S_{\nu_1}^* S_{\mu_2} = S_\mu^*$,

$$xy = S_{\mu_1} f S_{\nu_1}^* S_{\mu_2} g S_{\nu_2}^* = S_{\mu_1} f S_\mu^* g S_{\nu_2}^* = S_{\mu_1} f (\sigma_{\omega_\mu} g) S_{\nu_2 \mu}^*.$$

Since $f \in C_0(\Gamma \setminus X)$, we have $xy \in I_X$. It completes the proof. ∎

Proposition 3.6 *For any ω-invariant subset X of Γ, we have $X_{I_X} = X$.*

Proof By the definition of I_X, we get $X_{I_X} \subset X$. Let us assume $X_{I_X} \subsetneqq X$. Then there exists $f \in I_X \cap C_0(\Gamma)$ such that $f(\gamma_0) = 1$ for some $\gamma_0 \in X$. Since $f \in I_X$, there exist $f_k \in C_0(\Gamma \setminus X)$, $\mu_k, \nu_k \in \mathcal{W}_n$ $(k = 1, 2, \ldots, K)$ such that

$$\Big\| f - \sum_{k=1}^{K} S_{\mu_k} f_k S_{\nu_k}^* \Big\| < \frac{1}{2}.$$

From this inequality, we will derive a contradiction.

Since X is ω-invariant, there exists $i \in \{1, 2, \ldots, n\}$ such that $\gamma_0 - m\omega_i \in X$ for any $m \in \mathbb{N}$. Take $j \in \{1, 2, \ldots, n\}$ with $j \neq i$. Set $M = \max\bigl\{|\mu_k|, |\nu_k| \bigm| k = 1, 2, \ldots, K\bigr\}$. Then, $S_j^* S_i^{*M} f S_i^M S_j = \sigma_{(M\omega_i+\omega_j)} f$ and $S_j^* S_i^{*M} S_{\mu_k} f_k S_{\nu_k}^* S_i^M S_j$ is either 0 or $\sigma_{(m_k\omega_i+\omega_j)} f_k$ for some $m_k \leq M$. Therefore, from

$$\Bigl\| S_j^* S_i^{*M} \Bigl(f - \sum_{k=1}^{K} S_{\mu_k} f_k S_{\nu_k}^* \Bigr) S_i^M S_j \Bigr\| < \frac{1}{2},$$

we get

$$\Bigl\| \sigma_{(M\omega_i+\omega_j)} f - \sum_k \sigma_{(m_k\omega_i+\omega_j)} f_k \Bigr\| < \frac{1}{2}.$$

By evaluating at $\gamma_0 - M\omega_i - \omega_j$, we find $k \in \mathbb{N}$ such that $f_k\bigl(\gamma_0 - (M - m_k)\omega_i\bigr) \neq 0$. It contradicts the fact that $f_k \in C_0(\Gamma \setminus X)$ and $\gamma_0 - m\omega_i \in X$ for any $m \in \mathbb{N}$. Therefore we are done. ■

By Proposition 3.6, the map $I \mapsto X_I$ from the set of gauge invariant ideals I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ to the set of ω-invariant subsets of Γ is surjective. Now, we turn to showing that this map is injective (Proposition 3.13). The method we use here is inspired by [C1].

Let I be an ideal that is not $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. We investigate the quotient $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ by an ideal I. Since $I \cap C_0(\Gamma) = C_0(\Gamma \setminus X_I)$, a C^*-subalgebra $C_0(\Gamma)/\bigl(I \cap C_0(\Gamma)\bigr)$ of $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ is isomorphic to $C_0(X_I)$. We will consider $C_0(X_I)$ as a C^*-subalgebra of $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$. No confusion should occur by using the same symbols $S_1, S_2, \ldots, S_n \in M\bigl((\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I\bigr)$ as the ones in $M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ for denoting the isometries of $\mathcal{O}_n$ which is naturally embedded into $M\bigl((\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I\bigr)$.

For an ω-invariant set X, we can define a $*$-homomorphism $\sigma_{\omega_\mu} \colon C_0(X) \to C_0(X)$ for $\mu \in \mathcal{W}_n$. This map σ_{ω_μ} is always surjective, but it is injective only in the case that $X - \omega_\mu \subset X$, which is equivalent to $X - \omega_\mu = X$. One can easily verify the following.

Lemma 3.7 *Let I be an ideal that is not $\mathcal{O}_n \rtimes_{\alpha^\omega} G$.*

(i) *For $\mu, \nu \in \mathcal{W}_n$ and $f \in C_0(X_I)$, $S_\mu f S_\nu^* \in (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ is zero if and only if $f = 0$.*
(ii) *For $\mu \in \mathcal{W}_n$ and $f \in C_0(X_I)$, we have $f S_\mu = S_\mu \sigma_{\omega_\mu} f$.*
(iii) $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I = \overline{\operatorname{span}}\,\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n, f \in C_0(X_I)\}$.

We define a C^*-subalgebra of $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$, which corresponds to the AF-core for Cuntz algebras.

Definition 3.8 Let I be an ideal that is not $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. We define C^*-subalgebras $\mathcal{F}_I^{(k)}$ ($k \in \mathbb{N}$) and $\mathcal{F}_I$ of $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ by

$$\mathcal{F}_I^{(k)} = \overline{\operatorname{span}}\,\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n^{(k)}, f \in C_0(X_I)\},$$
$$\mathcal{F}_I = \overline{\operatorname{span}}\,\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n, |\mu| = |\nu|, f \in C_0(X_I)\}.$$

When $I = 0$, we write simply $\mathcal{F}^{(k)}, \mathcal{F}$ for $\mathcal{F}_0^{(k)}, \mathcal{F}_0$.

Lemma 3.9 *Let I be an ideal that is not $\mathcal{O}_n \rtimes_{\alpha^\omega} G$.*

(i) *The C^*-subalgebra $\mathcal{F}_I^{(k)}$ of $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ is isomorphic to $\mathbb{M}_{n^k} \otimes C_0(X_I)$ for $k \in \mathbb{N}$.*
(ii) *$\mathcal{F}_I^{(k)} \subset \mathcal{F}_I^{(k+1)}$ and the inductive limit of $\mathcal{F}_I^{(k)}$ is $\mathcal{F}_I$.*

Proof (i) Since the set $\mathcal{W}_n^{(k)}$ has n^k elements, we may use $\{e_{\mu,\nu}\}_{\mu,\nu \in \mathcal{W}_n^{(k)}}$ for denoting the matrix units of $\mathbb{M}_{n^k}$. For $x_1 = S_{\mu_1} f_1 S_{\nu_1}^*$, $x_2 = S_{\mu_2} f_2 S_{\nu_2}^* \in \mathcal{F}_I^{(k)}$, we have $x_1^* = S_{\nu_1} \overline{f_1} S_{\mu_1}^*$ and $x_1 x_2 = \delta_{\nu_1,\mu_2} S_{\mu_1} f_1 f_2 S_{\nu_2}^*$. Thus the map

$$\mathbb{M}_{n^k} \otimes C_0(X_I) \ni e_{\mu,\nu} \otimes f \mapsto S_\mu f S_\nu^* \in \mathcal{F}_I^{(k)}$$

defines a $*$-homomorphism. By the definition of $\mathcal{F}_I^{(k)}$, it is surjective. It is injective by Lemma 3.7(i). Thus $\mathbb{M}_{n^k} \otimes C_0(X_I) \cong \mathcal{F}_I^{(k)}$.

(ii) Since $S_\mu f S_\nu^* = \sum_{i=1}^n S_\mu S_i \sigma_{\omega_i} f S_i^* S_\nu^*$, we have $\mathcal{F}_I^{(k)} \subset \mathcal{F}_I^{(k+1)}$. The latter part is trivial by the definitions of $\mathcal{F}_I^{(k)}$ and $\mathcal{F}_I$. ■

Definition 3.10 A linear map E from some C^*-algebra A onto a C^*-subalgebra B of A is called a *conditional expectation* if $\|E\| \leq 1$ and $E(x) = x$ for any $x \in B$. A conditional expectation E is called *faithful* if $E(x) = 0$ implies $x = 0$ for a positive element x of A.

The following proposition essentially appeared in [C1].

Proposition 3.11 *For $i = 1, 2$, let E_i be a conditional expectation from a C^*-algebra A_i onto a C^*-subalgebra B_i of A_i. Let $\varphi\colon A_1 \to A_2$ be a $*$-homomorphism with $\varphi \circ E_1 = E_2 \circ \varphi$. If the restriction of φ on B_1 is injective and E_1 is faithful, then φ is injective.*

Proof Let x be a positive element of $\ker \varphi \subset A_1$. Since $\varphi \circ E_1 = E_2 \circ \varphi$, we have $\varphi\big(E_1(x)\big) = 0$. Since $E_1(x) \in B_1$ and φ is injective on B_1, we have $E_1(x) = 0$. Then $x = 0$ since E_1 is faithful. Thus $\ker \varphi = \{0\}$ which means that φ is injective. ■

For an ideal I which is invariant under the gauge action β, we can extend the gauge action on $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ to one on $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$, which is also denoted by β.

Lemma 3.12 *Let I be a gauge invariant ideal that is not $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. Then,*

$$E_I\colon (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I \ni x \mapsto \int_{\mathbb{T}} \beta_t(x)\, dt \in (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$$

is a faithful conditional expectation onto $\mathcal{F}_I$, where dt is the normalized Haar measure on $\mathbb{T}$.

Proof For $x \in (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$, we have

$$\|E_I(x)\| = \Big\| \int_{\mathbb{T}} \beta_t(x)\, dt \Big\| \leq \int_{\mathbb{T}} \|\beta_t(x)\|\, dt = \|x\|.$$

Thus $\|E_I\| \leq 1$. One can see that $E_I(x) = 0$ implies $x = 0$ for a positive element $x \in (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$.

For $\mu, \nu \in \mathcal{W}_n$ and $f \in C_0(X_I)$,

$$E_I(S_\mu f S_\nu^*) = \int_{\mathbb{T}} \beta_t(S_\mu f S_\nu^*)\, dt = \int_{\mathbb{T}} t^{|\mu|-|\nu|}(S_\mu f S_\nu^*)\, dt = \delta_{|\mu|,|\nu|} S_\mu f S_\nu^*.$$

Therefore $E_I(x) = x$ for any $x \in \operatorname{span}\big\{ S_\mu f S_\nu^* \;\big|\; \mu, \nu \in \mathcal{W}_n, |\mu| = |\nu|, f \in C_0(X_I)\big\}$, thus for any $x \in \mathcal{F}_I$ by the continuity of E_I. By the above computation, $E_I(x) \in \mathcal{F}_I$ for $x \in \operatorname{span}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n, f \in C_0(X_I)\}$, which is dense in $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ by Lemma 3.7(iii). Therefore, the image of E_I is $\mathcal{F}_I$ by the continuity of E_I. We have shown that E_I is a faithful conditional expectation onto $\mathcal{F}_I$. ■

Proposition 3.13 *For any gauge invariant ideal I, we have $I_{X_I} = I$.*

Proof When $I = \mathcal{O}_n \rtimes_{\alpha^\omega} G$, we have $X_I = \varnothing$. Thus $I_{X_I} = \mathcal{O}_n \rtimes_{\alpha^\omega} G$. Let I be a gauge invariant ideal that is not $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and set $J = I_{X_I}$. By the definition, $J \subset I$. Hence there is a surjective $*$-homomorphism $\pi\colon (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/J \to (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$. By Lemma 3.9, the restriction of π on $\mathcal{F}_J^{(k)}$ is an isomorphism from $\mathcal{F}_J^{(k)}$ onto $\mathcal{F}_I^{(k)}$ and so the restriction of π on $\mathcal{F}_J$ is an isomorphism from $\mathcal{F}_J$ onto $\mathcal{F}_I$. By Lemma 3.12, there are faithful conditional expectations $E_J\colon (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/J \to \mathcal{F}_J$ and $E_I\colon (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I \to \mathcal{F}_I$. Since $E_I\big(\pi(x)\big) = \pi\big(E_J(x)\big)$ for any $x \in \operatorname{span}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n, f \in C_0(X)\}$, which is dense in $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/J$, we have $E_I \circ \pi = \pi \circ E_J$. By Proposition 3.11, π is injective. Therefore $I_{X_I} = I$. ■

Theorem 3.14 *The maps $I \mapsto X_I$ and $X \mapsto I_X$ induce a one-to-one correspondence between the set of gauge invariant ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and the set of ω-invariant subsets of Γ.*

Proof Combine Proposition 3.6 and Proposition 3.13. ■

4 Simplicity and Primitivity of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$

In this section, we give necessary and sufficient conditions for $\omega \in \Gamma^n$ that the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ becomes simple or primitive.

Proposition 4.1 *Let I be an ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. Then, $I = \mathcal{O}_n \rtimes_{\alpha^\omega} G$ if and only if $X_I = \varnothing$.*

Proof The "only if" part is trivial. The "if" part follows from Proposition 2.3. ■

For an ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, we have $I_{X_I} \subset I$. In general there exists an ideal I such that $I_{X_I} \neq I$ (see Proposition 5.26). However if X_I satisfies a certain condition, then $I = I_{X_I}$ (Theorem 4.5).

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Definition 4.2 An ω-invariant subset X of Γ is said to be *bad* if there exists $\gamma \in X$ such that there is only one element i with $\gamma - \omega_i \in X$ in $\{1, 2, \dots, n\}$ and this element i satisfies that $m\omega_i = 0$ for some positive integer m. An ω-invariant subset X of Γ is said to be *good* if X is not bad.

Note that $\varnothing$ is a good ω-invariant set.

Lemma 4.3 *An ω-invariant subset X of Γ is good if and only if for any $\gamma \in X$, one of the following two conditions is satisfied:*

(i) *There exists $i \in \{1, 2, \dots, n\}$ such that $\gamma - m\omega_i \in X$ and $\gamma - m\omega_i \neq \gamma$ for any positive integer m.*
(ii) *There exist $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ such that $\gamma - m\omega_i - \omega_j \in X$ for any positive integer m.*

Proof When X is bad, there exists $\gamma \in X$ such that there is only one element i with $\gamma - \omega_i \in X$ in $\{1, 2, \dots, n\}$ and this element i satisfies that $m\omega_i = 0$ for some positive integer m. This $\gamma \in X$ satisfies neither condition in the statement.

Let us assume that X is good and that $\gamma \in X$ does not satisfy the condition (i). We will prove that $\gamma \in X$ satisfies the condition (ii). Since X is ω-invariant, there exists $i \in \{1, 2, \dots, n\}$ such that $\gamma - m\omega_i \in X$ for any positive integer m. Since $\gamma \in X$ does not satisfy the condition (i), there exists a positive integer K with $K\omega_i = 0$. Since X is good, there exists $j \in \{1, 2, \dots, n\}$ with $j \neq i$ such that $\gamma - \omega_j \in X$. For any positive integer m, if we take $l \in \mathbb{N}$ so that $lK - m \geq 0$, we have

$$\gamma - \omega_j - m\omega_i = \gamma - \omega_j + (lK - m)\omega_i \in X.$$

Thus $\gamma \in X$ satisfies the condition (ii). ∎

Proposition 4.4 *Let I be an ideal that is not $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. If X_I is a good ω-invariant set, then there exists a unique conditional expectation E_I from $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ onto $\mathcal{F}_I$ such that $E_I(S_\mu f S_\nu^*) = \delta_{|\mu|,|\nu|} S_\mu f S_\nu^*$ for $\mu, \nu \in \mathcal{W}_n$, $f \in C_0(X_I)$.*

Proof Let $\mu_l, \nu_l \in \mathcal{W}_n$ and $f_l \in C_0(X_I)$ be given for $l = 1, 2, \dots, L$. Then $x = \sum_{l=1}^{L} S_{\mu_l} f_l S_{\nu_l}^*$ is an element of

$$\operatorname{span}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n, f \in C_0(X_I)\} \subset (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I.$$

Set $k = \max\{\, |\mu_l|, |\nu_l| \bigm| l = 1, 2, \dots, L\}$. We may assume that if $|\mu_l| = |\nu_l|$, then $|\mu_l| = |\nu_l| = k$. Let $x_0 = \sum_{|\mu_l| = |\nu_l|} S_{\mu_l} f_l S_{\nu_l}^*$. Since $x_0 \in \mathcal{F}_I^{(k)} \cong \mathbb{M}_{n^k} \otimes C_0(X_I)$, there exists $\gamma_0 \in X_I$ such that $\|x_0\| = \|\sum_{|\mu_l| = |\nu_l|} S_{\mu_l} f_l(\gamma_0) S_{\nu_l}^*\|$. We will prove that $\|x_0\| \leq \|x\|$.

Since X_I is a good ω-invariant set, $\gamma_0 \in X_I$ satisfies one of the two conditions in Lemma 4.3. We first consider the case that $\gamma_0 \in X_I$ satisfies the condition (i) in Lemma 4.3, that is, there exists $i \in \{1, 2, \dots, n\}$ such that $\gamma_0 - m\omega_i \in X$ and $\gamma_0 - m\omega_i \neq \gamma_0$ for any positive integer m. We can find a neighborhood U of $\gamma_0 - k\omega_i \in$

X_I such that $U \cap (U + m\omega_i) = \varnothing$ for $m = 1, 2, \dots, k$. Choose a function f with $0 \le f \le 1$ satisfying $f(\gamma_0 - k\omega_i) = 1$ and the support of f is contained in U. Set $u = \sum_{\mu \in \mathcal{W}_n^{(k)}} S_\mu S_i^k f^{1/2} S_\mu^* \in (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$. Since

$$u^* u = \sum_{\mu,\nu \in \mathcal{W}_n^{(k)}} S_\mu f^{1/2} S_i^{*k} S_\mu^* S_\nu S_i^k f^{1/2} S_\nu^* = \sum_{\mu \in \mathcal{W}_n^{(k)}} S_\mu f S_\mu^*,$$

$u^* u$ is an element of $\mathcal{F}_I^{(k)}$ which corresponds to the element $1 \otimes f$ under the isomorphism $\mathcal{F}_I^{(k)} \cong \mathbb{M}_{n^k} \otimes C_0(X_I)$. Thus we have $\|u^* u\| = \sup_{\gamma \in X_I} |f(\gamma)| = 1$, and so $\|u\| = 1$. When $|\mu_l| \ne |\nu_l|$, for any $\mu, \nu \in \mathcal{W}_n^{(k)}$, $(S_i^{*k} S_\mu^*) S_{\mu_l} S_{\nu_l}^* (S_\nu S_i^k)$ is either zero, S_i^m or S_i^{*m} with some $0 < m \le k$. In the case that $(S_i^{*k} S_\mu^*) S_{\mu_l} S_{\nu_l}^* (S_\nu S_i^k) = S_i^m$, we have

$$(f^{1/2} S_i^{*k} S_\mu^*) S_{\mu_l} S_{\nu_l}^* (S_\nu S_i^k f^{1/2}) = S_i^m (\sigma_{m\omega_1} f^{1/2}) f^{1/2} = 0.$$

Similarly, we have $(f^{1/2} S_i^{*k} S_\mu^*) S_{\mu_l} S_{\nu_l}^* (S_\nu S_i^k f^{1/2}) = 0$ in the case that

$$(S_i^{*k} S_\mu^*) S_{\mu_l} S_{\nu_l}^* (S_\nu S_i^k) = S_i^{*m}.$$

Hence if $|\mu_l| \ne |\nu_l|$, then $u^* S_{\mu_l} f_l S_{\nu_l}^* u = 0$. When $|\mu_l| = |\nu_l| = k$, we have

$$u^* S_{\mu_l} f_l S_{\nu_l}^* u = \sum_{\mu,\nu \in \mathcal{W}_n^{(k)}} (S_\mu f^{1/2} S_i^{*k} S_\mu^*) S_{\mu_l} f_l S_{\nu_l}^* (S_\nu S_i^k f^{1/2} S_\nu^*) = S_{\mu_l} f (\sigma_{k\omega_i} f_l) S_{\nu_l}^*.$$

Hence $u^* x u = \sum_{|\mu_l| = |\nu_l|} S_{\mu_l} f (\sigma_{k\omega_i} f_l) S_{\nu_l}^*$. Thus we have

$$\begin{aligned} \|u^* x u\| &\ge \Big\| \sum_{|\mu_l| = |\nu_l|} S_{\mu_l} f(\gamma_0 - k\omega_i) \sigma_{k\omega_i} f_l(\gamma_0 - k\omega_i) S_{\nu_l}^* \Big\| \\ &= \Big\| \sum_{|\mu_l| = |\nu_l|} S_{\mu_l} f_l(\gamma_0) S_{\nu_l}^* \Big\| = \|x_0\|. \end{aligned}$$

Therefore when the condition (i) is satisfied, we have $\|x_0\| \le \|u^* x u\| \le \|x\|$.

Next we consider the case that there exist $i, j \in \{1, 2, \dots, n\}$ with $i \ne j$ such that $\gamma_0 - m\omega_i - \omega_j \in X$ for any positive integer m. Set $u = \sum_{\mu \in \mathcal{W}_n^{(k)}} S_\mu S_i^k S_j S_\mu^* \in \mathcal{O}_n \subset M(\mathcal{O}_n \rtimes_{\alpha^\omega} G/I)$. Since

$$u^* u = \sum_{\mu,\nu \in \mathcal{W}_n^{(k)}} (S_\mu S_j^* S_i^{*k} S_\mu^*)(S_\nu S_i^k S_j S_\nu^*) = \sum_{\mu \in \mathcal{W}_n^{(k)}} S_\mu S_\mu^* = 1,$$

u is an isometry. Since $S_j^* S_i^{*k} S_\mu^* S_\nu S_i^k S_j = \delta_{\mu,\nu}$ for $\mu,\nu \in \mathcal{W}_n$ such that $|\mu|,|\nu| \le k$, we have $u^* S_{\mu_l} S_{\nu_l}^* u = \delta_{|\mu_l|,|\nu_l|} S_{\mu_l} S_{\nu_l}^*$, for $l = 1, 2, \dots, L$. Therefore,

$$\begin{aligned} u^* x u &= \sum_{l=1}^{L} u^* S_{\mu_l} f_l S_{\nu_l}^* u \\ &= \sum_{l=1}^{L} u^* S_{\mu_l} S_{\nu_l}^* u (\sigma_{k\omega_i+\omega_j-\omega_{\nu_l}} f_l) \\ &= \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} S_{\nu_l}^* (\sigma_{k\omega_i+\omega_j-\omega_{\nu_l}} f_l) \\ &= \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} (\sigma_{k\omega_i+\omega_j} f_l) S_{\nu_l}^*. \end{aligned}$$

Since $\gamma_0 - k\omega_i - \omega_j \in X_I$, we have

$$\|u^* x u\| \ge \Big\| \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} \sigma_{k\omega_i+\omega_j} f_l(\gamma_0 - k\omega_i - \omega_j) S_{\nu_l}^* \Big\| = \Big\| \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} f_l(\gamma_0) S_{\nu_l}^* \Big\| = \|x_0\|.$$

Therefore also for the case that the condition (ii) is satisfied, we have $\|x_0\| \le \|x\|$.

Suppose x is expressed in two ways: $x = \sum_{l=1}^{L} S_{\mu_l} f_l S_{\nu_l}^* = \sum_{l=1}^{L'} S_{\mu_l'} f_l' S_{\nu_l'}^*$. Let $y = \sum_{l=1}^{L} S_{\mu_l} f_l S_{\nu_l}^* - \sum_{l=1}^{L'} S_{\mu_l'} f_l' S_{\nu_l'}^*$ and $y_0 = \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} f_l S_{\nu_l}^* - \sum_{|\mu_l'|=|\nu_l'|} S_{\mu_l'} f_l' S_{\nu_l'}^*$. Since $\|y_0\| \le \|y\|$ and $y = x - x = 0$, we get $y_0 = 0$. Thus $\sum_{|\mu_l|=|\nu_l|} S_{\mu_l} f_l S_{\nu_l}^* = \sum_{|\mu_l'|=|\nu_l'|} S_{\mu_l'} f_l' S_{\nu_l'}^*$ which means that x_0 does not depend on expressions of x. Hence we can define a norm-decreasing linear map E_I by

$$\begin{aligned} E_I \colon \operatorname{span}\{S_\mu f S_\nu^* \mid \mu,\nu \in \mathcal{W}_n, f \in C_0(X_I)\} &\ni x \\ \mapsto x_0 \in \operatorname{span}\big\{ S_\mu f S_\nu^* \bigm| \mu,\nu \in \mathcal{W}_n, |\mu| = |\nu|, f \in C_0(X_I)\big\}. \end{aligned}$$

Since E_I is norm-decreasing and $\operatorname{span}\{S_\mu f S_\nu^* \mid \mu,\nu \in \mathcal{W}_n, f \in C_0(X_I)\}$ is dense in $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$, we can extend E_I on $(\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ with $\|E_I\| \le 1$ whose image is contained in $\mathcal{F}_I$. Since $E_I(x) = x$ for $x \in \operatorname{span}\big\{ S_\mu f S_\nu^* \bigm| \mu,\nu \in \mathcal{W}_n, |\mu| = |\nu|, f \in C_0(X_I)\big\}$, which is dense in $\mathcal{F}_I$, we get $E_I(x) = x$ for any $x \in \mathcal{F}_I$.

Therefore E_I is a conditional expectation onto $\mathcal{F}_I$. Uniqueness follows from the condition $E_I(S_\mu f S_\nu^*) = \delta_{|\mu|,|\nu|} S_\mu f S_\nu^*$ for $\mu,\nu \in \mathcal{W}_n$ and $f \in C_0(X)$. ∎

When an ideal I such that X_I is good is gauge invariant, the conditional expectation E_I defined in Proposition 4.4 coincides with the one in Lemma 3.12 by uniqueness. Actually any ideal I such that X_I is good is gauge invariant.

Theorem 4.5 *Let I be an ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ such that X_I is good. Then we have $I_{X_I} = I$, and so I is gauge invariant.*

Proof If $X_I = \varnothing$, then $I = \mathcal{O}_n \rtimes_{\alpha^\omega} G$ so $I_{X_I} = I$. Let I be an ideal that is not $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, and set $J = I_{X_I}$. By the same way as in the proof of Proposition 3.13, there exists a surjective $*$-homomorphism $\pi\colon (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/J \to (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I$ whose restriction on $\mathcal{F}_J$ is an isomorphism from $\mathcal{F}_J$ onto $\mathcal{F}_I$. By Proposition 4.4, there exists a conditional expectation $E_I\colon (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/I \to \mathcal{F}_I$. Since $E_I\big(\pi(x)\big) = \pi\big(E_J(x)\big)$ for all x in a dense subset

$$\operatorname{span}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n, f \in C_0(X)\} \subset (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/J,$$

we have $E_I \circ \pi = \pi \circ E_J$ where $E_J\colon (\mathcal{O}_n \rtimes_{\alpha^\omega} G)/J \to \mathcal{F}_J$ is a faithful conditional expectation defined in Lemma 3.12. By Proposition 3.11, π is injective. Therefore $I = I_{X_I}$. ∎

When an ω-invariant set X is bad, there exists an ideal I with $X_I = X$ which is not gauge invariant (Proposition 5.26). As a special case of Theorem 4.5, we get the following.

Proposition 4.6 *Let I be an ideal of the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. Then $I = 0$ if and only if $X_I = \Gamma$.*

Proof The "only if" part is trivial. The "if" part follows from Theorem 4.5 since Γ is a good ω-invariant set. ∎

Definition 4.7 For a non-empty subset $\mathbb{I}$ of $\{1, 2, \ldots, n\}$, we denote by $\Omega_{\mathbb{I}}$ the closed semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$ and $-\omega_i$ for $i \in \mathbb{I}$.

For a non-empty subset $\mathbb{I}$ of $\{1, 2, \ldots, n\}$, the set $\Omega_{\mathbb{I}}$ is ω-invariant. In [Ki], A. Kishimoto found a necessary and sufficient condition for $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ to become simple. Now we can reprove it.

Theorem 4.8 (*cf.* **[Ki, Theorem 4.4]**) *The following conditions for $\omega \in \Gamma^n$ are equivalent:*

(i) *The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is simple.*
(ii) *Any ω-invariant subset of Γ must be $\varnothing$ or Γ.*
(iii) *$\Omega_{\{i\}} = \Gamma$ for any $i = 1, 2, \ldots, n$.*

Proof (i) $\Leftrightarrow$ (ii): Combine Proposition 4.1 and Proposition 4.6.

(ii) $\Rightarrow$ (iii): Since $\Omega_{\{i\}}$ is a non-empty ω-invariant subset, we have $\Omega_{\{i\}} = \Gamma$ for any i.

(iii) $\Rightarrow$ (ii): Let X be a non-empty ω-invariant subset. Let us choose an element $\gamma_0 \in X$. There exists $i \in \{1, 2, \ldots, n\}$ such that $\gamma_0 + \gamma \in X$ for any $\gamma \in \Omega_{\{i\}}$. Since $\Omega_{\{i\}}$ is Γ, we get $X = \Gamma$. ∎

Now we turn to determining for which $\omega \in \Gamma^n$ the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ becomes primitive. An ideal I of a C^*-algebra A is called *primitive* if I is a kernel of some irreducible representation. A C^*-algebra A is called *primitive* if 0 is a primitive ideal. When a C^*-algebra A is separable, an ideal I of A is primitive if and only if I is prime, *i.e.* $I_1 \cap I_2 \subset I$ implies $I_1 \subset I$ or $I_2 \subset I$ for ideals I_1, I_2 of A.

Definition 4.9 An ω-invariant set X is called *prime* if for any ω-invariant sets X_1, X_2 with $X \subset X_1 \cup X_2$, either $X \subset X_1$ or $X \subset X_2$ holds.

Proposition 4.10 *If an ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is primitive, then X_I becomes a prime ω-invariant set.*

Proof Let I be a primitive ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. Assume that two ω-invariant subsets X_1, X_2 of Γ satisfy $X_I \subset X_1 \cup X_2$. Then $I_{X_1} \cap I_{X_2} \subset I_{X_I} \subset I$. Since I is prime, either $I_{X_1} \subset I$ or $I_{X_2} \subset I$. Hence either $X_1 \supset X_I$ or $X_2 \supset X_I$. Therefore X_I is prime. ∎

In general, the converse of Proposition 4.10 is not true (see Corollary 5.3 and Proposition 5.43).

Proposition 4.11 *For a non-empty ω-invariant set X, the following are equivalent:*

(i) *X is prime.*
(ii) *For any $\gamma_0, \gamma_1 \in X$ and any neighborhoods U_0 and U_1 of γ_0 and γ_1 respectively, there exist $\gamma \in X$ and $\mu, \nu \in \mathcal{W}_n$ such that $\gamma + \omega_\mu \in U_0$ and $\gamma + \omega_\nu \in U_1$.*
(iii) *For any $\gamma_0, \gamma_1 \in X$, there exist sequences $\mu_1, \mu_2, \dots$ and $\nu_1, \nu_2, \dots$ in $\mathcal{W}_n$ such that $\gamma_0 - \omega_{\mu_k}, \gamma_1 - \omega_{\nu_k} \in X$ for any k and $\lim_{k\to\infty}\big((\gamma_0 - \omega_{\mu_k}) - (\gamma_1 - \omega_{\nu_k})\big) = 0$.*
(iv) *$X = \gamma + \Omega_{\mathbb{I}}$ for some $\gamma \in \Gamma$ and non-empty $\mathbb{I} \subset \{1, 2, \dots, n\}$.*

Proof (i) $\Rightarrow$ (ii): Let X be a non-empty prime ω-invariant set, γ_0, γ_1 elements of X, and U_0, U_1 neighborhoods of γ_0, γ_1 respectively. Set two open sets Y_0, Y_1 by

$$Y_0 = \bigcup_{k=0}^{\infty} \bigcap_{\mu \in \mathcal{W}_n^{(k)}} \bigcup_{\nu \in \mathcal{W}_n} (U_0 + \omega_\mu - \omega_\nu), \quad Y_1 = \bigcup_{k=0}^{\infty} \bigcap_{\mu \in \mathcal{W}_n^{(k)}} \bigcup_{\nu \in \mathcal{W}_n} (U_1 + \omega_\mu - \omega_\nu).$$

One can easily see that $\gamma - \omega_i \in Y_0$ for any $\gamma \in Y_0$ and any $i = 1, 2, \dots, n$, and that if $\gamma - \omega_i \in Y_0$ for any $i = 1, 2, \dots, n$, then $\gamma \in Y_0$. Thus the closed set $\Gamma \setminus Y_0$ is ω-invariant. Similarly, $\Gamma \setminus Y_1$ is ω-invariant. Since $\gamma_0 \in Y_0$ and $\gamma_1 \in Y_1$, neither $\Gamma \setminus Y_0$ nor $\Gamma \setminus Y_1$ contains X. Since X is prime, $(\Gamma \setminus Y_0) \cup (\Gamma \setminus Y_1)$ does not contain X. Therefore we get $\gamma' \in X$ with $\gamma' \in Y_0 \cap Y_1$. Thus for $j = 0, 1$, there exist $k_j \in \mathbb{N}$ satisfying that for any $\mu_j \in \mathcal{W}_n^{(k_j)}$, there exists $\nu_j \in \mathcal{W}_n$ with $\gamma' \in U_j + \omega_{\mu_j} - \omega_{\nu_j}$. Let $k \in \mathbb{N}$ be an integer with $k \geq k_0, k_1$. Since X is ω-invariant, there exists $\mu' \in \mathcal{W}_n^{(k)}$ with $\gamma' - \omega_{\mu'} \in X$. For $j = 0, 1$, there exist $\mu_j, \mu_j' \in \mathcal{W}_n$ with $\mu' = \mu_j \mu_j'$ and $|\mu_j| = k_j$. Thus we get $\nu_j \in \mathcal{W}_n$ with $\gamma' \in U_j + \omega_{\mu_j} - \omega_{\nu_j}$ for $j = 0, 1$. Set $\gamma = \gamma' - \omega_{\mu'} \in X$, $\mu = \nu_0 \mu_0'$ and, $\nu = \nu_1 \mu_1'$. Then, we have $\gamma + \omega_\mu \in U_0$ and $\gamma + \omega_\nu \in U_1$.

(ii) $\Rightarrow$ (iii): Let γ_0, γ_1 be elements of an ω-invariant set X. Let $U_1, U_2, \dots$ be a fundamental system of neighborhoods of 0. From (ii), for any $k = 1, 2, \dots$, there exist $\lambda_k \in X$ and $\mu_k, \nu_k \in \mathcal{W}_n$ such that $\lambda_k + \omega_{\mu_k} \in U_k + \gamma_0$ and $\lambda_k + \omega_{\nu_k} \in U_k + \gamma_1$. Replacing $\{k\}$ by a subsequence if necessary, we may assume that the number of i's appearing in μ_k and the one appearing in ν_k increase for any $i = 1, 2, \dots, n$. For any positive integer k, we have $\gamma_0 - \omega_{\mu_k} \in X$ because $\gamma_0 - \omega_{\mu_k} = \lim_{l\to\infty}(\lambda_l + \omega_{\mu_l} - \omega_{\mu_k})$

and $\lambda_l + \omega_{\mu_l} - \omega_{\mu_k} \in X$ when $l \geq k$. Similarly we have $\gamma_1 - \omega_{\nu_k} \in X$ for any positive integer k. Since $\lim_{k\to\infty}(\gamma_0 - \omega_{\mu_k} - \lambda_k) = 0$ and $\lim_{k\to\infty}\big(\lambda_k - (\gamma_1 - \omega_{\nu_k})\big) = 0$, we have $\lim_{k\to\infty}\big((\gamma_0 - \omega_{\mu_k}) - (\gamma_1 - \omega_{\nu_k})\big) = 0$.

(iii) $\Rightarrow$ (iv): Take $\gamma_0 \in X$ arbitrarily. From (iii), the countable set $X' = \{\gamma \in X \mid \gamma = \gamma_0 - \omega_\mu + \omega_\nu$ for some $\mu, \nu \in \mathcal{W}_n\}$ is dense in X. Denote all the elements of X' by $\{\lambda_1, \lambda_2, \dots\}$. Let $U_1, U_2, \dots$ be a fundamental system of neighborhoods of 0. Let us choose a bijection $\mathbb{Z}^+ \ni k \mapsto (m_k, l_k) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ where $\mathbb{Z}^+$ is the set of positive integers. Thus we have $\{(\lambda_{m_k}, U_{l_k})\}_{k=1}^\infty = \{(\lambda_m, U_l)\}_{m,l=1}^\infty$. By (iii), for a positive integer k, we can recursively find $\mu_k, \nu_k \in \mathcal{W}_n$ satisfying that $\gamma_0 - \sum_{j=1}^k \omega_{\mu_j} \in X$ and $\gamma_0 - \sum_{j=1}^k \omega_{\mu_j} - (\lambda_{m_k} - \omega_{\nu_k}) \in U_{l_k}$. Since an element $\gamma_0 - \sum_{j=1}^k \omega_{\mu_j} + \omega_\nu$ is in X for any positive integer k and any $\nu \in \mathcal{W}_n$, we have

$$\overline{\Big\{\gamma_0 - \sum_{j=1}^k \omega_{\mu_j} + \omega_\nu \mid k \in \mathbb{Z}^+, \nu \in \mathcal{W}_n\Big\}} \subset X.$$

Since the set of the left hand side above contains X' which is dense in X, the inclusion above is actually an equality. Let $\mathbb{I}$ be the set of $i \in \{1, 2, \dots, n\}$ such that the number of i's appearing in $\mu_1\mu_2\cdots\mu_k$ goes to infinity when k goes to infinity. For $i \notin \mathbb{I}$, let n_i be the limit of the number of i's appearing in $\mu_1\mu_2\cdots\mu_k$ when k goes to infinity. Set $\gamma = \gamma_0 - \sum_{i\notin\mathbb{I}} n_i\omega_i$. Then, one can see that $X = \gamma + \Omega_{\mathbb{I}}$.

(iv) $\Rightarrow$ (i): Let X be an ω-invariant set such that $X = \gamma + \Omega_{\mathbb{I}}$ for some $\gamma \in \Gamma$ and non-empty $\mathbb{I} \subset \{1, 2, \dots, n\}$. Take ω-invariant sets X_1, X_2 with $X \subset X_1 \cup X_2$. Since $\gamma - k(\sum_{i\in\mathbb{I}}\omega_i) \in X$ for any positive integer k, either X_1 or X_2, say X_1, contains $\gamma - k(\sum_{i\in\mathbb{I}}\omega_i)$ for infinitely many k. Then, X_1 contains $\gamma + \gamma'$ for any γ' in the (algebraic) semigroup generated by $\omega_1, \omega_2, \dots, \omega_n$ and $-\omega_i$ for $i \in \mathbb{I}$. Since X_1 is closed, $X_1 \supset \gamma + \Omega_{\mathbb{I}} = X$. Thus X is prime. ∎

We will use the equivalence (i) $\Leftrightarrow$ (iv) in Proposition 4.11 most often. The condition (ii) or (iii) in Proposition 4.11 can be considered as an analogue of maximal tails in [BPRS].

Theorem 4.12 *The following conditions for $\omega \in \Gamma^n$ are equivalent:*

(i) *The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is primitive.*
(ii) *Γ is a prime ω-invariant set.*
(iii) *The closed group generated by $\omega_1, \omega_2, \dots, \omega_n$ is equal to Γ.*

Proof (i) $\Rightarrow$ (ii): This follows from Proposition 4.10.

(ii) $\Rightarrow$ (i): It suffices to show that 0 is prime. Let I_1, I_2 be ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ with $I_1 \cap I_2 = 0$. We have $X_{I_1} \cup X_{I_2} = X_{I_1\cap I_2} = \Gamma$. Since Γ is prime, either $X_{I_1} \supset \Gamma$ or $X_{I_2} \supset \Gamma$. If $X_{I_1} \supset \Gamma$ hence $X_{I_1} = \Gamma$, then $I_1 = 0$ by Proposition 4.6. Similarly if $X_{I_2} \supset \Gamma$, then $I_2 = 0$. Thus 0 is prime and so $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is a primitive C^*-algebra.

(ii) $\Rightarrow$ (iii): By Proposition 4.11, there exist $\gamma \in \Gamma$ and non-empty $\mathbb{I} \subset \{1, 2, \dots, n\}$ with $\Gamma = \gamma + \Omega_{\mathbb{I}}$. The closed group generated by $\omega_1, \omega_2, \dots, \omega_n$ is equal to Γ because it contains $\Omega_{\mathbb{I}}$ and $\Omega_{\mathbb{I}} = \Gamma - \gamma = \Gamma$.

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(iii) $\Rightarrow$ (ii): This follows from Proposition 4.11 since $\Gamma = \Omega_{\{1,2,\dots,n\}}$. ∎

One can prove the equivalence between (i) and (iii) in the above theorem by characterization of primitivity of crossed products in terms of the Connes spectrum due to D. Olesen and G. K. Pedersen [OP1] and the computation of the Connes spectrum of our actions α^ω due to A. Kishimoto [Ki].

5 The Ideal Structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$

In this section, we completely determine the ideal structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ (Theorem 5.2, Theorem 5.49). The ideal structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ depend on whether $\omega \in \Gamma^n$ satisfies the following condition:

Condition 5.1 For each $i \in \{1, 2, \dots, n\}$, one of the following two conditions is satisfied:

(i) For any positive integer k, $k\omega_i \neq 0$.

(ii) There exists $j \neq i$ such that $-\omega_j \in \Omega_{\{i\}}$.

This condition is an analogue of Condition (II) in the case of Cuntz-Krieger algebras [C2] or Condition (K) in the case of graph algebras [KPRR].

5.1 When ω Satisfies Condition 5.1

When ω satisfies Condition 5.1, all ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ are gauge invariant.

Theorem 5.2 *When ω satisfies Condition 5.1, every ω-invariant set is good. Hence any ideal is gauge invariant and there is an inclusion reversing one-to-one correspondence between the set of ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and the set of ω-invariant subsets of Γ.*

Proof Let X be an ω-invariant set and γ be an element of X. Since X is ω-invariant, there exists $i \in \{1, 2, \dots, n\}$ such that $\gamma + \gamma' \in X$ for any $\gamma' \in \Omega_{\{i\}}$. If $k\omega_i \neq 0$ for any positive integer k, then $\gamma \in X$ satisfies the condition (i) in Lemma 4.3 and if there exists $j \neq i$ such that $-\omega_j \in \Omega_{\{i\}}$, then $\gamma \in X$ satisfies the condition (ii) in Lemma 4.3. Hence X is a good ω-invariant set.

By Theorem 4.5, any ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ satisfies $I_{X_I} = I$ and is gauge invariant. The last part follows from Theorem 3.14. ∎

Corollary 5.3 *When ω satisfies Condition 5.1, an ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is primitive if and only if the ω-invariant set X_I is prime.*

Proof It follows from Theorem 5.2. ∎

5.2 When ω Does Not Satisfy Condition 5.1

From here until the end of this section, we assume that ω does not satisfy Condition 5.1, *i.e.* there exists $i \in \{1, 2, \dots, n\}$ such that $k\omega_i = 0$ for some positive integer k, and that $-\omega_j$ is not in the closed semigroup generated by $\omega_1, \omega_2, \dots, \omega_n$ and $-\omega_i$ for any $j \neq i$. Without loss of generality, we may assume $i = 1$. Let K be the smallest positive integer satisfying $K\omega_1 = 0$. Note that $-\omega_1$ is in the semigroup generated by $\omega_1, \omega_2, \dots, \omega_n$ and that the closed set X is ω-invariant if and only if $X + \omega_i \subset X$ for any i. Define $A_0 = \operatorname{span}\{S_1^k f S_1^{*l} \mid f \in C_0(\Gamma), k, l \in \mathbb{N}\}$ which is a $*$-subalgebra of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and denote its closure in $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ by A.

Lemma 5.4 *For any $x \in A$, the element $(1 - S_1S_1^*)x(1 - S_1S_1^*)$ is of the form $(1 - S_1S_1^*)f$ for some $f \in C_0(\Gamma)$.*

Proof One can easily verify the conclusion for $x \in A_0$. We have the conclusion for an arbitrary $x \in A$, because $\{(1 - S_1S_1^*)f \mid f \in C_0(\Gamma)\}$ is closed. ■

Lemma 5.5 *The C^*-algebra A is the universal C^*-algebra generated by $C_0(\Gamma)$ and $S_1 \in M(A)$, that is, for any C^*-algebra B, any $*$-homomorphism $\varphi\colon C_0(\Gamma) \to B$ and $u \in M(B)$ such that $u^*u = 1_{M(B)}$ and $\varphi(f)u = u\varphi(\sigma_{\omega_1} f)$ for $f \in C_0(\Gamma)$, there exists a unique $*$-homomorphism $\Phi\colon A \to B$ such that $\Phi(S_1^k f S_1^{*l}) = u^k\varphi(f)u^{*l}$ for $k, l \in \mathbb{N}$ and $f \in C_0(\Gamma)$.*

Proof Let $\tilde{A}$ be the universal C^*-algebra satisfying the condition in the statement of this lemma. We may consider $C_0(\Gamma)$ as a C^*-subalgebra of $\tilde{A}$ and denote by $u \in M(\tilde{A})$ the isometry satisfying $fu = u\sigma_{\omega_1} f$ for $f \in C_0(\Gamma) \subset \tilde{A}$. The C^*-algebra $\tilde{A}$ is the closure of the linear span of elements $u^k f u^{*l}$ for $k, l \in \mathbb{N}$ and $f \in C_0(\Gamma)$. There is a unique $*$-homomorphism $\Psi\colon \tilde{A} \to A$ such that $\Psi(u^k f u^{*l}) = S_1^k f S_1^{*l}$. Since $\operatorname{span}\{S_1^k f S_1^{*l} \mid f \in C_0(\Gamma), k, l \in \mathbb{N}\}$ is dense in A, Ψ is surjective. By the universality of $\tilde{A}$, there exists an action $\tilde{\beta}$ of $\mathbb{T}$ on $\tilde{A}$ such that $\tilde{\beta}_t(u^k f u^{*l}) = t^{k-l}u^k f u^{*l}$ for $t \in \mathbb{T}$. Define $\tilde{E}(x) = \int_{\mathbb{T}} \tilde{\beta}_t(x)\,dt$ for $x \in \tilde{A}$. Then $\tilde{E}$ becomes a faithful conditional expectation onto a C^*-subalgebra $\tilde{B} = \overline{\operatorname{span}}\{u^k f u^{*k} \mid f \in C_0(\Gamma), k \in \mathbb{N}\}$ of $\tilde{A}$. Since A is invariant under the gauge action β, we can define a conditional expectation E on A by $E(x) = \int_{\mathbb{T}} \beta_t(x)\,dt$. Obviously $\Psi \circ \tilde{E} = E \circ \Psi$. Let us define $\tilde{B}^{(k)} = \overline{\operatorname{span}}\{u^l f u^{*l} \mid f \in C_0(\Gamma), 0 \leq l \leq k\}$. Then we have

$$
\begin{aligned}
\tilde{B}^{(k)} &= \Big(\bigoplus_{l=0}^{k-1} \overline{\operatorname{span}}\{u^l(1 - uu^*)fu^{*l} \mid f \in C_0(\Gamma)\}\Big) \oplus \overline{\operatorname{span}}\{u^k f u^{*k} \mid f \in C_0(\Gamma)\} \\
&\cong \bigoplus_{l=0}^{k} C_0(\Gamma)
\end{aligned}
$$

and $\varinjlim \tilde{B}^{(k)} = \tilde{B}$. Clearly Ψ is injective on $\tilde{B}^{(k)}$, hence on $\tilde{B}$. By Proposition 3.11, $\tilde{A}$ is isomorphic to A via Ψ. Thus A is the universal C^*-algebra generated by $C_0(\Gamma)$ and $S_1 \in M(A)$. ■

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Remark 5.6 The C^*-algebra A is isomorphic to the Toeplitz algebra of the Hilbert module coming from the automorphism σ_{ω_1} of $C_0(\Gamma)$ [Pi], but we do not use this fact.

We will denote the elements of $\mathbb{Z}/K\mathbb{Z}$ by $0, 1, \dots, K-1$ and sometimes regard them as integers.

Definition 5.7 Let H be a separable Hilbert space whose complete orthonormal system is given by $\{\xi_{k,m} \mid k \in \mathbb{Z}/K\mathbb{Z}, m \in \mathbb{N}\}$. Let p_k be a projection onto the subspace generated by $\{\xi_{k,m}\}_{m\in\mathbb{N}}$ for $k \in \mathbb{Z}/K\mathbb{Z}$ and define $u \in B(H)$ by $u(\xi_{k,m}) = \xi_{k+1,m+1}$. Let us denote by T_K the C^*-algebra generated by $p_0, p_1, \dots, p_{K-1}$ and u.

One can easily see that the elements $p_0, p_1, \dots, p_{K-1}$ and u satisfy $\sum_{k=0}^{K-1} p_k = 1$, $u^*u = 1$, and $p_k u = u p_{k-1}$ for $k \in \mathbb{Z}/K\mathbb{Z}$, and that $T_K = \overline{\operatorname{span}}\{u^l p_k u^{*m} \mid k \in \mathbb{Z}/K\mathbb{Z}, l, m \in \mathbb{N}\}$. There is an action $\beta' \colon \mathbb{T} \curvearrowright T_K$ such that $\beta'_t(u) = tu$ and $\beta'_t(p_k) = p_k$. For $\lambda_0, \lambda_1, \dots, \lambda_{K-1} \in \mathbb{C}$ and $\theta \in \mathbb{T}$, the diagonal matrix and the unitary

$$\begin{pmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{K-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & \theta \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

are denoted by $\operatorname{diag}\{\lambda_0, \lambda_1, \dots, \lambda_{K-1}\} \in \mathbb{M}_K$ and $u_\theta \in \mathbb{M}_K$ respectively. The C^*-algebra T_K satisfies the following.

Proposition 5.8

(i) *For any non-zero $x \in T_K$, there exist $l, m \in \mathbb{N}$ with*

$$(1 - uu^*){u^*}^l x u^m (1 - uu^*) \neq 0.$$

(ii) *There exists a surjection $\pi \colon T_K \to C(\mathbb{T}, \mathbb{M}_K)$ with*

$$\pi\Big(\sum_{k=0}^{K-1} \lambda_k p_k\Big)(\theta) = \operatorname{diag}\{\lambda_0, \lambda_1, \dots, \lambda_{K-1}\}, \quad \pi(u)(\theta) = u_\theta.$$

(iii) *For $t \in \mathbb{T}$, we define a $*$-automorphism β''_t of $C(\mathbb{T}, \mathbb{M}_K)$ by*

$$\beta''_t(f)(\theta) = \operatorname{diag}\{1, t, t^2, \dots, t^{K-1}\} f(t^K\theta) \operatorname{diag}\{1, \bar{t}, \bar{t}^2, \dots, \bar{t}^{K-1}\}$$

for $f \in C(\mathbb{T}, \mathbb{M}_K)$. Then we have $\pi \circ \beta'_t = \beta''_t \circ \pi$.

(iv) *If an ideal J of T_K satisfies that $1 \in \pi(J)$, then $J = T_K$.*

Proof (i) For any $k \in \mathbb{Z}/K\mathbb{Z}$ and $m \in \mathbb{N}$, $u^m(1-uu^*)u^{*m}p_k$ is the projection onto the one dimensional subspace generated by $\xi_{k,m} \in H$. Hence, for any non-zero $x \in T_K$, there exist $k_1, k_2 \in \mathbb{Z}/K\mathbb{Z}$ and $m_1, m_2 \in \mathbb{N}$ such that

$$p_{k_1}u^{m_1}(1-uu^*)u^{*m_1}xu^{m_2}(1-uu^*)u^{*m_1}p_{k_2} \neq 0.$$

Thus, we get $(1-uu^*)u^{*m_1}xu^{m_2}(1-uu^*) \neq 0$.

(ii) For $k \in \mathbb{Z}/K\mathbb{Z}$, set $I_k = \overline{\operatorname{span}}\,\{u^l(1-uu^*)p_ku^{*m} \mid l, m \in \mathbb{N}\} \subset T_K$. Since

$$\bigl(u^l(1-uu^*)p_ku^{*m}\bigr)\bigl(u^{l'}(1-uu^*)p_{k'}u^{*m'}\bigr) = \delta_{k,k'}\delta_{m,l'}u^l(1-uu^*)p_ku^{*m'},$$

the set I_k is isomorphic to $\mathbb{K}$ for any $k \in \mathbb{Z}/K\mathbb{Z}$ and I_k is orthogonal to $I_{k'}$ if $k \neq k'$. One can easily see that $I = \bigoplus_{k=0}^{K-1} I_k$ becomes an ideal of T_K. Let us denote by π the quotient map from T_K onto T_K/I. We will prove that T_K/I is isomorphic to $C(\mathbb{T}, \mathbb{M}_K)$. Since $1-uu^* = \sum_{k=0}^{K-1}(1-uu^*)p_k \in I$, $\pi(u)$ is an unitary of T_K/I. One can verify that $e_{i,j} = \pi(p_i)\pi(u)^{i-j} = \pi(u)^{i-j}\pi(p_j)$ satisfies the axiom of matrix units of $\mathbb{M}_K$ for $i, j \in \mathbb{Z}/K\mathbb{Z}$. Thus $T_K/I \cong \mathbb{M}_K \otimes \pi(p_0)(T_K/I)\pi(p_0)$. Since

$$\begin{aligned} p_0T_Kp_0 &= \overline{\operatorname{span}}\,\{p_0u^lp_ku^{*m}p_0 \mid k \in \mathbb{Z}/K\mathbb{Z}, l, m \in \mathbb{N}\} \\ &= \overline{\operatorname{span}}\,\{p_0u^lu^{*m}p_0 \mid l, m \in \mathbb{N} \text{ with } l-m \in K\mathbb{Z}\}, \end{aligned}$$

$\pi(p_0)(T_K/I)\pi(p_0) = \pi(p_0T_Kp_0)$ is generated by one unitary $\pi(p_0u^Kp_0)$. Since $p_0T_Kp_0$ and I are globally invariant under the action β' of $\mathbb{T}$, we can define an action β' of $\mathbb{T}$ on $\pi(p_0)(T_K/I)\pi(p_0)$. Since $\beta'_t\bigl(\pi(p_0u^Kp_0)\bigr) = t^K\pi(p_0u^Kp_0)$, the spectrum of $\pi(p_0u^Kp_0)$ is $\mathbb{T}$. Therefore we have

$$\pi(p_0)(T_K/I)\pi(p_0) \cong C(\mathbb{T}).$$

Thus, we have $T_K/I \cong C(\mathbb{T}, \mathbb{M}_K)$ and one can easily verify that π is a desired surjection.

(iii) For $k \in \mathbb{Z}/K\mathbb{Z}$, we have $\pi \circ \beta'_t(p_k) = \beta''_t \circ \pi(p_k) = \pi(p_k)$. One can easily see that $\pi \circ \beta'_t(u)(\theta) = tu_\theta$. On the other hand,

$$\begin{aligned} \beta''_t \circ \pi(u)(\theta) &= \operatorname{diag}\{1, t, t^2, \dots, t^{K-1}\}\pi(u)(t^K\theta)\operatorname{diag}\{1, \bar{t}, \bar{t}^2, \dots, \bar{t}^{K-1}\} \\ &= \operatorname{diag}\{1, t, t^2, \dots, t^{K-1}\}u_{t^K\theta}\operatorname{diag}\{1, \bar{t}, \bar{t}^2, \dots, \bar{t}^{K-1}\} \\ &= tu_\theta. \end{aligned}$$

Therefore we have $\pi \circ \beta'_t = \beta''_t \circ \pi$.

(iv) Since $1 \in \pi(J)$, there exist $x_k \in I_k$ for each $k \in \mathbb{Z}/K\mathbb{Z}$ with $1 - \sum_{k=0}^{K-1} x_k \in J$. For $k \in \mathbb{Z}/K\mathbb{Z}$, there exists $y_k \in I_k$ such that $y_k \neq x_ky_k$ since I_k is not unital. For $k \in \mathbb{Z}/K\mathbb{Z}$, we have $(1 - \sum_{k=0}^{K-1} x_k)y_k = y_k - x_ky_k \neq 0$ which is in $J \cap I_k$. Since $I_k \cong \mathbb{K}$, $J \cap I_k \neq \{0\}$ implies $I_k \subset J$. Thus $1 \in J$ and so $J = T_K$. ∎

Proposition 5.9 *There is a unique $*$-homomorphism $\varphi\colon A \to C_0(\Gamma, T_K)$ such that $\varphi(S_1^l f S_1^{*m}) = u^l\big(\sum_{k=0}^{K-1}(\sigma_{k\omega_1} f)p_k\big)\, u^{*m}$. The map φ is injective and its image is*

$$\big\{ f \in C_0(\Gamma, T_K) \mid f(\gamma + \omega_1) = \Phi\big(f(\gamma)\big) \text{ for any } \gamma \in \Gamma\big\} \subset C_0(\Gamma, T_K),$$

where Φ is a $*$-automorphism of T_K satisfying $\Phi(u) = u$ and $\Phi(p_k) = p_{k-1}$.

Proof First note that $C_0(\Gamma) \ni f \mapsto \sum_{k=0}^{K-1}(\sigma_{k\omega_1} f)p_k \in C_0(\Gamma, T_K)$ is an injective $*$-homomorphism. Since this map and $u \in M\big(C_0(\Gamma, T_K)\big)$ satisfy the condition in Lemma 5.5, there exists a unique map $\varphi\colon A \to C_0(\Gamma, T_K)$ such that

$$\varphi(S_1^l f S_1^{*m}) = u^l\Big(\sum_{k=0}^{K-1}(\sigma_{k\omega_1} f)p_k\Big)\, u^{*m}.$$

As we saw in the proof of Lemma 5.5, there exists a faithful conditional expectation E from A onto the C^*-subalgebra B of A which is an inductive limit of $C_0(\Gamma)^k$. If one defines a conditional expectation E' on $C_0(\Gamma, T_K)$ by $E'(f)(\gamma) = \int_{\mathbb{T}} \beta'_t\big(f(\gamma)\big)\, dt$, then one can easily see that E, E' and φ satisfy the condition in Proposition 3.11. Hence φ is injective. Since the C^*-subalgebra $\{f \in C_0(\Gamma, T_K) \mid f(\gamma + \omega_1) = \Phi\big(f(\gamma)\big)$ for any $\gamma \in \Gamma\}$ is the closed linear span of $u^l\sum_{k=0}^{K-1}(\sigma_{k\omega_1} f)p_k u^{*m}$ for $l, m \in \mathbb{N}$ and $f \in C_0(\Gamma)$, this subalgebra is the image of φ. ∎

Definition 5.10 For $\gamma \in \Gamma$, we denote by $\varphi_\gamma\colon A \to T_K$ the composition of the map $\varphi\colon A \to C_0(\Gamma, T_K)$ in Proposition 5.9 and the evaluation map at $\gamma \in \Gamma$.

For $(\gamma, \theta) \in \Gamma \times \mathbb{T}$, we define $\psi_{\gamma,\theta}\colon A \to \mathbb{M}_K$ by the composition of $\varphi_\gamma\colon A \to T_K$, $\pi\colon T_K \to C(\mathbb{T}, \mathbb{M}_K)$ in Proposition 5.8 and the evaluation map at $\theta \in \mathbb{T}$.

Explicitly, we have

$$\varphi_\gamma(S_1^l f S_1^{*m}) = u^l\Big(\sum_{k=0}^{K-1} f(\gamma + k\omega_1)p_k\Big)\, u^{*m} \in T_K,$$

$$\psi_{\gamma,\theta}(S_1^l f S_1^{*m}) = u_\theta^l \operatorname{diag}\{f(\gamma), f(\gamma + \omega_1), \dots, f(\gamma + (K-1)\omega_1)\}u_\theta^{*m} \in \mathbb{M}_K.$$

As we saw in Proposition 5.9, we have $\varphi_{\gamma+\omega_1} = \Phi \circ \varphi_\gamma$ for any $\gamma \in \Gamma$ and one can easily see that for any $(\gamma, \theta) \in \Gamma \times \mathbb{T}$, $\psi_{\gamma+\omega_1,\theta}(x) = u_\theta^*\psi_{\gamma,\theta}(x)u_\theta$ for $x \in A$. For any $t \in \mathbb{T}$ and any $\gamma \in \Gamma$, we have $\varphi_\gamma \circ \beta_t = \beta'_t \circ \varphi_\gamma$.

Denote by Γ' the quotient of Γ by the subgroup generated by ω_1, which is isomorphic to $\mathbb{Z}/K\mathbb{Z}$. We denote by $[\gamma]$ and $[U]$ the images in Γ' of $\gamma \in \Gamma$ and $U \subset \Gamma$ respectively. We use the symbol $([\gamma], \theta)$ for denoting elements of $\Gamma' \times \mathbb{T}$.

Definition 5.11 For an ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, we define the closed subset Y_I of $\Gamma' \times \mathbb{T}$ by

$$Y_I = \{([\gamma], \theta) \in \Gamma' \times \mathbb{T} \mid \psi_{\gamma,\theta}(x) = 0 \text{ for all } x \in A \cap I\}.$$

Note that $\psi_{\gamma,\theta}(x) = 0$ if and only if $\psi_{\gamma+\omega_1,\theta}(x) = 0$.

Definition 5.12 A subset Y of $\Gamma' \times \mathbb{T}$ is called *ω-invariant* if Y is a closed set satisfying that $([\gamma + \omega_i], \theta') \in Y$ for any $i \neq 1$, any $\theta' \in \mathbb{T}$ and any $([\gamma], \theta) \in Y$.

To show that the closed set Y_I is ω-invariant for an ideal I, we need the following lemma.

Lemma 5.13 *For any $x \in A$, $(\gamma, \theta) \in \Gamma \times \mathbb{T}$, and $i \neq 1$, we have*

$$\int_{\mathbb{T}} E\big(\psi_{\gamma+\omega_i,t}(x)\big)\, dt = \lim_{m\to\infty} \psi_{\gamma,\theta}(S_i^* S_1^{*\,mK} x S_1^{mK} S_i),$$

where E is the conditional expectation from $\mathbb{M}_K$ to its C^*-subalgebra of diagonal matrices.

Proof Take $(\gamma, \theta) \in \Gamma \times \mathbb{T}$, and $i \neq 1$. First we consider an element $x = S_1^k S_1^{*\,l} f \in A_0$ for $f \in C_0(\Gamma)$ and $k, l \in \mathbb{N}$. We have

$$\int_{\mathbb{T}} E\big(\psi_{\gamma+\omega_i,t}(x)\big)\, dt = \int_{\mathbb{T}} E\big(u_t^{k-l}\psi_{\gamma+\omega_i,\theta}(f)\big)\, dt,$$

here note that $\psi_{\gamma+\omega_i,t}(f)$ does not depend on $t \in \mathbb{T}$. When $k - l$ is not a multiple of K, we have $E\big(u_t^{k-l}\psi_{\gamma+\omega_i,\theta}(f)\big) = 0$. When $k - l = mK$ for some integer m, we have $E\big(u_t^{k-l}\psi_{\gamma+\omega_i,\theta}(f)\big) = t^m \psi_{\gamma+\omega_i,\theta}(f)$. Since $\int_{\mathbb{T}} t^m \psi_{\gamma+\omega_i,\theta}(f)\, dt = \delta_{m,0}\psi_{\gamma+\omega_i,\theta}(f)$, we get

$$\int_{\mathbb{T}} E\big(\psi_{\gamma+\omega_i,t}(x)\big)\, dt = \delta_{k,l}\psi_{\gamma+\omega_i,\theta}(f).$$

On the other hand, for any positive integer m satisfying $mK \geq k, l$, we have

$$\psi_{\gamma,\theta}(S_i^* S_1^{*\,mK} x S_1^{mK} S_i) = \delta_{k,l}\psi_{\gamma+\omega_i,\theta}(f)$$

because $S_i^* S_1^{*\,mK} x S_1^{mK} S_i = \delta_{k,l}\sigma_{\omega_i} f \in A$. By the linearity of the equation, for any $x \in A_0$, there exists a positive integer M such that

$$\int_{\mathbb{T}} E\big(\psi_{\gamma+\omega_i,t}(x)\big)\, dt = \psi_{\gamma,\theta}(S_i^* S_1^{*\,mK} x S_1^{mK} S_i)$$

for $m \geq M$. Approximating x by elements of A_0, we have

$$\int_{\mathbb{T}} E\big(\psi_{\gamma+\omega_i,t}(x)\big)\, dt = \lim_{m\to\infty} \psi_{\gamma,\theta}(S_i^* S_1^{*\,mK} x S_1^{mK} S_i)$$

for an arbitrary $x \in A$. ■

Proposition 5.14 *For any ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, the closed set Y_I is ω-invariant.*

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Proof Take $([\gamma],\theta)\in Y_I$, $i\neq 1$ and $\theta'\in\mathbb{T}$. By Lemma 5.13, for any positive element x of $A\cap I$,

$$\int_{\mathbb{T}} E\bigl(\psi_{\gamma+\omega_i,t}(x)\bigr)\,dt=\lim_{m\to\infty}\psi_{\gamma,\theta}(S_i^*S_1^{*mK}xS_1^{mK}S_i)=0$$

since $S_i^*S_1^{*mK}xS_1^{mK}S_i\in A\cap I$. Hence $E\bigl(\psi_{\gamma+\omega_i,\theta'}(x)\bigr)=0$. Since E is faithful, $\psi_{\gamma+\omega_i,\theta'}(x)=0$ for any $x\in A\cap I$. It implies that $([\gamma+\omega_i],\theta')\in Y_I$. Thus Y_I is ω-invariant. ∎

For an ideal I of $\mathcal{O}_n\rtimes_{\alpha^\omega}G$, the closed set X_I, defined in Definition 3.1, is determined from Y_I as follows:

Proposition 5.15 *For an ideal I of $\mathcal{O}_n\rtimes_{\alpha^\omega}G$, we have*

$$X_I=\{\gamma\in\Gamma\mid([\gamma],\theta)\in Y_I \text{ for some } \theta\in\mathbb{T}\}.$$

Proof When $\gamma\notin X_I$, there exists $f\in C_0(\Gamma)\cap I\subset A\cap I$ with $f(\gamma)\neq 0$. Then for any $\theta\in\mathbb{T}$, $\psi_{\gamma,\theta}(f)\neq 0$. Thus $([\gamma],\theta)\notin Y_I$ for any $\theta\in\mathbb{T}$. Conversely assume $\gamma\in\Gamma$ satisfies $([\gamma],\theta)\notin Y_I$ for any $\theta\in\mathbb{T}$. Then the ideal $J=\varphi_\gamma(A\cap I)$ of T_K satisfies $1\in\pi(J)$ where π is the surjection in Proposition 5.8. By Proposition 5.8(iv), we have $\varphi_\gamma(A\cap I)=T_K$. Hence there exists $x\in A\cap I$ with $\varphi_\gamma(x)=1$. By Proposition 5.9, φ induces the isomorphism from A to

$$\bigl\{f\in C_0(\Gamma,T_K)\mid f(\gamma+\omega_1)=\Phi\bigl(f(\gamma)\bigr)\text{ for any }\gamma\in\Gamma\bigr\}\subset C_0(\Gamma,T_K).$$

Therefor we can find $y\in A$ such that $xy\in C_0(\Gamma)$ and $\varphi_\gamma(xy)=1$. Since $xy\in C_0(\Gamma)\cap I$, we have $\gamma\notin X_I$. ∎

We get the ω-invariant set Y_I from an ideal I of $\mathcal{O}_n\rtimes_{\alpha^\omega}G$. Conversely, from an ω-invariant set Y, we can construct the ideal I_Y of $\mathcal{O}_n\rtimes_{\alpha^\omega}G$.

Definition 5.16 Let Y be an ω-invariant subset of $\Gamma'\times\mathbb{T}$. We define the subset $Y|_\Gamma$ of Γ by

$$Y|_\Gamma=\{\gamma\in\Gamma\mid\text{there exist } i\neq 1 \text{ and } \theta\in\mathbb{T}\text{ such that }([\gamma-\omega_i],\theta)\in Y\},$$

Since $\mathbb{T}$ is compact, the set

$$X_i=\{\gamma\in\Gamma\mid\text{there exists }\theta\in\mathbb{T}\text{ such that }([\gamma-\omega_i],\theta)\in Y\}$$

is closed for $i=2,3,\dots,n$. Thus $Y|_\Gamma=\bigcup_{i=2}^n X_i$ is a closed set of Γ. Since Y is ω-invariant, we have $([\gamma],\theta)\in Y$ for any $\gamma\in Y|_\Gamma$ and any $\theta\in\mathbb{T}$.

Definition 5.17 For an ω-invariant subset Y of $\Gamma'\times\mathbb{T}$, we define $J_Y\subset A$ and $I_Y\subset\mathcal{O}_n\rtimes_{\alpha^\omega}G$ by

$$J_Y=\{x\in A\mid\psi_{\gamma,\theta}(x)=0\text{ for }([\gamma],\theta)\in Y,\text{ and }\varphi_\gamma(x)=0\text{ for }\gamma\in Y|_\Gamma\},$$
$$I_Y=\overline{\mathrm{span}}\,\{S_\mu xS_\nu^*\mid\mu,\nu\in\mathcal{W}_n,x\in J_Y\}.$$

Clearly by the definition, J_Y is an ideal of A. To see that I_Y is an ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, we need the following lemmas.

Lemma 5.18 *For any $x \in A$ and $i \neq 1$, we have the following.*

(i) $(1 - S_1 S_1^* - S_i S_i^*) x S_i = 0$.
(ii) $\lim_{m\to\infty} S_1^m {S_1^*}^m x S_i = 0$.
(iii) $x S_i = \sum_{k=0}^{\infty} S_1^k S_i \big(S_i^* ({S_1^*}^k x) S_i \big)$.

Proof (i) and (ii): It suffices to prove them for an element of A_0, and this is done just by computation.

(iii): By (i), we have $S_1^k (1 - S_1 S_1^* - S_i S_i^*)({S_1^*}^k x) S_i = 0$ for any $k \in \mathbb{N}$. Taking a summation, we get

$$\Big(1 - S_1^m {S_1^*}^m - \sum_{k=0}^{m-1} S_1^k S_i S_i^* {S_1^*}^k \Big) x S_i = 0.$$

By (ii), we have $x S_i = \sum_{k=0}^{\infty} S_1^k S_i S_i^* {S_1^*}^k x S_i$. ∎

Lemma 5.19 *Let Y be an ω-invariant subset of $\Gamma' \times \mathbb{T}$. For any $x \in J_Y$ and $i \neq 1$, we have $S_i^* x S_i \in J_Y$.*

Proof Since $S_i = (1 - S_1 S_1^*) S_i$, we have $S_i^* x S_i = S_i^* (1 - S_1 S_1^*) x (1 - S_1 S_1^*) S_i$. By Lemma 5.4, $(1 - S_1 S_1^*) x (1 - S_1 S_1^*) = (1 - S_1 S_1^*) f$ for some $f \in C_0(\Gamma)$. Hence $S_i^* x S_i = \sigma_{\omega_i} f$. Since x is in J_Y, so is $(1 - S_1 S_1^*) f$. Let $([\gamma], \theta) \in Y$ be given. Since $\gamma + \omega_i \in Y|_\Gamma$, we have $\varphi_{\gamma+\omega_i}((1 - S_1 S_1^*) f) = 0$. It implies that $(1 - uu^*) \sum_{k=0}^{K-1} f(\gamma + \omega_i + k\omega_1) p_k = 0$, and so $f(\gamma + \omega_i + k\omega_1) = 0$ for $k = 0, 1, \ldots, K-1$. Therefore, we have

$$\begin{aligned}\psi_{\gamma,\theta}(S_i^* x S_i) &= \psi_{\gamma,\theta}(\sigma_{\omega_i} f) \\ &= \operatorname{diag}\big\{ f(\gamma + \omega_i), f(\gamma + \omega_i + \omega_1), \ldots, f\big(\gamma + \omega_i + (K-1)\omega_1\big) \big\} = 0.\end{aligned}$$

Similarly, if $\gamma \in Y|_\Gamma$, then $\gamma + \omega_i \in Y|_\Gamma$, and so

$$\varphi_\gamma(S_i^* x S_i) = \varphi_\gamma(\sigma_{\omega_i} f) = \sum_{k=0}^{K-1} f(\gamma + \omega_i + k\omega_1) p_k = 0.$$

Therefore $S_i^* x S_i \in J_Y$. ∎

Proposition 5.20 *For an ω-invariant subset Y of $\Gamma' \times \mathbb{T}$, I_Y is an ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$.*

Proof By definition, I_Y is a $*$-invariant closed subspace of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. To prove that I_Y is an ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, it suffices to show that for any $\mu, \nu \in \mathcal{W}_n$, $x \in J_Y$, the products of $y = S_\mu x S_\nu^* \in I_Y$ and S_i, S_i^* $(i = 1, 2, \ldots, n)$, or $f \in C_0(\Gamma)$ are in I_Y. It is clear that $y S_i^* = S_\mu x S_\nu^* S_i^* \in I_Y$ and $y f = S_\mu x(\sigma_{\omega_\nu} f) S_\nu^* \in I_Y$. It is also clear that $y S_i \in I_Y$ when $\nu \neq \varnothing$ or $i = 1$. Hence all we have to do is to prove $S_\mu x S_i \in I_Y$ for $\mu \in \mathcal{W}_n$,

$x \in J_Y$ and $i \neq 1$. By Lemma 5.18, we have $S_\mu x S_i = \sum_{k=0}^{\infty} S_\mu S_1^k S_i \big(S_i^* (S_1^{*k} x) S_i \big)$. By Lemma 5.19, we have $S_i^* (S_1^{*k} x) S_i \in J_Y$ for any positive integer k. Therefore $S_\mu x S_i \in I_Y$. We are done. ∎

We will show that an ideal I_Y satisfies that $Y_{I_Y} = Y$ for any ω-invariant subset Y of $\Gamma' \times \mathbb{T}$.

Lemma 5.21 *For an ω-invariant subset Y of $\Gamma' \times \mathbb{T}$, we have $I_Y \cap A = J_Y$.*

Proof By the definition of I_Y, we have $I_Y \cap A \supset J_Y$. We will prove the other inclusion. Take $x \in I_Y \cap A$. For an arbitrary $\varepsilon > 0$, there exist $\mu_l, \nu_l \in \mathcal{W}_n$ and $x_l \in J_Y$ for $l = 1, 2, \ldots, L$ such that $\|x - \sum_{l=1}^{L} S_{\mu_l} x_l S_{\nu_l}^*\| < \varepsilon$. Take a positive integer m such that $m \geq |\mu_l|, |\nu_l|$ for $l = 1, 2, \ldots, L$. Then, $\|S_1^{*m} x S_1^m - \sum_{l=1}^{L} x_l'\| < \varepsilon$ where $x_l' = S_1^{*m} S_{\mu_l} x_l S_{\nu_l}^* S_1^m$ for $l = 1, 2, \ldots, L$. Since $x_l' \in J_Y$, we have $\|\psi_{\gamma,\theta}(S_1^{*m} x S_1^m)\| < \varepsilon$ for $([\gamma], \theta) \in Y$. Since $\psi_{\gamma,\theta}(S_1)$ is a unitary, we have $\|\psi_{\gamma,\theta}(x)\| < \varepsilon$ for arbitrary $\varepsilon > 0$. Hence, we have $\psi_{\gamma,\theta}(x) = 0$ for any $([\gamma], \theta) \in Y$.

Let γ be an element of Γ. Assume $\varphi_\gamma(x) \neq 0$ and we will prove that $\gamma \notin Y|_\Gamma$. By Proposition 5.8 (i), there exist $k, l \in \mathbb{N}$ satisfying $(1 - uu^*) u^{*k} \varphi_\gamma(x) u^l (1 - uu^*) \neq 0$. Set $y = (1 - S_1 S_1^*) S_1^{*k} x S_1^l (1 - S_1 S_1^*) \in I_Y$. Then there exists $f \in C_0(\Gamma)$ with $y = (1 - S_1 S_1^*) f$. Since $\varphi_\gamma(y) \neq 0$, there exists an integer k with $0 \leq k \leq K - 1$ such that $f(\gamma + k\omega_1) \neq 0$. Therefore, for any $i \neq 1$ and any $\theta \in \mathbb{T}$, we have $\psi_{\gamma-\omega_i,\theta}(S_i^* y S_i) = \psi_{\gamma-\omega_i,\theta}(\sigma_{\omega_i} f) \neq 0$. By the former part of this proof, we have $([\gamma - \omega_i], \theta) \notin Y$ for any $i \neq 1$ and any $\theta \in \mathbb{T}$ because $S_i^* y S_i \in I_Y \cap A$. It implies that $\gamma \notin Y|_\Gamma$.

Therefore $x \in J_Y$. We have proved the inclusion $I_Y \cap A \subset J_Y$ and so $I_Y \cap A = J_Y$. ∎

Lemma 5.22 *Let Y be an ω-invariant subset of $\Gamma' \times \mathbb{T}$. For any $([\gamma_0], \theta_0) \notin Y$, there exists $x_0 \in J_Y$ such that $\psi_{\gamma_0,\theta_0}(x_0) \neq 0$.*

Proof Since $([\gamma_0], \theta_0) \notin Y$, we have $\gamma_0 \notin Y|_\Gamma$. Since $Y|_\Gamma$ and Y are closed, there exist a neighborhood $U \subset \Gamma$ of γ_0 and a neighborhood $V \subset \mathbb{T}$ of θ_0 such that $U + \omega_1 = U$, $Y|_\Gamma \cap U = \varnothing$ and $Y \cap ([U] \times V) = \varnothing$. Take $g \in C(\mathbb{T})$ whose support is contained in V and satisfying $g(\theta_0) = 1$. The C^*-algebra $C^*(S_1^K)$ generated by S_1^K in $M(A)$ is isomorphic to the Toeplitz algebra. There exists an element $x \in C^*(S_1^K) \subset M(A)$ such that $\Psi_\theta(x) = g(\theta)$ where Ψ_θ is the $*$-homomorphism from $C^*(S_1^K)$ to $\mathbb{C}$ determined by $\Psi_\theta(S_1^K) = \theta$ for $\theta \in \mathbb{T}$. Take $f \in C_0(\Gamma)$ whose support is contained in U and satisfying $f(\gamma_0) = 1$ and set $x_0 = xf \in A$. We have $x_0 \in J_Y$ since $\varphi_\gamma(x_0) = 0$ when $\gamma \notin U$ and $\psi_{\gamma,\theta}(x_0) = 0$ when $(\gamma, \theta) \notin U \times V$. Thus we get $x_0 \in J_Y$ with $\psi_{\gamma_0,\theta_0}(x_0) \neq 0$. ∎

Proposition 5.23 *For any ω-invariant subset Y of $\Gamma' \times \mathbb{T}$, we have $Y_{I_Y} = Y$.*

Proof Combine Lemma 5.21 and Lemma 5.22. ∎

By Proposition 5.23, the map $I \mapsto Y_I$ from the set of ideals I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ to the set of ω-invariant subsets of $\Gamma' \times \mathbb{T}$ is surjective. We will prove this map is injective in the next subsection. We conclude this subsection by proving some results on I_Y.

Proposition 5.24 *Let Y be an ω-invariant subset of $\Gamma' \times \mathbb{T}$. For $t \in \mathbb{T}$, set $\rho_t(Y) = \{([\gamma], \theta) \in \Gamma' \times \mathbb{T} \mid ([\gamma], t\theta) \in Y\}$. Then $\rho_t(Y)$ becomes ω-invariant and $\beta_t(I_Y) = I_{\rho_{t^K}(Y)}$ where β is the gauge action. In particular, if $t^K = 1$ then $\beta_t(I_Y) = I_Y$.*

Proof By Proposition 5.8 (iii), $\psi_{\gamma,\theta}\big(\beta_t(x)\big) = 0$ if and only if $\psi_{\gamma,t^K\theta}(x) = 0$ for $x \in A$. Since $\rho_{t^K}(Y)|_\Gamma = Y|_\Gamma$, we have

$$\begin{aligned}\beta_t(J_Y) &= \big\{x \in A \mid \psi_{\gamma,\theta}\big(\beta_t(x)\big) = 0 \text{ for } ([\gamma], \theta) \in Y, \varphi_\gamma\big(\beta_t(x)\big) = 0 \text{ for } \gamma \in Y|_\Gamma\big\}\\ &= \big\{x \in A \mid \psi_{\gamma,t^K\theta}(x) = 0 \text{ for } ([\gamma], \theta) \in Y, \varphi_\gamma\big(\beta_t(x)\big) = 0 \text{ for } \gamma \in \rho_{t^K}(Y)|_\Gamma\big\}\\ &= J_{\rho_{t^K}(Y)}.\end{aligned}$$

Hence, $\beta_t(I_Y) = I_{\rho_{t^K}(Y)}$. ■

A relation between I_Y and I_X is the following.

Proposition 5.25

(i) *For an ω-invariant subset X of Γ, $Y = [X] \times \mathbb{T}$ is an ω-invariant subset of $\Gamma' \times \mathbb{T}$ and $I_Y = I_X$.*
(ii) *For an ω-invariant subset Y of $\Gamma' \times \mathbb{T}$, $X = \{\gamma \in \Gamma \mid ([\gamma], \theta) \in Y$ for some $\theta \in \mathbb{T}\}$ is an ω-invariant subset of Γ and $I_X = \bigcap_{t\in\mathbb{T}} \beta_t(I_Y)$.*

Proof (i) It is easy to see that $Y = [X] \times \mathbb{T}$ becomes an ω-invariant set. Noting that $Y_{I_Y} = Y$ by Proposition 5.23, we have $X_{I_Y} = X$ from Proposition 5.15. By Proposition 5.24, I_Y is a gauge invariant ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. Therefore $I_Y = I_X$.

(ii) Noting that $Y_{I_Y} = Y$, we have $X_{I_Y} = X$ from Proposition 5.15. Hence

$$\Big(\bigcap_{t\in\mathbb{T}} \beta_t(I_Y)\Big) \cap C_0(\Gamma) = \bigcap_{t\in\mathbb{T}} \beta_t\big(I_Y \cap C_0(\Gamma)\big) = C_0(\Gamma \setminus X).$$

Since the ideal $\bigcap_{t\in\mathbb{T}} \beta_t(I_Y)$ is gauge invariant, we have $I_X = \bigcap_{t\in\mathbb{T}} \beta_t(I_Y)$. ■

Proposition 5.26 *Let X be an ω-invariant subset of Γ and set $X' = \bigcup_{i=2}^n (X + \omega_i)$ which becomes an ω-invariant set satisfying $X' \subset X$. The set X is a bad ω-invariant set if and only if $X' \subsetneqq X$. When X is bad, the set $Y = ([X] \times \{1\}) \cup ([X'] \times \mathbb{T})$ becomes an ω-invariant subset of $\Gamma' \times \mathbb{T}$ satisfying $Y \subsetneqq [X] \times \mathbb{T}$. Any closed set Y' satisfying $Y \subset Y' \subset [X] \times \mathbb{T}$ is ω-invariant and satisfies $X_{I_{Y'}} = X$.*

Proof If X is good, then for any $\gamma \in X$ there exists $i \neq 1$ with $\gamma - \omega_i \in X$. Hence $X' = X$. Conversely, if $X' = X$, then for any $\gamma \in X$ there exists $i \neq 1$ with $\gamma - \omega_i \in X$. Hence $\gamma \in X$ satisfies the condition (ii) in Lemma 4.3. Therefore X is good. When $X' \subsetneqq X$, it is easy to see that any closed set Y' satisfying $Y \subset Y' \subset [X] \times \mathbb{T}$ is ω-invariant. The last statement follows from Proposition 5.15. ■

By Proposition 5.26, we can find many ideals I with $X_I = X$ if X is a bad ω-invariant subset of Γ (note that a bad ω-invariant set exists whenever ω does not satisfy Condition 5.1).

5.3 Primitive Ideals

Now, we turn to showing that $I_{Y_I} = I$ for any ideal I (Theorem 5.49). To see this, we examine the primitive ideal space of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. Let P be a primitive ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. By Proposition 4.10, the closed set X_P of Γ is prime. Hence there exist non-empty subset $\mathbb{I}$ of $\{1, 2, \dots, n\}$ and $\gamma_0 \in \Gamma$ such that $X_P = \gamma_0 + \Omega_{\mathbb{I}}$ by Proposition 4.11.

Proposition 5.27 *If a non-empty subset $\mathbb{I}$ is not $\{1\}$, then $X = \gamma_0 + \Omega_{\mathbb{I}}$ is a good ω-invariant set for any $\gamma_0 \in \Gamma$.*

Proof There is $i \in \mathbb{I}$ which is not 1. For any $\gamma \in X$, we have $\gamma - m\omega_1 - \omega_i \in X$ for any positive integer m. Hence, X is good by Lemma 4.3. ■

If a primitive ideal P satisfies $X_P = \gamma_0 + \Omega_{\mathbb{I}}$ for $\mathbb{I} \neq \{1\}$, then $P = I_{X_P}$ by Theorem 4.5. Conversely for any $\gamma_0 \in \Gamma$ and $\mathbb{I} \neq \{1\}$, the ideal $I_{\gamma_0 + \Omega_{\mathbb{I}}}$ is primitive.

Lemma 5.28 *Let $\gamma_0 \in \Gamma$, $\mathbb{I} \neq \{1\}$, and $X = \gamma_0 + \Omega_{\mathbb{I}}$. If an ideal I satisfies $X \subset X_I$, then $I \subset I_X$.*

Proof Let us write $P = I_X$. We will first show that $X_{I+P} = X$. Clearly, $X_{I+P} \subset X$. To derive a contradiction, let us assume $X_{I+P} \neq X$. Choose $\gamma \in X$ with $\gamma \notin X_{I+P}$. Then there exists $f \in (I + P) \cap C_0(\Gamma)$ with $f \geq 0$ and $f(\gamma) = 1$. Let us denote $f = x_1 + y_1$ where $x_1 \in I$ and $y_1 \in P$. Take $i \in \mathbb{I}$ with $i \neq 1$. For $m \in \mathbb{N}$, let us define $u_m = \sum_{\mu \in \mathcal{W}_n^{(m)}} S_\mu S_1^{mK} S_i S_\mu^* \in M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$. We have $u_m^* u_m = 1$ for any $m \in \mathbb{N}$. By checking for elements which are finite sums of $S_\mu f S_\nu^*$, one can prove $\lim_{m\to\infty} u_m^* x u_m = \sigma_{\omega_i}\big(E(x)\big)$ for any $x \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$ where E is the conditional expectation onto $\mathcal{F}$ and σ_{ω_i} is an automorphism of $\mathcal{F}$ coming from the shift of $\mathcal{F}^{(k)} \cong C_0(\Gamma) \otimes \mathbb{M}_{n^k}$. Set $x_2 = \sigma_{\omega_i}\big(E(x_1)\big)$ and $y_2 = \sigma_{\omega_i}\big(E(y_1)\big)$. Then we have $x_2 \in I$, $y_2 \in P$ and $\sigma_{\omega_i}(f) = x_2 + y_2$. For sufficiently large integer k which is a multiple of K, one can find $x_3 \in I \cap \mathcal{F}^{(k)}$, $y_3 \in P \cap \mathcal{F}^{(k)}$ with $\|x_3 - x_2\| < 1/2$, $\|y_3 - y_2\| < 1/2$. Note that $I \cap \mathcal{F}^{(k)} \cong C_0(\Gamma \setminus X_I) \otimes \mathbb{M}_{n^k}$ and $P \cap \mathcal{F}^{(k)} \cong C_0(\Gamma \setminus X) \otimes \mathbb{M}_{n^k}$. Let φ be an evaluation map at $\gamma - \omega_i \in \Gamma$ from $\mathcal{F}^{(k)} \cong C_0(\Gamma) \otimes \mathbb{M}_{n^k}$ to $\mathbb{M}_{n^k}$. Since $\gamma - \omega_i \in X \subset X_I$, we have $\varphi(x_3) = 0$ and $\varphi(y_3) = 0$. Since $\sigma_{\omega_i}(f) = \sum_{\mu \in \mathcal{W}_n^{(k)}} S_\mu \sigma_{\omega_i + \omega_\mu}(f) S_\mu^* \in \mathcal{F}^{(k)}$, we have $\varphi\big(\sigma_{\omega_i}(f)\big) = \sum_{\mu \in \mathcal{W}_n^{(k)}} f(\gamma + \omega_\mu) S_\mu S_\mu^*$. Since $f \geq 0$, we have $\varphi\big(\sigma_{\omega_i}(f)\big) \geq f(\gamma + k\omega_1) S_1^k S_1^{*k} = S_1^k S_1^{*k}$. Therefore $\big\|\varphi\big(\sigma_{\omega_i}(f)\big)\big\| \geq 1$ which contradicts the fact that $\|\sigma_{\omega_i}(f) - x_3 - y_3\| < 1$. Hence $X_{I+P} = X$.

Since X is a good ω-invariant set, $X_{I+P} = X$ implies $I + P = I_X = P$. Therefore, $I \subset P$. ■

Proposition 5.29 *Let $\gamma_0 \in \Gamma$, $\mathbb{I} \neq \{1\}$, and $X = \gamma_0 + \Omega_{\mathbb{I}}$. The ideal $P = I_X$ is primitive.*

Proof Since $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is separable, it suffices to show that P is prime. Let I_1, I_2 be ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ with $I_1 \cap I_2 \subset P$. Then $X_{I_1} \cup X_{I_2} \supset X$. Since X is prime, either $X_{I_1} \supset X$ or $X_{I_2} \supset X$. By Lemma 5.28, we have either $I_1 \subset P$ or $I_2 \subset P$. Thus, P is prime. ■

Next we will determine all primitive ideals P with $X_P = \gamma_0 + \Omega_{\{1\}}$ for some $\gamma_0 \in \Gamma$. Note that $\gamma_0 + \Omega_{\{1\}}$ is a bad ω-invariant set. Let $\Omega = \{\omega_\mu \mid \mu \in \mathcal{W}_n\}$ which is the semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$. Using that $\omega \in \Gamma^n$ does not satisfy Condition 5.1, we can show that Ω has no accumulation point.

Proposition 5.30 *For any $\gamma \in \Gamma$, there exists a neighborhood U of γ with $(U \setminus \{\gamma\}) \cap \Omega = \varnothing$.*

Proof To derive a contradiction, assume that there exists $\gamma \in \Gamma$ such that $(U \setminus \{\gamma\}) \cap \Omega \neq \varnothing$ for any neighborhood U of γ. One can find $\mu_0, \mu_1, \ldots, \mu_k, \ldots \in \mathcal{W}_n$ with $\lim_{k\to\infty} \omega_{\mu_k} = \gamma$ and $\omega_{\mu_k} \neq \gamma$ for any $k \in \mathbb{N}$. Replacing $\{k\}$ by a subsequence if necessary, we may assume the number of i's appearing in μ_k does not decrease for any $i = 2, 3, \ldots, n$. There exists $i_0 \neq 1$ such that the number of i_0's appearing in μ_k goes to infinity since $\omega_{\mu_k} \neq \gamma$ for any $k \in \mathbb{N}$. Replacing $\{k\}$ by a subsequence if necessary, we may assume the number of i_0's appearing in μ_k increases strictly. We get $\lim_{k\to\infty}(\omega_{\mu_k} - \omega_{\mu_{k-1}} - \omega_{i_0}) = \gamma - \gamma - \omega_{i_0} = -\omega_{i_0}$. Since $\omega_{\mu_k} - \omega_{\mu_{k-1}} - \omega_{i_0} \in \Omega \subset \Omega_{\{1\}}$, we have $-\omega_{i_0} \in \Omega_{\{1\}}$. It contradicts the assumption for ω. ■

Corollary 5.31 *We have $\Omega_{\{1\}} = \Omega$ and Ω is a discrete set.*

By the corollary above, we can define the following.

Definition 5.32 For $([\gamma], \theta) \in \Gamma' \times \mathbb{T}$, we set

$$Y_{([\gamma],\theta)} = \{([\gamma], \theta)\} \cup \big(([\gamma + \Omega] \setminus \{[\gamma]\}) \times \mathbb{T}\big)$$

which is an ω-invariant closed subset of $\Gamma' \times \mathbb{T}$. We write $P_{([\gamma],\theta)}$ for denoting $I_{Y_{([\gamma],\theta)}}$.

We will show that $P_{([\gamma],\theta)}$ is a primitive ideal for any $([\gamma], \theta) \in \Gamma' \times \mathbb{T}$. To see this, we need the following proposition. This will be used to determine the topology of primitive ideal space of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. Let us define a subset $\mathcal{W}_n^+$ of $\mathcal{W}_n$ by

$$\mathcal{W}_n^+ = \{(i_1, i_2, \ldots, i_k) \in \mathcal{W}_n \mid i_k \neq 1\} \cup \{\varnothing\}.$$

Proposition 5.33 *Let X be a compact subset of Γ such that $X \cap (X + \gamma) = \varnothing$ for any $\gamma \in \Omega \setminus \{0\}$. If we set $X_1 = X + \Omega$ and $X_2 = X + \big(\Omega \setminus \{0, \omega_1, \ldots, (K-1)\omega_1\}\big)$, then X_1 and X_2 become ω-invariant sets and $I_{X_2}/I_{X_1} \cong \mathbb{K} \otimes C(X \times \mathbb{T})$.*

Proof Since X is compact and Ω is closed, $X_1 = X + \Omega$ becomes closed. The same reason shows that X_2 is closed. It is easy to see that both X_1 and X_2 satisfy the conditions of ω-invariance. Note that $X_1 \setminus X_2$ is a disjoint union of compact sets $X, X + \omega_1, \ldots, X + (K-1)\omega_1$. Since $fS_i = S_i \sigma_{\omega_i} f = 0$ for $i \neq 1$ and $f \in C(X_1 \setminus X_2) \subset I_{X_2}/I_{X_1}$, we have

$$S_1 f S_1^* = \sigma_{-\omega_1}(f) S_1 S_1^* = \sigma_{-\omega_1} f - \sigma_{-\omega_1} f \sum_{i=2}^{n} S_i S_i^* = \sigma_{-\omega_1} f.$$

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Therefore $I_{X_2}/I_{X_1} = \overline{\operatorname{span}}\,\{S_\mu f S_\nu^* \mid \mu,\nu \in \mathcal{W}_n, f \in C(X)\}$. For $(k,\mu),(l,\nu) \in \mathbb{Z}/K\mathbb{Z} \times \mathcal{W}_n^+$, let us define $e_{(k,\mu),(l,\nu)} = S_\mu S_1^k \chi S_1^{*l} S_\nu^* \in I_{X_2}/I_{X_1}$ where χ is a characteristic function of X. Then $\{e_{(k,\mu),(l,\nu)}\}$ satisfies the relation of matrix units and $\sum_{(k,\mu)\in\mathbb{Z}/K\mathbb{Z}\times\mathcal{W}_n^+} e_{(k,\mu),(k,\mu)} = 1$ (strictly). Since $e_{(0,\varnothing),(0,\varnothing)} = \chi$, we have $I_{X_2}/I_{X_1} \cong \mathbb{K} \otimes B$ where $B = \chi(I_{X_2}/I_{X_1})\chi$. We have

$$B = \overline{\operatorname{span}}\,\{\chi S_\mu f S_\nu^* \chi \mid \mu,\nu \in \mathcal{W}_n, f \in C(X)\} = \overline{\operatorname{span}}\,\{(S_1^K)^m f \mid m \in \mathbb{Z}, f \in C(X)\}.$$

Since B is generated by $C(X)$ and a unitary $S_1^K\chi$ which commute with each other and since B is globally invariant under the gauge action, we have $B \cong C(X \times \mathbb{T})$. Therefore we get $I_{X_2}/I_{X_1} \cong \mathbb{K} \otimes C(X \times \mathbb{T})$. ■

Let us choose $\gamma_0 \in \Gamma$ and fix it. Set $X_1 = \gamma_0 + \Omega$ and $X_2 = \gamma_0 + \big(\Omega \setminus \{0, \omega_1, \ldots, (K-1)\omega_1\}\big)$ which are ω-invariant subsets of Γ by Proposition 5.33. Since $[X_1] \times \mathbb{T} \supset Y_{([\gamma_0],\theta_0)} \supset [X_2] \times \mathbb{T}$, we have $I_{X_1} \subset P_{([\gamma_0],\theta_0)} \subset I_{X_2}$ for any $\theta_0 \in \mathbb{T}$. Taking $X = \{\gamma_0\}$ in Proposition 5.33, we get an isomorphism $I_{X_2}/I_{X_1} \cong \mathbb{K} \otimes C(\mathbb{T})$ which sends $S_1^K\chi$ to $p \otimes z$ where $\chi \in C_0(X_1 \setminus X_2)$ is a characteristic function of γ_0, $p \in \mathbb{K}$ is a minimal projection corresponding to χ, and z is the canonical generator of $C(\mathbb{T})$.

Lemma 5.34 *Under the isomorphism $I_{X_2}/I_{X_1} \cong \mathbb{K}\otimes C(\mathbb{T})$ above, we have $P_{([\gamma_0],\theta_0)}/I_{X_1} \cong \mathbb{K} \otimes C_0(\mathbb{T} \setminus \{\theta_0\})$ for any $\theta_0 \in \mathbb{T}$.*

Proof Since $P_{([\gamma_0],\theta_0)}/I_{X_1}$ is an ideal of I_{X_2}/I_{X_1}, all we have to do is to show

$$(P_{([\gamma_0],\theta_0)}/I_{X_1}) \cap C^*(S_1^K\chi) \cong p \otimes C_0(\mathbb{T} \setminus \{\theta_0\}).$$

For $\theta \in \mathbb{T}$, the map $\psi_{\gamma_0,\theta} \colon A \to \mathbb{M}_K$ vanishes on $A \cap I_{X_1}$. Hence we can define the map $\psi'_\theta \colon A/(A \cap I_{X_1}) \to \mathbb{M}_K$ so that $\psi'_\theta \circ \pi' = \psi_{\gamma_0,\theta}$ where π' is the canonical surjection from A to $A/(A \cap I_{X_1})$. The image of $C^*(S_1^K\chi) \subset A/(A \cap I_{X_1})$ under ψ'_θ is contained in $\mathbb{C}1 \subset \mathbb{M}_K$. One can show that the map $\psi'_\theta \colon C^*(S_1^K\chi) \to \mathbb{C}1$ is isomorphic to the evaluation map at $\theta \in \mathbb{T}$ from $p \otimes C_0(\mathbb{T}) \subset \mathbb{K} \otimes C_0(\mathbb{T})$ under the isomorphism $C^*(S_1^K\chi) \cong p \otimes C_0(\mathbb{T})$. Noting that $(P_{([\gamma_0],\theta_0)}/I_{X_1}) \cap C^*(S_1^K\chi) = (J_{Y_{([\gamma_0],\theta_0)}}/I_{X_1}) \cap C^*(S_1^K\chi)$ by Lemma 5.21, we get the desired isomorphism $(P_{([\gamma_0],\theta_0)}/I_{X_1}) \cap C^*(S_1^K\chi) \cong C_0(\mathbb{T} \setminus \{\theta_0\})$. ■

To prove $P_{([\gamma_0],\theta_0)}$ is primitive, we need the following observation which is inspired by [aHR]. Let $H = l^2((\mathbb{Z}/K\mathbb{Z}) \times \mathcal{W}_n^+)$ be a Hilbert space whose complete orthonormal system is given by $\{\xi_{k,\mu} \mid k \in \mathbb{Z}/K\mathbb{Z}, \mu \in \mathcal{W}_n^+\}$. For $i = 1, 2, \ldots, n$, let us define $T_i \in B(H)$ by

$$T_i(\xi_{k,\mu}) = \begin{cases} \xi_{k+1,\varnothing} & (\text{if } i = 1, \mu = \varnothing), \\ \xi_{k,i\mu} & (\text{otherwise}). \end{cases}$$

For $\gamma \in \Omega$, let us denote by $Q_\gamma \in B(H)$ a projection onto $\overline{\operatorname{span}}\,\{\xi_{k,\mu} \mid k\omega_1 + \omega_\mu = \gamma\}$. One can easily see the following.

Lemma 5.35

(i) *For* $i = 1, 2, \ldots, n$, $T_i^* T_i = 1$ *and* $\sum_{i=1}^n T_i T_i^* = 1$.
(ii) $\sum_{\gamma \in \Omega} Q_\gamma = 1$ (*strongly*).
(iii) *For* $i = 1, 2, \ldots, n$ *and* $\gamma \in \Omega$, $Q_\gamma T_i = \begin{cases} T_i Q_{\gamma - \omega_i} & (\textit{if } \gamma - \omega_i \in \Omega), \\ 0 & (\textit{otherwise}). \end{cases}$

By this lemma, there exists a unique $*$-homomorphism $\varphi_1 \colon \mathcal{O}_n \rtimes_{\alpha^\omega} G \to B(H)$ with $\varphi_1(S_i) = T_i$ and $\varphi_1(f) = \sum_{\gamma \in \Omega} f(\gamma_0 + \gamma) Q_\gamma$ for $f \in C_0(\Gamma)$.

Lemma 5.36 *We have* $\varphi_1(I_{X_1}) = 0$ *and* $\varphi_1(I_{X_2}) = K(H)$.

Proof Since $\varphi_1(f) = 0$ for any $f \in C_0(\Gamma \setminus X_1)$, we have $\varphi_1(I_{X_1}) = 0$. From $I_{X_2} = \overline{\operatorname{span}}\, \{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n, f \in C_0(\Gamma \setminus X_2)\}$ and $Q_{k\omega_1} = T_1^k Q_0 T_1^{*k}$, we get

$$\begin{aligned} \varphi_1(I_{X_2}) &= \overline{\operatorname{span}}\, \{T_\mu Q_{k\omega_1} T_\nu^* \mid \mu, \nu \in \mathcal{W}_n, k \in \mathbb{Z}/K\mathbb{Z}\} \\ &= \overline{\operatorname{span}}\, \{T_\mu Q_0 T_\nu^* \mid \mu, \nu \in \mathcal{W}_n\}. \end{aligned}$$

Writing $T_\mu = T_{\mu'} T_1^l$ and $T_\nu = T_{\nu'} T_1^m$ where $\mu', \nu' \in \mathcal{W}_n^+$ and $l, m \in \mathbb{N}$, we see that $T_\mu Q_0 T_\nu^*$ is a one rank operator from $\xi_{m',\nu'}$ to $\xi_{l',\mu'}$ where $m', l' \in \mathbb{Z}/K\mathbb{Z}$ are images of $m, l \in \mathbb{N}$ respectively. Therefore $\varphi_1(I_{X_2}) = K(H)$. ∎

Since $\varphi_1(\mathcal{O}_n \rtimes_{\alpha^\omega} G) \supset K(H)$, φ_1 is an irreducible representation. Hence $\ker \varphi_1$ is a primitive ideal. We will prove that $\ker \varphi_1 = P_{([\gamma_0], \theta_0)}$ for some $\theta_0 \in \mathbb{T}$. For $t \in \mathbb{T}$, let us define a unitary $U_t \in B(H)$ by $U_t(\xi_{k,\mu}) = t^{|\mu|} \xi_{k,\mu}$. One can easily see the following.

Lemma 5.37

(i) $U_t Q_\gamma U_t^* = Q_\gamma$ *for* $\gamma \in \Omega$.
(ii) $U_t T_i U_t^* = t T_i$ *for* $i = 2, 3, \ldots, n$.
(iii) $U_t T_1 U_t^* = t T_1 + (1 - t) V$ *where* $V \in B(H)$ *is defined by* $V(\xi_{k,\mu}) = \delta_{\mu,\varnothing} \xi_{k+1,\varnothing}$.

For $t \in \mathbb{T}$, let us define a $*$-automorphism β_t' of $B(H)$ by $\beta_t'(x) = U_t x U_t^*$ for $x \in B(H)$. Since $\beta_t'\big(K(H)\big) = K(H)$ for any $t \in \mathbb{T}$, we can extend the $*$-automorphism β_t' of $B(H)$ to one of $B(H)/K(H)$ which is also denoted by β_t'. Let us denote by $\varphi_2 \colon \mathcal{O}_n \rtimes_{\alpha^\omega} G \to B(H)/K(H)$ the composition of φ_1 and a natural surjection $\pi \colon B(H) \to B(H)/K(H)$.

Lemma 5.38 *For any* $t \in \mathbb{T}$, *we have* $\beta_t' \circ \varphi_2 = \varphi_2 \circ \beta_t$ *where* β *is the gauge action on* $\mathcal{O}_n \rtimes_{\alpha^\omega} G$.

Proof The only non-trivial part is $\beta_t'\big(\pi(T_1)\big) = t\pi(T_1)$ for $t \in \mathbb{T}$ which follows from the fact that V in Lemma 5.37(iii) is a compact operator. ∎

Lemma 5.39 *We have* $\ker \varphi_2 = I_{X_2}$.

Proof By Lemma 5.38, $\ker\varphi_2$ is a gauge invariant ideal. For $\gamma \in \Omega$, Q_γ becomes a compact operator if and only if $\gamma = k\omega_1$ for some $k \in \mathbb{Z}/K\mathbb{Z}$. Hence $X_{\ker\varphi_2} = \gamma_0 + \big(\Omega \setminus \{0, \omega_1, \dots, (K-1)\omega_1\}\big) = X_2$. Therefore we have $\ker\varphi_2 = I_{X_2}$. ■

Proposition 5.40 *For any $\theta \in \mathbb{T}$, $P_{([\gamma_0],\theta)}$ is a primitive ideal.*

Proof By Lemma 5.36 and Lemma 5.39, we have $I_{X_1} \subset \ker\varphi_1 \subset I_{X_2}$. Since $\ker\varphi_1$ is primitive, the ideal $\ker\varphi_1/I_{X_1}$ of I_{X_2}/I_{X_1} is also primitive. Hence we have $\ker\varphi_1/I_{X_1} \cong \mathbb{K} \otimes C_0(\mathbb{T} \setminus \{\theta_0\})$ for some $\theta_0 \in \mathbb{T}$. By Lemma 5.34, we have $\ker\varphi_1 = P_{([\gamma_0],\theta_0)}$. Thus $P_{([\gamma_0],\theta_0)}$ is a primitive ideal. For an arbitrary $\theta \in \mathbb{T}$, there exists $t \in \mathbb{T}$ with $\theta = t^K\theta_0$. Hence $P_{([\gamma_0],\theta)} = \beta_t(P_{([\gamma_0],\theta_0)})$ is also primitive. ■

In fact, we can prove $\ker\varphi_1 = P_{([\gamma_0],1)}$, although we omit the proof because we do not need it.

Proposition 5.41 *For $\gamma_0 \in \Gamma$, the set of all primitive ideals P satisfying $X_P = \gamma_0 + \Omega$ is $\{P_{([\gamma_0],\theta)} \mid \theta \in \mathbb{T}\}$.*

Proof By Proposition 5.40, the ideal $P = P_{([\gamma_0],\theta)}$ is primitive with $X_P = \gamma_0 + \Omega$ for any $\theta \in \mathbb{T}$. Let P be a primitive ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ with $X_P = \gamma_0 + \Omega$ for some $\gamma_0 \in \Gamma$. Then, $I_{X_1} \subset P$ and $I_{X_2} \not\subset P$ where $X_1 = \gamma_0 + \Omega$ and $X_2 = \gamma_0 + \big(\Omega \setminus \{0, \omega_1, \dots, (K-1)\omega_1\}\big)$. The set of all primitive ideals P satisfying $I_{X_1} \subset P$ and $I_{X_2} \not\subset P$ corresponds to the set of primitive ideals of $I_{X_2}/I_{X_1} \cong \mathbb{K} \otimes C(\mathbb{T})$ bijectively (see, for example, [D]). Hence there is no primitive ideal P satisfying $X_P = \gamma_0 + \Omega$ other than $\{P_{([\gamma_0],\theta)} \mid \theta \in \mathbb{T}\}$. ■

Now we can describe the primitive ideal space of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$.

Lemma 5.42 *Let $\gamma_1, \gamma_2 \in \Gamma$ and $\mathbb{I}_1, \mathbb{I}_2$ be non-empty sets of $\{1, 2, \dots, n\}$. Then $I_{\gamma_1+\Omega_{\mathbb{I}_1}} = I_{\gamma_2+\Omega_{\mathbb{I}_2}}$ if and only if $\Omega_{\mathbb{I}_1} = \Omega_{\mathbb{I}_2}$ and $\gamma_1 - \gamma_2 \in \Omega_{\mathbb{I}_1} \cap (-\Omega_{\mathbb{I}_1})$.*

Proof Obviously $I_{\gamma_1+\Omega_{\mathbb{I}_1}} = I_{\gamma_2+\Omega_{\mathbb{I}_2}}$ is equivalent to $\gamma_1+\Omega_{\mathbb{I}_1} = \gamma_2+\Omega_{\mathbb{I}_2}$. If $\Omega_{\mathbb{I}_1} = \Omega_{\mathbb{I}_2}$ and $\gamma_1-\gamma_2 \in \Omega_{\mathbb{I}_1}\cap(-\Omega_{\mathbb{I}_1})$, then $\gamma_1+\Omega_{\mathbb{I}_1} = \gamma_2+\Omega_{\mathbb{I}_2}$. Conversely assume $\gamma_1+\Omega_{\mathbb{I}_1} = \gamma_2+\Omega_{\mathbb{I}_2}$ and denote it by X. Then we have $\Omega_{\mathbb{I}_1} = \Omega_{\mathbb{I}_2}$ because $\Omega_{\mathbb{I}_j} = \{\gamma \in \Gamma \mid X + \gamma \subset X\}$ for $j = 1, 2$. Hence we get $\gamma_1 - \gamma_2 \in \Omega_{\mathbb{I}_1} \cap (-\Omega_{\mathbb{I}_1})$. The proof is complete. ■

For non-empty sets $\mathbb{I}_1, \mathbb{I}_2$ of $\{1, 2, \dots, n\}$, we write $\mathbb{I}_1 \sim \mathbb{I}_2$ if $\Omega_{\mathbb{I}_1} = \Omega_{\mathbb{I}_2}$. Let us choose and fix representative elements of each equivalence classes of $\sim$ and denote by $\mathcal{I}$ the set of them. Note that $\{1\} \in \mathcal{I}$ because $\mathbb{I} \sim \{1\}$ if and only if $\mathbb{I} = \{1\}$. For each $\mathbb{I} \in \mathcal{I}$, we define a topological space $\Gamma_{\mathbb{I}}$ by $\Gamma_{\mathbb{I}} = \Gamma/\big(\Omega_{\mathbb{I}} \cap (-\Omega_{\mathbb{I}})\big)$ if $\mathbb{I} \neq \{1\}$ and $\Gamma_{\{1\}} = \Gamma' \times \mathbb{T}$. For $[\gamma] \in \Gamma_{\mathbb{I}}$ with $\mathbb{I} \neq \{1\}$, we define a primitive ideal $P_{[\gamma]}$ by $I_{\gamma+\Omega_{\mathbb{I}}}$. Note that if $[\gamma] = [\gamma']$ in $\Gamma_{\mathbb{I}}$, then $I_{\gamma+\Omega_{\mathbb{I}}} = I_{\gamma'+\Omega_{\mathbb{I}}}$. For $([\gamma], \theta) \in \Gamma_{\{1\}} = \Gamma' \times \mathbb{T}$, the ideal $P_{([\gamma],\theta)}$ is defined in Definition 5.32.

Proposition 5.43 *The map $\coprod_{\mathbb{I}\in\mathcal{I}} \Gamma_{\mathbb{I}} \ni y \mapsto P_y \in \operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ is bijective where $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ is the primitive ideal space of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$.*

Proof The map is injective by Lemma 5.42 and surjective by Proposition 5.41. ∎

The primitive ideal space $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ is a topological space whose closed sets are given by $\{P \in \operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G) \mid I \subset P\}$ for ideals I. We will investigate which subset of $\coprod_{\mathbb{I}\in\mathcal{I}} \Gamma_{\mathbb{I}}$ corresponds to a closed subset of $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$.

Lemma 5.44 *For any $\gamma_0 \in \Gamma$, there exists a compact neighborhood X of γ_0 such that $X \cap (X+\gamma) = \varnothing$ for any $\gamma \in \Omega \setminus \{0\}$.*

Proof By Proposition 5.30, there exists a neighborhood U of 0 with $\gamma \notin U$ for any $\gamma \in \Omega \setminus \{0\}$. Take a compact neighborhood V of 0 with $V - V \subset U$. Then $X = \gamma_0 + V$ is a desired compact neighborhood of γ_0. ∎

Lemma 5.45 *We have $X + \Omega \not\supset \gamma + \Omega_{\mathbb{I}}$ for any $\gamma \in \Gamma$, $\mathbb{I} \neq \{1\}$, and any compact set X of Γ.*

Proof To derive a contradiction, assume $X + \Omega \supset \gamma + \Omega_{\mathbb{I}}$ for some $\gamma \in \Gamma$, $\mathbb{I} \neq \{1\}$, and some compact set X of Γ. Take $i \in \mathbb{I}$ with $i \neq 1$. For any $k \in \mathbb{N}$, the element $\gamma - k\omega_i$ is in $\gamma + \Omega_{\mathbb{I}}$. Hence there exists $\gamma_k \in X$ and $\mu_k \in \mathcal{W}_n$ with $\gamma - k\omega_i = \gamma_k + \omega_{\mu_k}$ for any $k \in \mathbb{N}$. Since X is compact, there exists a subsequence $\{k_l\}_{l\in\mathbb{N}}$ of $\mathbb{N}$ so that γ_{k_l} converges to some point. Thus $\{\omega_{\mu_{k_l}} + k_l\omega_i\}_{l\in\mathbb{N}}$ becomes a convergent sequence. By the same argument as in the proof of Proposition 5.30, we can show that $-\omega_i \in \Omega = \Omega_{\{1\}}$. This contradicts the assumption for ω. ∎

Lemma 5.46 *Let Y be a subset of $\Gamma' \times \mathbb{T}$. If for any $[\gamma] \in \Gamma'$, there exists a compact neighborhood $[X_\gamma]$ of $[\gamma]$ such that $Y \cap ([X_\gamma] \times \mathbb{T})$ is closed set, then Y is closed.*

Proof Take a net $\{([\gamma_\lambda], \theta_\lambda)\}$ in Y converging to $([\gamma], \theta) \in \Gamma' \times \mathbb{T}$. Eventually $[\gamma_\lambda] \in [X_\gamma]$ because $[X_\gamma]$ is a neighborhood of $[\gamma]$. Then $([\gamma], \theta) \in Y$ since $Y \cap ([X_\gamma] \times \mathbb{T})$ is closed. Thus Y is closed. ∎

Lemma 5.47 *For any ω-invariant subset X of Γ, we have $I_X = \bigcap_{y\in[X]\times\mathbb{T}} P_y$.*

Proof By Proposition 5.24, the ideal $I = \bigcap_{y\in[X]\times\mathbb{T}} P_y$ is gauge invariant. Hence $I = I_X$ because $C_0(\Gamma) \cap I = \bigcap_{y\in[X]\times\mathbb{T}} \big(C_0(\Gamma) \cap P_y\big) = \bigcap_{\gamma\in X} C_0\big(\Gamma \setminus (\gamma + \Omega)\big) = C_0(\Gamma \setminus X)$. ∎

Proposition 5.48 *Let Y be a subset of $\coprod_{\mathbb{I}\in\mathcal{I}} \Gamma_{\mathbb{I}}$ and set $Y_{\mathbb{I}} = Y \cap \Gamma_{\mathbb{I}}$ for $\mathbb{I} \in \mathcal{I}$. The set $P_Y = \{P_y \mid y \in Y\}$ is closed in $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ if and only if $Y_{\{1\}}$ is an ω-invariant set of $\Gamma_{\{1\}} = \Gamma' \times \mathbb{T}$ and $Y_{\mathbb{I}} = \{[\gamma] \in \Gamma_{\mathbb{I}} \mid [\gamma + \Omega_{\mathbb{I}}] \times \mathbb{T} \subset Y_{\{1\}}\}$ for any $\mathbb{I} \in \mathcal{I}$ with $\mathbb{I} \neq \{1\}$.*

Proof Let us take a subset $Y = \coprod_{\mathbb{I}\in\mathcal{I}} Y_{\mathbb{I}}$ of $\coprod_{\mathbb{I}\in\mathcal{I}} \Gamma_{\mathbb{I}}$. If $Y_{\{1\}}$ is an ω-invariant subset of $\Gamma_{\{1\}} = \Gamma' \times \mathbb{T}$, then we can define the ideal $I_{Y_{\{1\}}}$. One can easily see that $\{([\gamma], \theta) \in \Gamma_{\{1\}} \mid I_{Y_{\{1\}}} \subset P_{([\gamma],\theta)}\} = Y_{\{1\}}$ and that for $\mathbb{I} \neq \{1\}$, $I_{Y_{\{1\}}} \subset I_{\gamma+\Omega_{\mathbb{I}}}$ if and only if

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$[\gamma + \Omega_{\mathbb{I}}] \times \mathbb{T} \subset Y_{\{1\}}$. Therefore if Y satisfies the condition in the statement, then P_Y is closed in $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$.

Conversely, assume P_Y is closed, *i.e.* there exists an ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ so that $Y = \{y \in \coprod_{\mathbb{I} \in \mathfrak{I}} \Gamma_{\mathbb{I}} \mid I \subset P_y\}$. We first show that $Y_{\{1\}}$ is ω-invariant. Take $\gamma_0 \in \Gamma$ arbitrarily. By Lemma 5.44, there exists a compact neighborhood X of γ_0 such that $X \cap (X + \gamma) = \varnothing$ for any $\gamma \in \Omega \setminus \{0\}$. If we set $X_1 = X + \Omega$ and $X_2 = X + \big(\Omega \setminus \{0, \omega_1, \dots, (K-1)\omega_1\}\big)$, then $I_{X_2}/I_{X_1} \cong \mathbb{K} \otimes C(X \times \mathbb{T})$ by Proposition 5.33. The subset $\{P \in \operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G) \mid I_{X_1} \subset P, I_{X_2} \not\subset P\}$ of $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ is homeomorphic to $\operatorname{Prim}(I_{X_2}/I_{X_1}) \cong X \times \mathbb{T}$. By Lemma 5.45, $X_1 \not\supset \gamma + \Omega_{\mathbb{I}}$ for any $\gamma \in \Gamma$ and for any $\mathbb{I} \neq \{1\}$. Hence

$$\Big\{x \in \coprod_{\mathbb{I} \in \mathfrak{I}} \Gamma_{\mathbb{I}} \;\Big|\; I_{X_1} \subset P_x, I_{X_2} \not\subset P_x\Big\} = [X] \times \mathbb{T} \subset \Gamma_{\{1\}}.$$

Therefore $[X] \times \mathbb{T} \ni x \mapsto P_x \in \operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ is a homeomorphism from $[X] \times \mathbb{T}$ whose topology is the relative topology of $\Gamma' \times \mathbb{T}$ to

$$\{P \in \operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G) \mid I_{X_1} \subset P, I_{X_2} \not\subset P\} \subset \operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$$

(note that X is homeomorphic to $[X]$). The set $Y \cap ([X] \times \mathbb{T}) \subset \Gamma_{\{1\}}$ is closed in $[X] \times \mathbb{T}$ because P_Y is closed. By Lemma 5.46, the subset $Y_{\{1\}}$ is closed in $\Gamma_{\{1\}}$. If $([\gamma], \theta) \in Y_{\{1\}}$, then $([\gamma + \omega_i], \theta') \in Y_{\{1\}}$ for any $i \in \{2, 3, \dots, n\}$ and $\theta' \in \mathbb{T}$ because $P_{([\gamma],\theta)} \subset P_{([\gamma+\omega_i],\theta')}$. Therefore $Y_{\{1\}}$ is an ω-invariant subset of $\Gamma_{\{1\}}$. Take $\mathbb{I} \in \mathfrak{I}$ with $\mathbb{I} \neq \{1\}$ and $[\gamma] \in \Gamma_{\mathbb{I}}$. Since $I_{\gamma+\Omega_{\mathbb{I}}} = \bigcap_{y \in [\gamma+\Omega_{\mathbb{I}}] \times \mathbb{T}} P_y$ by Lemma 5.47, the element $[\gamma]$ is in $Y_{\mathbb{I}}$ if and only if $[\gamma + \Omega_{\mathbb{I}}] \times \mathbb{T} \subset Y_{\{1\}}$. Therefore Y satisfies the condition in the statement. ∎

By the proposition above, we get the following.

Theorem 5.49 *When ω does not satisfy Condition 5.1, there is an inclusion reversing one-to-one correspondence between the set of ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and the set of ω-invariant subsets of $\Gamma' \times \mathbb{T}$. Hence for any ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, we have $I = I_{Y_I}$.*

Proof There is a one-to-one correspondence between the set of ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and the closed subset of $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$. By Proposition 5.48, the closed subset of $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ corresponds bijectively to the set of ω-invariant subsets of $\Gamma' \times \mathbb{T}$. ∎

6 The Strong Connes Spectrum and the K-Groups of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$

As a consequence of knowing all ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, we can compute the strong Connes spectrum of the action $\alpha^\omega \colon G \curvearrowright \mathcal{O}_n$. We recall the definition of the strong Connes spectrum.

Definition 6.1 Let $\alpha \colon G \curvearrowright A$ be an action of an abelian group G, whose dual group is Γ, on a C^*-algebra A. The strong Connes spectrum $\tilde{\Gamma}(\alpha)$ of α is defined by

$$\tilde{\Gamma}(\alpha) = \{\gamma \in \Gamma \mid \hat{\alpha}_\gamma(I) \subset I, \text{ for any ideal } I \text{ of } A \rtimes_\alpha G\},$$

where $\hat{\alpha}\colon \Gamma \curvearrowright A \rtimes_\alpha G$ is the dual action of α.

For each action α, the strong Connes spectrum $\tilde{\Gamma}(\alpha)$ is a closed subsemigroup of Γ. We remark that in the original paper [Ki], A. Kishimoto defined the strong Connes spectrum in a different way and proved that his definition is equivalent to the definition above (see, [Ki, Lemma 3.4]).

In our setting, the dual actions $\hat{\alpha^{\omega}}\colon \Gamma \curvearrowright \mathcal{O}_n \rtimes_{\alpha^\omega} G$ are characterized by

$$\hat{\alpha^{\omega}}_\gamma(S_\mu f S_\nu^*) = S_\mu \sigma_\gamma f S_\nu^* \text{ for } \mu, \nu \in \mathcal{W}_n,\ f \in C_0(\Gamma) \text{ and } \gamma \in \Gamma.$$

Proposition 6.2 *Let ω be an element of Γ^n. The strong Connes spectrum $\tilde{\Gamma}(\alpha^\omega)$ of the action α^ω is $\bigcap_{i=1}^n \Omega_{\{i\}}$.*

Proof First we consider the case that ω satisfies Condition 5.1. Since the correspondence between ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and ω-invariant subsets of Γ is one-to-one by Theorem 5.2, $\hat{\alpha^{\omega}}_\gamma(I) \subset I$ if and only if $X_I - \gamma \supset X_I$ for an ideal I and $\gamma \in \Gamma$. For any $i \in \{1, 2, \dots, n\}$, the set $\Omega_{\{i\}}$ is an ω-invariant set satisfying $\{\gamma \in \Gamma \mid \Omega_{\{i\}} + \gamma \subset \Omega_{\{i\}}\} = \Omega_{\{i\}}$. Therefore $\tilde{\Gamma}(\alpha^\omega) \subset \bigcap_{i=1}^n \Omega_{\{i\}}$. We have $X \supset X + \bigcap_{i=1}^n \Omega_{\{i\}}$ for any ω-invariant set X because for any $x \in X$ there exists i with $x + \Omega_{\{i\}} \subset X$. We have $\tilde{\Gamma}(\alpha^\omega) \supset \bigcap_{i=1}^n \Omega_{\{i\}}$. Thus $\tilde{\Gamma}(\alpha^\omega) = \bigcap_{i=1}^n \Omega_{\{i\}}$ in the case that ω satisfies Condition 5.1.

Next we consider the case that ω does not satisfy Condition 5.1. In this case, the set $\bigcap_{i=1}^n \Omega_{\{i\}}$ coincides with $\Omega = \{\omega_\mu \mid \mu \in \mathcal{W}_n\}$. Since Ω is an ω-invariant subset of Γ and $\{\gamma \in \Gamma \mid \widehat{\alpha^{\omega}}_\gamma(I_\Omega) \subset I_\Omega\} = \{\gamma \in \Gamma \mid \Omega + \gamma \subset \Omega\} = \Omega$, we have $\tilde{\Gamma}(\alpha^\omega) \subset \Omega$. For any ω-invariant subset Y of $\Gamma' \times \mathbb{T}$, we have $([\gamma + \omega_\mu], \theta) \in Y$ for any $\mu \in \mathcal{W}_n$ and any $([\gamma], \theta) \in Y$. Since the correspondence between ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and ω-invariant subsets of $\Gamma' \times \mathbb{T}$ is one-to-one by Theorem 5.49, we have $\widehat{\alpha^{\omega}}_\gamma(I) \subset I$ for any ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and for any $\gamma \in \Omega$. Hence $\tilde{\Gamma}(\alpha^\omega) \supset \Omega$. Therefore also in the case that ω does not satisfy Condition 5.1, we have $\tilde{\Gamma}(\alpha^\omega) = \bigcap_{i=1}^n \Omega_{\{i\}}$. ∎

Remark 6.3 The inclusion $\tilde{\Gamma}(\alpha^\omega) \subset \bigcap_{i=1}^n \Omega_{\{i\}}$ had been already proved by A. Kishimoto [Ki].

The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is a Cuntz-Pimsner algebra. Let $E = C_0(\Gamma)^n$ be a right $C_0(\Gamma)$ module. The left $C_0(\Gamma)$ module structure of E is given by

$$f \cdot (f_1, f_2, \dots, f_n) = \big(\sigma_{\omega_1}(f) f_1, \sigma_{\omega_2}(f) f_2, \dots, \sigma_{\omega_n}(f) f_n\big) \in E$$

for $f \in C_0(\Gamma)$ and $(f_1, f_2, \dots, f_n) \in E$.

Proposition 6.4 *The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_E$.*

Proof The inclusion $C_0(\Gamma) \hookrightarrow \mathcal{O}_n \rtimes_{\alpha^\omega} G$ and $E \ni (f_1, f_2, \dots, f_n) \mapsto \sum_{i=1}^n S_i f_i \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$ satisfies the conditions in [Pi, Theorem 3.12] (for example, the condition (4) is equivalent to saying that $\sum_{i=1}^n S_i \sigma_{\omega_i}(f) S_i^* = f$ for any $f \in C_0(\Gamma)$). Hence there exists a $*$-homomorphism $\varphi\colon \mathcal{O}_E \to \mathcal{O}_n \rtimes_{\alpha^\omega} G$ which is surjective since $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is generated by $\{\sum_{i=1}^n S_i f_i \mid f_i \in C_0(\Gamma)\}$. One can show that φ is injective by using Proposition 3.11. Thus $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is isomorphic to $\mathcal{O}_E$. ■

The ideal structures of Cuntz-Pimsner algebras were investigated in [KPW] when Hilbert bimodules are finitely generated. Our Hilbert bimodule E is finitely generated if and only if the group G is discrete. When G is discrete, we can know the detailed structure of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ without using the result in [KPW] (see subsection 7.2). Thanks to considering our algebra $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ as a Cuntz-Pimsner algebra, we can compute the K-groups of it by [Pi, Theorem 4.9].

Proposition 6.5 *Let ω be an element of Γ^n. The following sequence is exact:*

$$\begin{array}{ccccc}
K_0\big(C_0(\Gamma)\big) & \xrightarrow{\mathrm{id}-\sum_{i=1}^n(\sigma_{\omega_i})_*} & K_0\big(C_0(\Gamma)\big) & \xrightarrow{\iota_*} & K_0(\mathcal{O}_n \rtimes_{\alpha^\omega} G) \\
\uparrow & & & & \downarrow \\
K_1(\mathcal{O}_n \rtimes_{\alpha^\omega} G) & \xleftarrow{\iota_*} & K_1\big(C_0(\Gamma)\big) & \xleftarrow{\mathrm{id}-\sum_{i=1}^n(\sigma_{\omega_i})_*} & K_1\big(C_0(\Gamma)\big),
\end{array}$$

where ι is the embedding $\iota\colon C_0(\Gamma) \hookrightarrow \mathcal{O}_n \rtimes_{\alpha^\omega} G$.

Proof Let us denote by $\mathcal{T}_n$ the Cuntz-Toeplitz algebra, which is generated by n isometries $T_1, T_2, \dots, T_n$ satisfying $\sum_{i=1}^n T_i T_i^* < 1$. There is a surjection $\pi\colon \mathcal{T}_n \to \mathcal{O}_n$ with $\pi(T_i) = S_i$ for $i = 1, 2, \dots, n$. The kernel of π is isomorphic to $\mathbb{K}$. If we define an action $\bar{\alpha}^\omega\colon G \curvearrowright \mathcal{T}_n$ by $\bar{\alpha}^\omega_t(T_i) = \langle t \,|\, \omega_i \rangle T_i$ for $t \in G$ and $i = 1, 2, \dots, n$, then the kernel of π is invariant under this action and $\pi \circ \bar{\alpha}^\omega_t = \alpha^\omega_t \circ \pi$ for any $t \in G$. Hence there exists a short exact sequence

$$0 \to \mathbb{K} \rtimes_{\bar{\alpha}^\omega} G \to \mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G \to \mathcal{O}_n \rtimes_{\alpha^\omega} G \to 0.$$

One can see that $\mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G$ is isomorphic to $\mathcal{T}_E$ in a similar way to Proposition 6.4. The C^*-algebra $\mathbb{K} \rtimes_{\bar{\alpha}^\omega} G$ is isomorphic to $\mathbb{K} \otimes C_0(\Gamma)$. The subalgebra $\mathbb{C}1 \rtimes_{\bar{\alpha}^\omega} G$ of $\mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G$ is isomorphic to $C_0(\Gamma)$. The inclusion $C_0(\Gamma) \hookrightarrow \mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G$ induces a KK-equivalence between $C_0(\Gamma)$ and $\mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G$ whose inverse is given by a Kasparov bimodule

$$(\mathcal{E}_+ \oplus \mathcal{E}_+, \pi_0 \oplus \pi_1, T) \in KK\big(\mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G, C_0(\Gamma)\big)$$

where $\mathcal{E}_+ = \bigoplus_{k=0}^\infty E^{\otimes k}$ is a right $C_0(\Gamma)$-module, $\pi_0\colon \mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G \to L(\mathcal{E}_+)$ is the natural representation, $\pi_1\colon \mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G \to L(\bigoplus_{k=1}^\infty E^{\otimes k}) \subset L(\mathcal{E}_+)$ is the representation obtained from the universal property of $\mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G$ and $T \in L(\mathcal{E}_+ \oplus \mathcal{E}_+)$ is the odd operator defined by $T(\xi \oplus \zeta) = \zeta \oplus \xi$ (for the detail, see Section 4 in [Pi]). To show that the 6-term exact sequence obtained from the short exact sequence above is the desired one, it suffices to see that the element $\big(\mathcal{E}_+ \oplus \mathcal{E}_+, (\pi_0 \circ \varphi) \oplus (\pi_1 \circ \varphi), T\big) \in KK\big(C_0(\Gamma), C_0(\Gamma)\big)$ coincides with $\mathrm{id} - \sum_{i=1}^n (\sigma_{\omega_i})_*$ where $\varphi\colon C_0(\Gamma) \to \mathcal{T}_n \rtimes_{\bar{\alpha}^\omega} G$ is given by $\varphi(f) = (1 - \sum_{i=1}^n T_i T_i^*) f$ (note that $1 - \sum_{i=1}^n T_i T_i^*$ is a minimal projection of the kernel of π which is isomorphic to $\mathbb{K}$). A routine computation shows that $\pi_0 \circ \varphi$ vanishes on $\bigoplus_{k=1}^{\infty} E^{\otimes k}$ and $\pi_1 \circ \varphi$ vanishes on $\bigoplus_{k=2}^{\infty} E^{\otimes k}$ and on $E^{\otimes 0}$ $(= C_0(\Gamma))$. Thus

$$\begin{aligned}
\big(\mathcal{E}_+ \oplus \mathcal{E}_+, (\pi_0 \circ \varphi) \oplus (\pi_1 \circ \varphi), T\big) &= \big(\mathcal{E}_+ \oplus \mathcal{E}_+, (\pi_0 \circ \varphi) \oplus (\pi_1 \circ \varphi), 0\big) \\
&= (\mathcal{E}_+, \pi_0 \circ \varphi, 0) - (\mathcal{E}_+, \pi_1 \circ \varphi, 0) \\
&= \big(C_0(\Gamma), \pi_0 \circ \varphi, 0\big) - (E, \pi_1 \circ \varphi, 0) \\
&= \big(C_0(\Gamma), \mathrm{id}, 0\big) - \sum_{i=1}^{n} \big(C_0(\Gamma), \sigma_{\omega_i}, 0\big) \\
&= \mathrm{id} - \sum_{i=1}^{n} (\sigma_{\omega_i})_*. \qquad \blacksquare
\end{aligned}$$

7 Examples and Remarks

7.1 When G is Compact

When G is compact, its dual group Γ becomes discrete. In this case, for any $\omega \in \Gamma^n$ the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ becomes a graph algebra of some skew product graph which is row-finite (see [KP]) and a part of our results here has been already proved in [BPRS]. There are many graph algebras which are not isomorphic to $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, and it should be interesting to determine the ideal structures of such algebras. Our technique here may help. We may consider our C^*-algebras $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ as a continuous counterpart of graph algebras. It seems to be interesting to define and examine graph algebras of continuous graphs (see [Ka3, Ka4, Ka5]).

7.2 When G is Discrete

When G is discrete, its dual group Γ becomes compact. Let us choose $\omega \in \Gamma^n$ and fix it. Let us denote by $\bar{\Omega}$ a closed semigroup generated by $\omega_1, \omega_2, \dots, \omega_n$.

Proposition 7.1 *When G is discrete, we have $-\omega_i \in \bar{\Omega}$ for $i = 1, 2, \dots, n$.*

Proof Let us take $i \in \{1, 2, \dots, n\}$. Since Γ is compact, a sequence $\{k\omega_i\}_{k=1}^{\infty}$ has a subsequence $\{k_l\omega_i\}_{l=1}^{\infty}$ which converges to some element in Γ. For any l, we have $(k_{l+1} - k_l - 1)\omega_i \in \Omega$ because $k_{l+1} > k_l$. Hence $-\omega_i = \lim_{l\to\infty}(k_{l+1} - k_l - 1)\omega_i \in \bar{\Omega}$. ∎

The following are easy consequences of above proposition.

Corollary 7.2 *Any $\omega \in \Gamma^n$ satisfies Condition 5.1.*

Corollary 7.3 *The set $\bar{\Omega}$ becomes a closed subgroup of Γ and the set of all ω-invariant subsets of Γ is one-to-one correspondent to the set of all closed subset of $\Gamma/\bar{\Omega}$.*

By two corollaries above, the set of all ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is one-to-one correspondence to the set of all closed subset of $\Gamma/\bar{\Omega}$. In fact, we can examine the ideal structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ directly and more structures of the crossed product. Let G' be a quotient of G by the closed subgroup

$$\begin{aligned}\{t \in G \mid \alpha_t^\omega = \mathrm{id}\} &= \{t \in G \mid \langle\, t \,|\, \omega_i \,\rangle = 1 \text{ for } i = 1, 2, \ldots, n\}\\ &= \{t \in G \mid \langle\, t \,|\, \gamma \,\rangle = 1 \text{ for any } \gamma \in \bar{\Omega}\}.\end{aligned}$$

Then the dual group of G' is naturally isomorphic to $\bar{\Omega}$. Since $\omega \in \bar{\Omega}^n$, we can define an action $\alpha^\omega \colon G' \curvearrowright \mathcal{O}_n$. The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G'$ is simple by Theorem 4.8 and purely infinite (see [KK2] or [Ka1]). The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ becomes a continuous field over the space $\Gamma/\bar{\Omega}$ whose fiber of any point is isomorphic to $\mathcal{O}_n \rtimes_{\alpha^\omega} G'$ (see [OP2]).

7.3 When $G = \mathbb{R}$

When G is the real group $\mathbb{R}$, its dual group Γ is also $\mathbb{R}$. We define three types for elements of $\mathbb{R}^n$.

Definition 7.4 Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n$. The element ω is said to be *of type* $(+)$ if $\omega_i > 0$ for all i or $\omega_i < 0$ for all i, and to be of type $(-)$ if there exist i, j such that $\omega_i < 0 < \omega_j$. Otherwise, the element ω is said to be *of type* (0).

Namely ω is of type (0) if and only if there exists $i \in \{1, 2, \ldots, n\}$ such that $\omega_i = 0$ and all the other ω_i's have the same sign. When ω is of type $(+)$ or $(-)$, the set $\Omega_{\{i\}}$ coincides with the closed group generated by $\omega_1, \omega_2, \ldots, \omega_n$ for any $i = 1, 2, \ldots, n$. An element $\omega \in \mathbb{R}^n$ is called *aperiodic* if the closed group generated by $\omega_1, \omega_2, \ldots, \omega_n$ is $\mathbb{R}$. By Theorem 4.8, we have the following.

Proposition 7.5 *For $\omega \in \mathbb{R}^n$, the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ is simple if and only if ω is aperiodic and of type $(+)$ or $(-)$.*

When ω is of type $(+)$ or $(-)$ and not aperiodic, the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ is isomorphic to a mapping torus whose fiber is the simple C^*-algebra $\mathcal{O}_n \rtimes_{\alpha^{\omega'}} \mathbb{T}$ where $\omega' = (\omega_1/K, \omega_2/K, \ldots, \omega_n/K) \in \mathbb{Z}^n$ and K is the (positive) generator of the closed group generated by $\omega_1, \omega_2, \ldots, \omega_n$ which is isomorphic to $\mathbb{Z}$. Hence in this case, $\operatorname{Prim}(\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}) \cong \mathbb{T}$ and the set of ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ corresponds to the set of closed sets of $\mathbb{T}$. The case that ω is of type (0) is more complicated. When ω is of type (0), the set $\Omega = \{\omega_\mu \mid \mu \in \mathcal{W}_n\}$ is closed and a closed set $X \subset \mathbb{R}$ is ω-invariant if and only if $X + \Omega \subset X$. We can prove the proposition below in a similar way to the proof of Proposition 5.33.

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Proposition 7.6 *Let $\omega \in \mathbb{R}^n$ be of type (0) with $\omega \neq (0, 0, \dots, 0)$ and X be an ω-invariant set. Set $X' = \bigcup_{\omega_i \neq 0}(X + \omega_i)$. Any closed set X_1 with $X' \subset X_1 \subset X$ is ω-invariant, and $I_{X_1}/I_X \cong \mathbb{K} \otimes C_0(X \setminus X_1) \otimes \mathcal{O}_k$ where k is the number of i with $\omega_i = 0$ and $\mathcal{O}_1 = C(\mathbb{T})$.*

One can easily see that an element $\omega \in \mathbb{R}^n$ does not satisfy Condition 5.1 if and only if ω is of type (0) and the number of i with $\omega_i = 0$ is 1.

Remark 7.7 When ω is of type $(+)$, the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ becomes stable and projectionless [KK1]. In the forthcoming paper [Ka1], we will show that $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ is AF-embeddable in this case. More generally, we will give one sufficient condition for crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ becomes AF-embeddable in [Ka1]. As a consequence of it, we will show that $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is either AF-embeddable or purely infinite when it is simple.

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