KNOT GROUPS WITH MANY KILLERS

DANIEL S. SILVER ✉, WILBUR WHITTEN and SUSAN G. WILLIAMS

Abstract

The group of any nontrivial torus knot, hyperbolic 2-bridge knot, or hyperbolic knot with unknotting number one contains infinitely many elements, none of which is the automorphic image of another, such that each normally generates the group.

1. Introduction

Let k be a knot in $\mathbb{S}^3$ with group $\pi_k = \pi_1(\mathbb{S}^3 \backslash k, *)$. For convenience, we choose the basepoint $*$ on the boundary of a regular neighborhood $k \times \mathbb{D}^2$. A *meridian* is an element $\mu \in \pi_k$ represented by a curve freely homotopic to $* \times \partial \mathbb{D}^2$. We orient k and insist that the curve have linking number $+1$ with the knot. Then μ is well defined up to conjugation.

Following [13], we say that an element of a group is a *killer* if the group modulo the element is trivial; in other words, the element normally generates the group. Obviously the image of a killer under any automorphism of the group is a killer. It is well known that the meridian of a knot group is a killer. In [15], Tsau gave an example of a knot group killer that is not the automorphic image of the meridian. He called such an element a 'nonalgebraic killer'. We prefer a less violent term, 'pseudo-meridian'.

DEFINITION 1.1. A *pseudo-meridian* of a knot group is an element that normally generates the group but is not an automorphic image of the meridian. Two pseudo-meridians are *equivalent* if one is the automorphic image of the other.

THEOREM 1.2. *Let k be a nontrivial 2-bridge knot or torus knot, or a hyperbolic knot with unknotting number one. Then π_k contains infinitely many pairwise nonconjugate pseudo-meridians.*

It is well known that the group of automorphisms of π_k modulo inner automorphisms is finite. This is a consequence of Mostow rigidity when k is hyperbolic (see [14, Ch. 5]). When k is a torus knot, the group has order two (see [6]). Hence, we obtain the following.

COROLLARY 1.3. *Let k be a nontrivial 2-bridge knot or torus knot, or a hyperbolic knot with unknotting number one. Then π_k contains infinitely many pairwise nonequivalent pseudo-meridians.*

Our interest in pseudo-meridians is motivated largely by a conjecture of Jonathan Simon (see, for example, [7]): a sequence $\pi_{k_1} \to \pi_{k_2} \to \cdots \to \pi_{k_n} \to \cdots$ of knot group epimorphisms must be isomorphisms, for sufficiently large n. Much has been written about Simon's conjecture (see [1] for numerous references). It was recently proven for all 2-bridge knots [1].

The image of a meridian under a knot group epimorphism is either a meridian or a pseudo-meridian. Examples for which the image is a pseudo-meridian are known [5, 12]. It is an open question whether in this case there is always another epimorphism between the groups that does map a meridian to a meridian.

2. Proof of Theorem 1.2

Assume that k is a nontrivial 2-bridge knot. From a 2-bridge diagram for k, we obtain a group presentation for π_k having the form $\langle x, y \mid r \rangle$, where x and y are meridians corresponding to arcs of a 2-bridge diagram for k. Without loss of generality, we can assume that the exponent sum of y in r is $+1$ (and, consequently, that of x is -1). We introduce a new generator a and defining relation $y = ax$, and use the relation to eliminate y from the presentation. We obtain a presentation of the form $\pi_k = \langle x, a \mid \prod x^{-k_i} a^{\epsilon_i} x^{k_i} \rangle$, where $\sum \epsilon_i = 1$.

Consider the elements $\mu_n = x(yx^{-1})^n = xa^n$, where $n \geq 0$. Killing μ_n introduces the relation $x = a^{-n}$. The relator above becomes $\prod a^{nk_i} a^{\epsilon_i} a^{-nk_i}$, which is simply a. Since $x = a^{-n}$, killing μ_n also kills x. Hence, μ_n is a killer, for any $n \geq 0$. We will show that the elements μ_n are pairwise nonconjugate, for sufficiently large n.

The group π admits nonabelian parabolic representations $\rho : \pi_k \to SL_2 \mathbb{C}$:

$$x \mapsto X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto Y = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}, \tag{2.1}$$

where ω is any root of a nontrivial polynomial $\Phi(w)$ defined in [10]. Every nonabelian parabolic representation is conjugate to one of this form. Denote the trace of $\rho(\mu_n) = X(YX^{-1})^n$ by τ_n. Some values of τ_n are given below.

n	t_n
0	2
1	2
2	$2(1 - \omega)$
3	$2(1 - 3\omega + \omega^2)$
4	$2(-1 + 6\omega - 5\omega^2 + \omega^3)$

(The values τ_0 and τ_1 are both 2 for an obvious reason: $\mu_0 = x$ while $\mu_1 = xyx^{-1}$ is conjugate to x.)

One can check that τ_n satisfies a linear recurrence relation:

$$\tau_n = (2 - \omega)\tau_{n-1} - \tau_{n-2}, \quad \tau_0 = \tau_1 = 2.$$

The characteristic equation of the recurrence is $x^2 + (\omega - 2)x + 1 = 0$. The roots of the equation are ℓ, ℓ^{-1} such that $\ell + \ell^{-1} = 2 - \omega$. It is a straightforward matter to solve:

$$\tau_n = 2 \cdot \frac{\ell^{n-\frac{1}{2}} + \ell^{\frac{1}{2}-n}}{\ell^{\frac{1}{2}} + \ell^{-\frac{1}{2}}}.$$

If $|\ell| \neq 1$, then the absolute value of τ_n grows exponentially, and hence the elements μ_n are pairwise nonconjugate, for n sufficiently large. The claim is established in this case.

Suppose that $\ell = e^{i\theta}$, for some θ. The expression for τ_n simplifies:

$$\tau_n = 2 \cdot \frac{\cos((n - \frac{1}{2})\theta)}{\cos(\frac{1}{2})\theta}.$$

Either τ_n is nonrepeating, and the claim is proved, or otherwise θ is rationally related to π.

Suppose that θ is rationally related to π; that is, suppose that $\ell = e^{i\theta}$, where $\theta \in \mathbb{Q}\pi$. We can use the relation $\ell + \ell^{-1} = 2 - \omega$ to solve for ω:

$$\omega = 2(1 - \cos\theta) = 4\sin^2\left(\frac{\theta}{2}\right) > 0.$$

If k is a torus knot, then the coefficients of $\Phi(w)$ are all positive (an easy consequence of the form of the 2-bridge knot group presentation). Since not all of the roots of such a polynomial can be positive, we can choose a root ω away from the positive real axis. Then the traces τ_n, $n \geq 1$, are nonrepeating. (In fact, all roots of $\Phi(w)$ are real and negative. We do not need this here.)

If k is hyperbolic, then for some choice of ω, ρ projects to the $\mathrm{PSL}_2\,\mathbb{C}$ representation corresponding to the hyperbolic structure. This ω cannot be real since π_k is a Kleinian group of finite covolume (see [9, Exercise 1.3, No. 1], for example). Again the traces τ_n, $n \geq 1$, are nonrepeating. A 2-bridge knot cannot be a satellite, so Theorem 1.2 is proved for 2-bridge knots.

Next suppose that k is a hyperbolic knot with unknotting number one. There exists a crossing in some diagram of k such that changing the crossing results in a diagram of the trivial knot. We regard this crossing as a 4-valent vertex with meridianal generators x, y, z, w, as in Figure 1. Suppose that k has a left-hand crossing here. (The right-hand case is similar.) Then π_k has a presentation with relations $xy = zw$ and $x = w$, together with Wirtinger generators and relations for the other arcs and
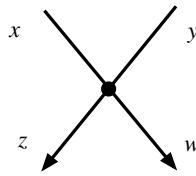
FIGURE 1. Crossing and generators.

crossings. Replacing $x = w$ with $y = z$ gives a presentation of $\mathbb{Z}$, the group of the trivial knot.

Let $\mu_n = xa^{-n}$, where $a = yx^{-1}$ and n is an arbitrary integer. A presentation for the quotient of π_k obtained by killing μ_n can be obtained as follows: add the new generator a and new relators $x = a^n$ and $y = a^{n+1}$. The relations at the crossing in Figure 1 force $z = a^{n+1}$ and $w = a^n$. Applying the same procedure in the given presentation of the unknot has the same result, so the two quotient groups are equal. However, the quotient of $\mathbb{Z}$ obtained by setting one generator equal to a^n and another equal to a^{n+1} is trivial. Hence, μ_n is a killer, for any n.

The elements x, y do not commute in π_k. (Otherwise, $y = z$ in π_k. As above, changing the crossing would not affect the group. However, the group would be trivial.) By [10], the discrete faithful $\mathrm{PSL}_2\,\mathbb{C}$ representation of π_k corresponding to the hyperbolic structure lifts to a parabolic representation $\rho : \pi_k \to \mathrm{SL}_2\,\mathbb{C}$ such that $\rho(x)$, $\rho(y)$ have the form in (2.1). The same argument as in the case of 2-bridge knots shows that the μ_n are pairwise nonconjugate.

Finally, assume that k is a nontrivial (p, q)-torus knot. Its group has a presentation of the form

$$\pi_k = \langle u, v \mid u^p = v^q \rangle.$$

We can assume without any loss of generality that p, q are relatively prime integers with $p > q > 0$.

There is an epimorphism $\chi : \pi_k \to \mathbb{Z}$ with $\chi(u) = q$ and $\chi(v) = p$. We can find integers r, s with

$$rp + sq = 1, \quad |r| < q, \ |s| < p. \tag{2.2}$$

Then $\chi(u^s v^r) = 1$. Note that by replacing r and s with $r - q$ and $s + p$, if necessary, we can assume that s is positive in condition (2.2). Clearly, $s = 1$ is impossible. Hence, $s > 1$.

Introduce new generators x, a and define relations $x = u^s v^r$ and $a = ux^{-q}$. Then

$$\pi_k = \langle x, u, v, a \mid u^p = v^q, x = u^s v^r, u = ax^q \rangle.$$

Using the relations above, eliminate u. We have

$$\pi_k = \langle x, v, a \mid (ax^q)^p = v^q, (ax^q)^{-s}x = v^r \rangle. \tag{2.3}$$

We claim that the element $\mu_n = xa^{-n}$ is a killer, for any integer n. Setting $x = a^n$ in the relations in presentation (2.3) gives

$$a^{p(nq+1)} = v^q, \quad a^{-s(nq+1)+n} = v^r,$$

and so

$$v^r q = a^{rp(nq+1)} = a^{-sq(nq+1)+nq}.$$

Since $rp + sq = 1$, we have $a^{nq+1} = a^{nq}$. Thus, $a = 1$ and so $x = 1$. It follows immediately that $v = 1$.

To show that the μ_n are pairwise nonconjugate, it suffices to show their images are pairwise nonconjugate in the quotient

$$\langle u, v \mid u^p = v^q = 1 \rangle = \mathbb{Z}/(p) * \mathbb{Z}/(q).$$

Rewriting xa^{-n} in terms of u and v yields

$$\mu_n = u^s v^r [u(u^s v^r)^{-q}]^{-n} = u^s v^r [(u^s v^r)^q u^{-1}]^n.$$

Since $s > 1$, cyclic reduction of this word will not reduce the number of occurrences of $v^{\pm 1}$, which is a monotone function of n. Hence, the killers μ_n are pairwise nonconjugate.
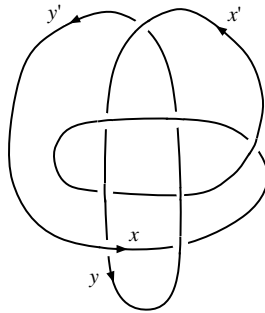
The proof of Theorem 1.2 is complete.

REMARK 2.1. Consider a knot diagram with a crossing, described by Figure 1, with the property that the elements $\mu_n = x(yx^{-1})^n$ are pairwise nonequivalent killers for arbitrarily large n. Introducing any number of full twists in the two arcs corresponding to z, w preserves this property, provided that the resulting knot is hyperbolic. Letting $x = a^n$ and $y = a^{n+1}$, as in the proof of Theorem 1.2, results in the same trivial quotient group. The above argument for hyperbolic knots applies. Hence, twisting in this manner produces many new examples of knot groups with infinitely many pairwise nonequivalent killers.

3. Examples and conjecture

EXAMPLE 3.1. Consider the diagram for the hyperbolic knot $k = 8_{20}$ in Figure 2, with certain Wirtinger generators indicated. Changing the crossing involving x and y produces a diagram of the unknot. Hence, by the proof of Theorem 1.2, the elements $\mu_n = x(yx^{-1})^n$ are nonequivalent pseudo-meridians, for sufficiently large n (in fact, for all $n > 1$).

From the proof of Theorem 1.2, one might wonder whether the elements $x(yx^{-1})^n$ are killers of any knot group whenever x and y are noncommuting meridians. (If so, then the conclusion of Theorem 1.2 would follow for any hyperbolic knot.) However, this is not the case for this example. If we choose meridians x' and y', then this word, according to GAP [11], is not a killer for $n = 2$.

EXAMPLE 3.2. Let $\tilde{k}$ be a hyperbolic knot embedded in a standard solid torus V in such a way that $V \setminus \tilde{k}$ is a hyperbolic manifold. The pair $(V, \tilde{k})$ is represented

FIGURE 2. The knot 8_{20}.

by a link diagram, one component representing $\tilde{k}$ and the other a meridian of V. Assume that changing some crossing unknots $\tilde{k}$ in V (equivalently, it produces a trivial 2-component link diagram). Twist knots other than the trivial and trefoil knots provide simple examples.

Let $\hat{k}$ be any knot. Form the satellite knot k with companion $\hat{k}$ and pattern $(V, \tilde{k})$ (see [2], for example). Changing a crossing of $\tilde{k}$ that unknots it in V also unknots the satellite knot k. As in the proof of Theorem 1.2, we produce a sequence $\mu_n = x(yx^{-1})^n$ of elements that are killers for $\pi_{\tilde{k}}$. They are also killers for π_k. This can be seen from the link diagram for the pattern. As in the proof of Theorem 1.2, the quotient group of $\pi_1(V \backslash \tilde{k})$ we obtain by setting $x = a^n$ and $y = a^{n+1}$ is unaffected if first we change the crossing and then make the substitution. However, changing the crossing produces a trivial 2-component link, and then the substitution kills x. Since $\pi_1(V \backslash \tilde{k})$ is a subgroup of π_k and since x normally generates π_k, the elements μ_n are killers of π_k.

Since $\tilde{k}$ is hyperbolic, the argument in the proof of Theorem 1.2 shows that, for n sufficiently large, no μ_n is the image of another under an automorphism of $\pi_{\tilde{k}}$.

Since $V \backslash \tilde{k}$ is a hyperbolic manifold, and since the characteristic submanifold of the exterior of k is unique up to ambient isotopy [3, 4], it follows that any autormorphism of π_k can be realized by a homeomorphism that leaves ∂V invariant, mapping the longitude $\lambda \in \pi_1(\partial V)$ to itself. The restriction to $V \backslash \tilde{k}$ induces an automorphism of $\pi_1(V \backslash \tilde{k})$ which induces an automorphism of $\pi_{\tilde{k}} \cong \pi_1(V \backslash \tilde{k})/\langle\langle \lambda \rangle\rangle$. Hence, the pseudo-meridians μ_n for $\pi_{\tilde{k}}$ that are pairwise nonequivalent are also pairwise nonequivalent pseudo-meridians for π_k.

CONJECTURE 3.3. *Every nontrivial knot group has infinitely many nonequivalent pseudo-meridians.*

References

[1] M. Boileau, S. Boyer, A. Reid and S. Wang, 'Simon's conjecture for 2-bridge knots', *Comm. Anal. Geom.*, to appear, arXiv:0903.2898.

[2] G. Burde and H. Zieschang, *Knots*, 2nd edn de Gruyter Studies in Mathematics, 5 (de Gruyter, Berlin, 2003).

[3] W. Jaco, *Lectures on Three-Manifold Topology*, CBMS Regional Conference Series in Mathematics, 43 (American Mathematical Society, Providence, RI, 1980).

[4] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics, 761 (Springer, Berlin, 1979).

[5] D. Johnson and C. Livingston, 'Peripherally specified homomorphs of knot groups', *Trans. Amer. Math. Soc.* **311** (1989), 135–146.

[6] A. Kawauchi, *A Survey of Knot Theory* (Birkhäuser, Basel, 1996).

[7] R. Kirby, 'Problems in low-dimensional topology', in: *Geometric Topology (Athens, GA, 1993).* (American Mathematical Society Publications, Providence, RI, 1997), pp. 35–473.

[8] P. B. Kronheimer and T. S. Mrowka, 'Witten's conjecture and property P', *Geom. Topol.* **8** (2004), 295–310.

[9] C. Maclachlan and A. W. Reid, *The Arithmetic of Hyperbolic 3-Manifolds* (Springer, New York, 2003).

[10] R. Riley, 'Parabolic representations of knot groups, I', *Proc. London Math. Soc.* **24** (1972), 217–242.

[11] M. Schönet *et al.*, GAP—Groups, Algorithms, and Programming — version 3 release 4 patchlevel 4. Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1997.

[12] D. S. Silver and W. Whitten, 'Knot group epimorphisms', *J. Knot Theory Ramifications* **15** (2006), 153–166.

[13] J. Simon, 'Wirtinger approximations and the knot groups of F^n in $\mathbb{S}^{n+2}$', *Pacific J. Math.* **90** (1980), 177–189.

[14] W. P. Thurston, *The Geometry and Topology of Three-Manifolds,* Volume 1 (Princeton University Press, Princeton, NJ, 1997).

[15] C. M. Tsau, 'Nonalgebraic killers of knot groups', *Proc. Amer. Math. Soc.* **95** (1985), 139–146.

DANIEL S. SILVER, Department of Mathematics and Statistics,
University of South Alabama, Mobile, AL 36688, USA
e-mail: silver@jaguar1.usouthal.edu

WILBUR WHITTEN, 1620 Cottontown Road, Forest, VA 24551, USA
e-mail: bjwcw@aol.com

SUSAN G. WILLIAMS, Department of Mathematics and Statistics,
University of South Alabama, Mobile, AL 36688, USA
e-mail: swilliam@jaguar1.usouthal.edu