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COMMUTING k-TUPLES OF SELF ADJOINT OPERATORS AND MATRIX MEASURES

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1. **Introduction.** In this note we give a generalization of a spectral representation theorem for self adjoint operators in a Hilbert space recently obtained by the author [1]. Our interest here is to develop corresponding results for a k-tuple $(T_1, \ldots, T_k)$ of commuting self adjoint operators.

In this section we present some definitions and terminology.

DEFINITION 1. Let $\{\mu_{ij}\}$, $1 \le i, j \le n$, be a family of complex valued set functions defined on the bounded Borel subsets of R^k. The family (μ_{ij}) will be called an $n \times n$ positive matrix measure if

(i) the matrix $(\mu_{ij}(M))$ is Hermitian positive semi-definite for each bounded Borel set $M \subset R^k$;

(ii) we have

$$\mu_{ij}\left(\bigcup_{m=1}^{\infty} M_m\right) = \sum_{m=1}^{\infty} \mu_{ij}(M_m), \qquad 1 \le i, j \le n,$$

for each sequence (M_m) of disjoint Borel sets with bounded union.

The theory of matrix measures defined on the bounded Borel sets of R^1 is given in detail in [2, pp. 1337–1346]. A careful study of these pages shows that all properties of and results concerning matrix measures on R^1 carry over to the R^k case. We shall have occasion to call on these results in the R^k setting.

DEFINITION 2. Let H be a Hilbert space and

$$T_i = \int_{-\infty}^{\infty} \lambda E_i(d\lambda), \qquad i = 1, 2, \ldots, k,$$

self adjoint operators in H. We say that the k-tuple $(T_1, \ldots, T_k)$ is a commuting k-tuple of self adjoint operators if for any choice of Borel sets $M_i \subset R$ we have

$$E_i(M_i)E_j(M_j) = E_j(M_j)E_i(M_i), \qquad 1 \le i, j \le k.$$

This definition follows Prugovečki [3, page 261] and we shall be leaning heavily on the theory of functions of such operators as outlined in the above reference (pp. 269–284).

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Specifically if E denotes the spectral measure on the Borel sets of R^k satisfying

$$E(M_1 \times M_2 \times \cdots \times M_k) = E_1(M_1)E_2(M_2) \cdots E_k(M_k)$$

whenever $M_i \subset R$, $i=1, \ldots, k$ are Borel sets and if $p(\lambda_1, \ldots, \lambda_k)$ is a Borel function on R^k, there exists a unique linear operator $p(T_1, \ldots, T_k)$ in H satisfying

$$(1) \qquad (p(T_1, \ldots, T_k)f, g) = \int_{R^k} p(\lambda)(E(d\lambda)f, g), \qquad g \in H.$$

Its domain of definition is given by

$$D(p(T_1, \ldots, T_k)) = \left\{ f \in H \;\Big|\; \int_{R^k} |p(\lambda)|^2 \,(E(d\lambda)f, f) < \infty \right\}.$$

Details of these results can be found in [3, pp. 269–284]. Our definition of $p(T_1, \ldots, T_k)$ differs slightly from that of Prugovečki in that given a Borel function $p(\lambda)$ he establishes the existence of a linear operator $p(T_1, \ldots, T_k)$ such that

$$(2) \qquad (g, p(T_1, \ldots, T_k)f) = \int_{R^k} p(\lambda)(g, E(d\lambda)f), \qquad g \in H.$$

Our definition can be obtained by replacing $p(\lambda)$ by $\overline{p(\lambda)}$ in (2). For consistency with our earlier work [1] we shall use definition (1) which also ensures that

$$(\alpha_1 p_1 + \alpha_2 p_2)(T_1, \ldots, T_k) = \alpha_1 p_1(T_1, \ldots, T_k) + \alpha_2 p_2(T_1, \ldots, T_k)$$

on the intersection of the domains of $p_1(T_1, \ldots, T_k)$ and $p_2(T_1, \ldots, T_k)$. Furthermore we shall restrict our attention to Borel functions p which are defined E almost everywhere on R^k. Under this condition it is easy to show that $D(p(T_1, \ldots, T_k))$ is dense in H, c.f. the corresponding result for a single self adjoint operator [2, Theorem 6, p. 1196]. The same argument also shows that $p(T_1, \ldots, T_k)$ is a closed operator in H.

2. **The spectral representation theorem.** Let H be a Hilbert space and $T_i = \int_{-\infty}^{\infty} \lambda E_i(d\lambda)$, $i=1, 2, \ldots, k$, a commuting k-tuple of self adjoint operators in H. Denote by E, the spectral measure defined on Borel subsets of R^k and satisfying

$$E(M_1 \times \cdots \times M_k) = E_1(M_1) \cdots E_k(M_k)$$

for all $M_1, \ldots, M_k$, Borel subsets of R^1.

For a finite set $N \subset H$, $N = \{f_1, \ldots, f_n\}$, H_N will denote the closure of the manifold in H consisting of all vectors of the form $p_1(T_1, \ldots, T_k)f_1 + \cdots + p_n(T_1, \ldots, T_k)f_n$ where p_i varies over Borel functions defined E-almost everywhere on R^k for which $f_i \in D(p_i(T_1, \ldots, T_k))$, $1 \le i \le n$.

The following Lemmas, Definition and Theorems have precise analogues in our earlier work dealing with a single self adjoint operator [1]. It is not our purpose here to reproduce verbatim existing arguments but rather to note that they carry over mutatis mutandis to the present situation. Accordingly, we take the liberty of stating these results without proof, referring the reader to the above reference.

LEMMA 1. *Let the complex valued measures* μ_{ij}, $1 \leq i, j \leq n$, *be defined by* $\mu_{ij}(\cdot) = (E(\cdot)f_i, f_j)$. *Then* (μ_{ij}) *is an* $n \times n$ *positive matrix measure on the Borel subsets of* R^k.

LEMMA 2. *Let* (μ_{ij}) *be as above. Define* μ *by* $\mu(\cdot) = \sum_{i=1}^n \mu_{ii}(\cdot)$. *Then* μ *is a totally finite regular measure on the Borel subsets of* R^k *and each* μ_{ij} *is absolutely continuous with respect to* μ. *Further,* H_N *is a Hilbert space which is unitarily equivalent to* $L^2(\mu_{ij})$.

LEMMA 3. *Let* $n \geq 1$ *be an integer. Then there is a collection* $\mathscr{L}$ *of finite subsets* $N \subset H$, *each containing n points such that*

$$H = \sum_{N \in \mathscr{L}} \oplus H_N.$$

For $f \in H$ we let f_N be its component in H_N, that is

$$f = \sum_{N \in \mathscr{L}} f_N, f_N \in H_N.$$

LEMMA 4. *For every Borel function p we have*

$$D(p(T_1, \ldots, T_k)) = \Big\{f \in H \,\big|\, f_N \in D(p(T_1, \ldots, T_k)), N \in \mathscr{L};$$
$$\sum_{N \in \mathscr{L}} \|p(T_1, \ldots, T_k)f_N\|^2 \leq \infty\Big\}$$

and $(p(T_1, \ldots, T_k)f)_N = p(T_1, \ldots, T_k)f_N$.

We now adopt the following notation:

$$N = \{f_1^N, f_2^N, \ldots, f_n^N\}, \mu_{ij}^N(\cdot) = (E(\cdot)f_i^N, f_j^N),$$

$$\mu_N = \sum_{i=1}^n \mu_{ii}^N, \mu_{ij}^N(M) = \int_M m_{ij}^N(t)\mu_N(dt).$$

LEMMA 5. *H is unitarily equivalent to* $\sum_{N \in \mathscr{L}} L^2(\mu_{ij}^N)$.

LEMMA 6. *For every Borel function k we have*

$$UD(p(T_1, \ldots, T_k)) = \left\{F \in \sum_N L^2(\mu_{ij}^N) \;\Big|\; \sum_N \int_{R^k} |p(t)|^2 \sum_{i,j=1}^n m_{ij}^N(t) F_{Ni}(t)\overline{F_{Nj}(t)}\mu_N(dt) < \infty\right\}$$

and

$$(Up(T_1, \ldots, T_k)f)_N(t) = p(t)(Uf)_N(t).$$

Here U denotes the unitary map $U: H \to \sum_N L^2(\mu_{ij}^N)$ established in the previous lemma. We have also used the notation for $F = \sum_N F_N \in \sum_N L^2(\mu_{ij}^N)$, $F_N = (F_{N1}, \ldots, F_{Nn})$.

DEFINITION 3. Let $(T_1, \ldots, T_k)$ be a commuting k-tuple of self adjoint operators on a Hilbert space H and let $\{(\mu_{ij}^N)\}$ be a family of finite $n \times n$ positive matrix measures on the Borel subsets of R^k and vanishing on the complement of the Cartesian product of the spectra of the operators $T_1, \ldots, T_k$. Let U be a unitary map between H and $\sum_N L^2(\mu_{ij}^N)$. The transformation U is an $n \times n$ spectral representation of H onto $\sum_N L^2(\mu_{ij}^N)$ relative to $(T_1, \ldots, T_k)$ if the following conditions are satisfied:

(a) for every Borel function p defined E-almost everywhere on R^k we have

$$UD(p(T_1,\ldots,T_k)) = \left\{F \in \sum_N L^2(\mu_{ij}^N) \mid \sum_N \int_{R^k} |p(t)|^2 \sum_{i,j=1}^{n} F_{Ni}(t)\overline{F_{Nj}(t)}\mu_{ij}^N(dt) < \infty\right\};$$

(b) $(Up(T_1, \ldots, T_k)f)_N(t) = p(t)(Uf)_N(t)$.

THEOREM 1. *Every Hilbert space admits an $n \times n$ spectral representation relative to an arbitrary commuting k-tuple of self adjoint operators defined in it, for any value of the integer n.*

THEOREM 2. *Let U be an $n \times n$ spectral representation of H onto $\sum_K L^2(\rho_{ij}^K)$ relative to a commuting k-tuple of self adjoint operators $(T_1, \ldots, T_k)$. Then to each K there corresponds a finite set $N \subset H$ such that $(\rho_{ij}^K) = (\mu_{ij}^N)$, H is a direct sum of subspaces H_N and U maps H_N onto $L^2(\mu_{ij}^N)$.*

3. **An Analytic Representation.** We now specialize our attention to the case $H = L^2(S, \Sigma, \nu)$ where (S, Σ, ν) is a positive measure space. We shall further assume that there is an expanding sequence of sets $S_n \subset S$, covering S, each of which has finite ν measure and such that for bounded Borel sets $M \subset R^k$, the range of $E(M)$ contains only functions which are ν-essentially bounded on each of the sets S_n, (c.f. [2], p. 1210).

This restriction on the operators $T_1, \ldots, T_k$ will be assumed to hold throughout this section.

LEMMA 7. *Under the above hypothesis there is for each $g \in L^2(S, \Sigma, \nu)$ a function W defined on the Cartesian product of S and R^k which is measurable with respect to the product of ν and the measure $\mu = (E(\cdot)g, g)$ and which has the property that for every bounded Borel set $M \subset R^k$ and every F in $L^2(\mu)$ we have*

(*i*) $\nu\text{-ess}\sup_{s \in S_n} \int_M |W(s, \lambda)|^2 \mu(d\lambda) < \infty,$

(*ii*) $(E(M)F(T_1, \ldots, T_k)g)(s) = \int_M W(s, \lambda)F(\lambda)\mu(d\lambda).$

Proof. Again we are in the situation of being able to refer to an existing result proved for the case of a single operator noticing that the argument used there is applicable to our present case. We refer the reader to [2, Lemma 9, p. 1211].

In like manner, we follow the argument of Dunford and Schwartz and with reference to [2, Theorem 11, p. 1213] claim

THEOREM 3. *Let (S, Σ, ν) be a positive measure space and let $\{S_n\}$ be an increasing sequence of sets of finite measure covering S. Let U be a 1×1 spectral representation of $L^2(S, \Sigma, \nu)$ onto $\sum_{N\in\mathscr{L}} L^2(\mu^N)$ relative to the commuting k-tuple of self adjoint operators $(T_1, \ldots, T_k)$. Suppose for each bounded Borel set $M\subset R^k$, the range of $E(M)$ contains only functions which are ν-essentially bounded on each of the sets S_n. Then for each element $N\in\mathscr{L}$ there is a function W_N defined on $S\times R^k$ and having the properties:*

(a) *W_N is measurable with respect to the product measure $\nu\times\mu^N$;*
(b) *for each bounded Borel set $M\subset R^k$ we have*

$$\nu\text{-ess}\sup_{s\in S_n}\int_M |W_N(s,\lambda)|^2\,\mu^N(d\lambda) < \infty, \qquad n\geq 1;$$

(c) $$(Uf)_N(\lambda) = \int_S f(s)\overline{W_N(s,\lambda)}\nu(ds), \qquad f\in L^2(S,\Sigma,\nu),$$

the integral existing in the mean square sense in $L^2(\mu^N)$.

This theorem gives an analytic representation of the spectral representation U as a collection of integral transforms in the case of a 1×1 spectral representation. The construction of an analytic representation for an $n\times n$ spectral representation is identical with that used for the theory of a single self adjoint operator. We refer the reader to [1, Lemmas 7, 8, Theorems 3, 4]. Accordingly we claim

THEOREM 4. *Let U be an $n\times n$ spectral representation of $L^2(S, \Sigma, \nu)$ onto $\sum_{N\in\mathscr{L}} L^2(\mu_{ij}^N)$ relative to the commuting k-tuple of self adjoint operators $(T_1, \ldots, T_k)$. Under the hypothesis of the previous theorem, for each $N\in\mathscr{L}$ there are functions W_N^i, $1\leq i\leq n$, defined on $S\times R^k$ such that*

(a) *W_N^i is measurable with respect to $\nu\times\mu_N$;*
(b) *for each bounded Borel set $M\subset R^k$ we have*

$$\nu\text{-ess}\sup_{s\in S_n}\int_M \sum_{i,j=1}^n m_{ij}^N(t)W_N^i(s,t)\overline{W_N^j(s,t)}\mu_N(dt) < \infty, \qquad n\geq 1;$$

(c) $$((Uf)_N(t)) = \left(\int_S f(s)\overline{W_N^i(s,t)}\nu(ds)\right)_{i=1}^n, \qquad f\in L^2(S,\Sigma,\nu),$$

the integrals existing in the mean square sense in $L^2(\mu_{ij}^N)$.

THEOREM 5. *With the notation of the previous theorem we have*

$$f(s) = \sum_{N\in L}\int_{R^k}\sum_{i,j=1}^n m_{ij}^N(t)(Uf_N)_i(t)W_N^i(s,t)\mu_N(dt),$$

$f\in L^2(S, \Sigma, \nu)$, the integrals existing in the mean square sense in $L^2(S, \Sigma, \nu)$ and the series converging in the norm of $L^2(S, \Sigma, \nu)$.

Our project is now complete. The reader will see that by reference to existing results and noting their immediate generalization we have been able to produce a spectral representation theory for commuting k-tuples of self adjoint operators.

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