A REMARK ON THE THEOREMS OF LUSIN AND EGOROFF

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(received June 20, 1963)

In this note we do not intend to establish new results but only to suggest a very simple proof of Lusin's theorem, direct for σ-finite regular measures, a proof that bypasses the usual procedure of first establishing this theorem for sets of finite measure only. The proposed proof utilizes the notion of subuniform convergence, a method which seems not yet to have been used, despite its simplicity and adaptability.[1] Simultaneously, a useful supplement to Egoroff's theorem will be obtained.

The note should be easily understood by first year graduate students and senior undergradutes.

TERMINOLOGY AND NOTATION. A sequence of extended real valued functions[2] $\{f_n\}$ defined on a topological space S is said to converge subuniformly on a set $A \subseteq S$ to a function f if every point $p \in S$ has a neighborhood G_p such that $f_n \to f$ uniformly on $A \cap G_p$.[3] Notation: $f_n \to f$ (subunif.) on A. A similar notation will be used for uniform (unif.) and "almost

1) A different proof was given by Schaerf in [6] and [7], for functions with values in any space satisfying the second axiom of countability. For the original theorem of Lusin, cf. [3]; [4] p. 159; [5] p. 72.

2) i.e., functions whose values are real numbers and (possibly) $\pm\infty$.

3) This is stronger than the usual variant of this concept as defined, e.g., in [4], p. 44, Ex. j.

Canad. Math. Bull. vol. 7, no. 2, April 1964

everywhere" (a.e.) convergence. In the sequel, m will denote a (non-negative) completely additive measure defined on a σ-field M of subsets of S. m is said to be <u>regular</u> if the measure mA of every set $A \in M$ is the infimum of the measures of all open measurable supersets of A; in this case S is referred to as a <u>regular measure space</u>. S and m are said to be <u>σ-finite</u> if $S = \bigcup_{k=1}^{\infty} G_k$ for some sequence of sets $G_k \in M$, with $mG_k < \infty$, $k = 1, 2, \ldots$. If, in addition, the sets G_k can be chosen to be open, we use the term σ°-finite instead (Schaerf).

THE THEOREMS. We shall first prove the supplement to Egoroff's theorem, mentioned above.[4)]

I. <u>Let the extended real valued functions</u> $f, f_1, f_2, \ldots$ <u>be defined and measurable on a σ°-finite measure space</u> S. <u>If</u> f <u>is a.e. finite and if</u> $f_n \to f$ <u>(a.e.) on</u> S, <u>then, for any real</u> $\varepsilon > 0$, <u>there is a measurable set</u> $A \subseteq S$ <u>such that</u> $m(S-A) < \varepsilon$ <u>and</u> $f_n \to f$ <u>(subunif.) on</u> A.

<u>Proof</u>. By assumption, $S = \bigcup_{k=1}^{\infty} G_k$ where the G_k are open measurable sets of finite measure. Define $D_1 = G_1$ and $D_k = G_k - \bigcup_{i=1}^{k-1} G_i$, $k = 2, 3, \ldots$. Then $S = \bigcup_{k=1}^{\infty} D_k$, with $D_k \in M$ and $mD_k < \infty$, $k = 1, 2, \ldots$. Therefore, given $\varepsilon > 0$, Egoroff's theorem yields, for each k, a measurable set $A_k \subseteq D_k$, with $m(D_k - A_k) < \varepsilon/2^k$ and $f_n \to f$ (unif.) on each A_k separately. Let $A = \bigcup_{k=1}^{\infty} A_k$. Then $A \in M$, and

$$m(S-A) = m\left(\bigcup_{k=1}^{\infty} D_k - \bigcup_{k=1}^{\infty} A_k\right) \leq m \bigcup_{k=1}^{\infty} (D_k - A_k) \leq \sum_{k=1}^{\infty} m(D_k - A_k)$$

4) For the original theorem of Egoroff, see [1]; [2] p. 88; or [4] p. 157; [5] p. 18.

https://doi.org/10.4153/CMB-1964-028-x Published online by Cambridge University Press

$< \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon$. It remains to show that $f_n \to f$ (subunif.) on A. For this purpose, fix any $p \in S$. Then p is in some G_k, call it $G_{\underline{k}}$. By definition, all sets D_k with $k > \underline{k}$ are disjoint from $G_{\underline{k}}$. A fortiori, so are the sets $A_k \subseteq D_k$ $(k > \underline{k})$. Hence $G_{\underline{k}} \cap \bigcup_{k>\underline{k}} A_k = \emptyset$ so that

$$(1) \qquad A \cap G_{\underline{k}} \subseteq \bigcup_{k=1}^{\underline{k}} A_k .$$

Now, as $f_n \to f$ (unif.) on each A_k, we also have $f_n \to f$ (unif.) on the finite union $\bigcup_{k=1}^{\underline{k}} A_k$ and, by (1), also on $A \cap G_{\underline{k}}$. Thus every point $p \in S$ has a neighborhood $G_{\underline{k}}$ with $f_n \to f$ (unif.) on $A \cap G_{\underline{k}}$, as required. This completes the proof.

The significance of this proposition lies in that it relaxes the finiteness restriction contained in Egoroff's theorem (namely the requirement that $mS < \infty$) to that of σ^o-finiteness, at the expense of replacing uniform convergence by its weaker subuniform variety. This is probably the most that can be done in this respect since, as is well known, the ordinary variant of Egoroff's theorem cannot be extended to σ-finite or σ^o-finite measures. (Cf. [4], p. 158, Ex. e - i.) It is now easy to obtain the σ-finite version of Lusin's theorem.

II. <u>If f is an a. e. finite extended real valued function, defined and measurable on a σ-finite regular measure space</u> S, <u>then, for every</u> $\varepsilon > 0$, <u>there is a closed set</u> $F \subseteq S$ <u>such that</u> $m(S-F) < \varepsilon$ <u>and</u> f <u>is continuous when restricted to</u> F.

For the proof, we note the almost obvious fact that the limit of every subuniformly convergent sequence of continuous functions on a set A is itself a continuous function on A, and that, for regular measures, the notions of σ-finiteness and σ^o-finiteness coincide. With these facts established, our Theorem II follows from I in exactly the same way as the original theorem of Lusin is deduced from Egoroff's theorem. To obtain it, it suffices, e.g., to verbally repeat the proof given in [4], pp. 159-160, with uniform convergence replaced by subuniform convergence. Thus the σ-finite version of Lusin's theorem can be obtained with no more effort than its finite variant, and with practically no change in its standard proof, once our Theorem I has been established.

FINAL REMARKS. For simplicity, we have limited ourselves to extended real valued functions. However, a look at the proof shows that the same method could be applied to measurable functions with values in any space which admits Egoroff's theorem[5] and an approximation of measurable functions by simple functions.[6] Indeed, these are the only prerequisites of the standard proof of Lusin's theorem (quoted above), and nothing more is required for our theorems I and II. These requirements can certainly be satisfied in separable pseudometric spaces and, more generally, in separable uniform spaces whose uniformity has a countable base[7] (since such spaces are pseudo-metrizable; cf. [8], p. 186). Thus this method of proof is sufficiently general (though probably less general than Schaerf's) and is so simple that it is easily adaptable to any course in measure theory.

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5) i.e., in any uniform space T such that Egoroff's theorem remains valid, with real functions replaced by functions taking values in T.

6) i.e., functions which take only a finite number of values (each value on some measurable set).

7) It is not necessary that such a space be a T_1-space.

https://doi.org/10.4153/CMB-1964-028-x Published online by Cambridge University Press

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