BULL. AUSTRAL. MATH. SOC. 16A70
VOL. 27 (1983), 221-224.

A COMMUTATIVITY THEOREM FOR SEMIPRIME RINGS

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Let R be a semiprime ring in which for each x in R there exists a positive integer $n = n(x) > 1$ such that $(xy)^n = x^n y^n$ for all y in R. Then R is commutative.

A theorem of Herstein [3] states that a ring R which satisfies the identity $(xy)^n = x^n y^n$, where n is a fixed positive integer greater than 1, must have nil commutator ideal. In [2] Bell proved that if R is an n-torsion-free ring with identity 1 and satisfies the two identities $(xy)^n = x^n y^n$ and $(xy)^{n+1} = x^{n+1} y^{n+1}$, then R is commutative. Recently, the author [1] proved that if R is $n(n-1)$-torsion-free ring with 1 and satisfies the identity $(xy)^n = x^n y^n$, then R is commutative. In this note, we consider rings which satisfy $(xy)^n = x^n y^n$ where n is a positive integer depending on x. In this direction we prove the following theorem which generalizes the above result of Herstein.

THEOREM. *Let R be a semiprime ring in which, for each x in R, there exists an integer $n = n(x) > 1$ such that $(xy)^n = x^n y^n$ for all y in R. Then R is commutative.*

In preparation for the proof of our main theorem, we first prove the following lemmas. Throughout, R will denote an associative ring.

LEMMA 1. *If R is a semiprime ring in which, for each x in R,*

Received 18 October 1982.

there exists a positive integer $n = n(x) > 1$ *such that* $(xy)^n = x^n y^n$ *for all* y *in* R *, then* R *has no nonzero nilpotent elements.*

Proof. Let $a \in R$ such that $a^2 = 0$. Using the hypothesis, there exists an integer $n = n(a) > 1$ such that $(ax)^n = a^n x^n$ for all x in R . This implies that $(ax)^n = 0$ for all x in R . If $aR \neq 0$, then the above shows that aR is a nonzero nil right ideal satisfying the identity $y^n = 0$ for all y in aR . So by Lemma 2.1.1 of [5], R has a nonzero nilpotent ideal. This is a contradiction since R is semiprime. Thus $aR = 0$, and hence $aRa = 0$. This implies that $a = 0$ since R is semiprime.

LEMMA 2. *If* R *is a prime ring in which, for each* x *in* R *, there exists an integer* $n = n(x) > 1$ *such that* $(xy)^n = x^n y^n$ *for all* y *in* R *, then* R *has no zero divisors.*

Proof. By Lemma 1 above, R has no nonzero nilpotent elements. So by Lemma 1.1.1 of [5], R has no zero divisors since it is prime with no nonzero nilpotent elements.

LEMMA 3. *If* R *is a prime ring in which, for each* x *in* R *, there exists an integer* $n = n(x) > 1$ *such that* $(xy)^n = x^n y^n$ *for all* y *in* R *, then for each* x *in* R *, there exists an integer* $n = n(x) > 1$ *such that*

(i) $(x^k y)^n = x^{kn} y^n$ *, for all* y *in* R *and all integers* $k \geq 1$ *, and*

(ii) $(yx^k)^n = x^{k(n-1)} y^n x^k$ *, for all* $y \in R$ *and all integers* $k \geq 1$.

Proof. Part *(i)* can easily be proved by induction on k . To prove *(ii)*, let $n = n(x) > 1$ such that $(x^k y)^n = x^{kn} y^n$, for all y in R and all integers $k \geq 1$. Note that $x^k (yx^k)^n = (x^k y)^n x^k = x^{kn} y^n x^k$, and hence $x^k \left((yx^k)^n - x^{k(n-1)} y^n x^k \right) = 0$, and now part *(ii)* of the lemma follows since, by Lemma 2, R has no zero divisors.

Proof of the theorem. Since R is a semiprime ring then it is isomorphic to a subdirect sum of prime rings R_α each of which, as a homomorphic image of R, satisfies the hypothesis of the theorem. So we may assume that R is prime. Let x and y be any two nonzero elements of R. Then by Lemma 3 *(i)*, there exist positive integers $n = n(x) > 1$, and $m = m(y) > 1$ such that

(1) $\left(x^k z\right)^n = x^{kn} z^n$, for all z in R and all integers $k \geq 1$,

(2) $\left(y^k z\right)^m = y^{km} z^m$, for all z in R and all integers $k \geq 1$.

Using (2) and Lemma 3 *(ii)* we get

(3) $$(yx)^{mn} = \left((yx)^m\right)^n = \left(y^m x^m\right)^n = x^{m(n-1)} y^{mn} x^m .$$

From Lemma 3 *(ii)*, $(yx)^n = x^{n-1} y^n x$, and hence

(4) $$\left((yx)^n\right)^m = \left(x^{n-1} y^n x\right)^m = x^{n-1} y^n x \cdot x^{n-1} y^n x \ldots x^{n-1} y^n x$$
$$= x^{n-1} \left(y^n x^n\right)^{m-1} y^n x .$$

Thus, using (4) and (2),

$$(yx)^{nm} x^{n-1} = \left(x^{n-1} \left(y^n x^n\right)^{m-1} y^n x\right) x^{n-1} = x^{n-1} \left(y^n x^n\right)^m = x^{n-1} y^{nm} x^{nm} .$$

Thus,

$$\left[(yx)^{nm} - x^{n-1} y^{nm} x^{nm-(n-1)}\right] x^{n-1} = 0 .$$

Using Lemma 2, R has no zero divisors, and hence

(5) $$(yx)^{nm} = x^{n-1} y^{nm} x^{nm-(n-1)} .$$

Now (3) and (5) imply that

(6) $$x^{n-1} y^{nm} x^{nm-(n-1)} = x^{m(n-1)} y^{mn} x^m .$$

Clearly $m(n-1) > (n-1)$, and $nm - (n-1) = n(m-1) + 1 > m$ since $m \geq 2$, and $n \geq 2$. So (6) implies that

(7) $$x^{n-1} \left[y^{nm} x^{nm-m-n+1} - x^{nm-m-n+1} y^{nm}\right] x^m = 0 .$$

Since R has no zero divisors, (7) implies that

$$y^{nm} x^{nm-m-n+1} = x^{nm-m-n+1} y^{nm} ,$$

and hence for any two elements x and y in R, there exist two positive integers $p = p(x, y) \geq 1$ and $q = q(x, y) \geq 1$ such that $y^p x^q = x^q y^p$, and therefore R is commutative by a theorem of Herstein [4].

COROLLARY. *If R is a ring in which, for each x in R, there exists an integer $n = n(x) > 1$ such that $(xy)^n = x^n y^n$ for each $y \in R$, then the commutator ideal of R is nil.*

Proof. To prove that the commutator ideal of R is nil it is enough to show that if R has no nonzero nil ideals then it is commutative. So we suppose that R has no nonzero nil ideals. Then R is a subdirect product of prime rings each of which, as a homomorphic image of R, satisfies the hypothesis of the corollary. So R is commutative by the above theorem.

References

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