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ON THE RESIDUAL FINITENESS OF POLYGONAL PRODUCTS OF NILPOTENT GROUPS

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ABSTRACT. In general polygonal products of finitely generated torsion-free nilpotent groups amalgamating cyclic subgroups need not be residually finite. In this paper we prove that polygonal products of finitely generated torsion-free nilpotent groups amalgamating maximal cyclic subgroups such that the amalgamated cycles generate an isolated subgroup in the vertex group containing them, are residually finite. We also prove that, for finitely generated torsion-free nilpotent groups, if the subgroups generated by the amalgamated cycles have the same nilpotency classes as their respective vertex groups, then their polygonal product is residually finite.

1. **Introduction.** Polygonal products of groups were introduced by A. Karrass, A. Pietrowski and D. Solitar [8]. They studied the subgroup structure of these products and applied the results to the Picard group $\mathrm{PSL}(2, Z[i])$ which is a polygonal product of A_4, the four group and two copies of S_3. In [4], Brunner, Frame, Lee and Wielenberg used their results to determine all the torsion-free subgroups of finite index in the Picard group. These products are also discussed by B. Fine in [6]. Allenby and Tang [2] studied the residual finiteness of polygonal products. They showed that polygonal products of finitely generated free abelian groups amalgamating cyclic subgroups with trivial intersections are residually finite. On the other hand, they constructed an example of a polygonal product of four finitely generated torsion-free nilpotent groups of class 2 amalgamating cyclic subgroups with trivial intersections, which is not residually finite. In this example, the amalgamated subgroups are not isolated subgroups [3] of their vertex groups. Moreover, the subgroups generated by the amalgamated subgroups in their respective vertex groups do not have the same nilpotency classes as their vertex groups. In this paper we show that if the subgroups generated by the two amalgamated cyclic subgroups are either isolated subgroups of their respective vertex groups (Theorem 3.6) or of the same nilpotency classes as their respective vertex groups (Theorem 4.6) then their polygonal products are residually finite.

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2. **Preliminaries.** Briefly polygonal products can be described as follows [2]: Let P be a polygon. Assign to each vertex v of P a vertex group G_v and to each edge e, an edge group G_e together with monomorphisms λ_e and ρ_e embedding G_e as a subgroup of the two vertex groups at the ends of e. The polygonal product of this system of groups is the group G with generators and relations those of the vertex groups together with the extra relations obtained by identifying $g_e\lambda_e$ and $g_e\rho_e$ for each $g_e \in G_e$.

Throughout this paper, we only consider the case when P is a square. The results can be extended to polygons with more than four vertices. However the case of triangles can be nasty, because the triangle groups so formed may not contain the vertex groups isomorphically (see [9], p. 525).

We shall adopt the following notations and terminology:

We use $N \triangleleft_f G$ to denote that N is a normal subgroup of finite index in G. RF means residually finite and $f.g.$ means finitely generated. If $N \triangleleft G$ and $\bar{G} = G/N$ then $\bar{x}$ denotes Nx for $x \in G$. If $G = A *_H B$ and $x \in G$, then $\|x\|$ denotes the free product length of x in G. $Z_i(G)$ denotes the ith term of the upper central series of G. For convenience, we let $Z(G) = Z_1(G)$. Let H be a subgroup of G, then H^G denotes the normal closure of H in G.

A group G is said to be *subgroup separable* (LERF) if for every $f.g.$ subgroup H of G and every $x \in G \setminus H$ there exists $N \triangleleft_f G$ such that $\bar{x} \notin \bar{H}$ in $\bar{G} = G/N$. If H is a subgroup of G and for every $x \in G \setminus H$, there exists $N \triangleleft_f G$ such that $\bar{x} \notin \bar{H}$ in $\bar{G} = G/N$, then we say G is *H-separable.*

A torsion-free group G is said to be *potent* if, for each positive integer n and each $1 \neq x \in G$, there exists $N \triangleleft_f G$ such that Nx has order exactly n in G/N.

Free groups and $f.g.$ torsion-free nilpotent groups are potent.

Let H be a subgroup of G. Then H is called a *retract* of G if there exists $N \triangleleft G$ such that $G = NH$ and $N \cap H = 1$.

We shall use the following results.

THEOREM 2.1 ([5], THEOREM 1). *The generalized free products of residually finite groups amalgamating retracts are* RF.

THEOREM 2.2 ([3], THEOREM 2.5). *Let G be a $f.g.$ torsion-free nilpotent group. If H is an isolated subgroup of G, then $\bigcap_{k=1}^{\infty} G^{p^k}H = H$ for all primes p.*

3. **Amalgamating maximal cyclic subgroups.** The example given in [2] showed that the polygonal products of $f.g.$ torsion-free nilpotent groups amalgamating cyclic groups need not be RF. However, under certain conditions, if the amalgamated subgroups are maximal cyclic subgroups then we can prove that the polygonal products are RF.

Throughout the following we shall adopt the following notation. $A = \langle a,b\rangle$, $B = \langle b,c\rangle$, $C = \langle c,d\rangle$ and $D = \langle d,a\rangle$ are torsion-free nilpotent groups with $\langle a\rangle \cap \langle b\rangle = \langle b\rangle \cap \langle c\rangle = \langle c\rangle \cap \langle d\rangle = \langle d\rangle \cap \langle a\rangle = 1$. A_0, B_0, C_0, D_0 are $f.g.$ torsion-free nilpotent groups containing A, B, C, D respectively such that $A_0 \cap B_0 = \langle b\rangle$, $B_0 \cap C_0 = \langle c\rangle$, $C_0 \cap D_0 = \langle d\rangle$ and $D_0 \cap A_0 = \langle a\rangle$.

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LEMMA 3.1. *Let* $\bar{A}_0 = A_0/A_0^{p^k}$, $\bar{B}_0 = B_0/B_0^{p^k}$, $\bar{C}_0 = C_0/C_0^{p^k}$ *and* $\bar{D}_0 = D_0/D_0^{p^k}$. *If* A, B, C, D *are isolated subgroups of* A_0, B_0, C_0, D_0 *respectively then the polygonal product* $\bar{P}_0$ *of* $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$, $\bar{D}_0$ *amalgamating* $\langle\bar{b}\rangle$, $\langle\bar{c}\rangle$, $\langle\bar{d}\rangle$ *and* $\langle\bar{a}\rangle$ *is* RF *for almost all primes* p.

PROOF. CASE 1. Let $A_0 = A$, $B_0 = B$, $C_0 = C$ and $D_0 = D$. Let $\tilde{A} = A/A^n$, $\tilde{B} = B/B^n$, $\tilde{C} = C/C^n$, and $\tilde{D} = D/D^n$. Let Q_n be the polygonal product of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ amalgamating $\langle\tilde{b}\rangle$, $\langle\tilde{c}\rangle$, $\langle\tilde{d}\rangle$ and $\langle\tilde{a}\rangle$. Since A, B, C, D are torsion-free nilpotent groups, we have $\langle\tilde{a}\rangle \cap \langle\tilde{b}\rangle^{\tilde{A}} = \langle\tilde{c}\rangle \cap \langle\tilde{b}\rangle^{\tilde{B}} = 1$ and $\langle\tilde{c}\rangle \cap \langle\tilde{d}\rangle^{\tilde{C}} = \langle\tilde{a}\rangle \cap \langle\tilde{d}\rangle^{\tilde{D}} = 1$. Let $\tilde{E} = \tilde{A} *_{\langle\tilde{b}\rangle} \tilde{B}$ and $\tilde{F} = \tilde{C} *_{\langle\tilde{d}\rangle} \tilde{D}$. Then $\tilde{E} = \langle\tilde{b}\rangle^{\tilde{E}} \cdot \tilde{H}$ and $\tilde{F} = \langle\tilde{d}\rangle^{\tilde{F}} \cdot \tilde{H}$ where $\tilde{H} = \langle\tilde{a}\rangle * \langle\tilde{c}\rangle$. Since $\langle\tilde{b}\rangle^{\tilde{E}} \cap \tilde{H} = \langle\tilde{d}\rangle^{\tilde{F}} \cap \tilde{H} = 1$, $\tilde{H}$ is a retract of both $\tilde{E}$ and $\tilde{F}$. By Theorem 2.1, Q_n is RF.

CASE 2. A_0, B_0, C_0, D_0 not necessarily equal to A, B, C, D respectively. Let p be a prime greater than the nilpotency classes of A_0, B_0, C_0, D_0 respectively; we have $A_0^{p^k} \cap A = A^{p^k}$, $B_0^{p^k} \cap B = B^{p^k}$, $C_0^{p^k} \cap C = C^{p^k}$ and $D_0^{p^k} \cap D = D^{p^k}$. Thus $\bar{A} \approx A/A^{p^k}$, $\bar{B} \approx B/B^{p^k}$, $\bar{C} \approx C/C^{p^k}$ and $\bar{D} \approx D/D^{p^k}$. Hence, by Case 1, the polygonal product $\bar{P}$ of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ amalgamating $\langle\bar{b}\rangle$, $\langle\bar{c}\rangle$, $\langle\bar{d}\rangle$, $\langle\bar{a}\rangle$ is RF. Since $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ are finite, it follows that $\bar{P}_0 \approx \Big(\big((\bar{P} *_{\bar{A}} \bar{A}_0) *_{\bar{B}} \bar{B}_0\big) *_{\bar{C}} \bar{C}_0\Big) *_{\bar{D}} \bar{D}_0$ is RF. ∎

LEMMA 3.2. *Let* G *be a f.g. torsion-free nilpotent group of class* c. *Let* $x, y \in G$ *such that* $\langle x\rangle \cap \langle y\rangle = 1$ *and* $H = \langle x, y\rangle$ *is an isolated subgroup of* G. *If* $g \in G \setminus \langle x\rangle\langle y\rangle$ *then, for every prime* $p > c$, *there exists an integer* n *such that* $g \notin G^{p^n}\langle x\rangle\langle y\rangle$.

PROOF. Since H is isolated, by Theorem 2.2, $\bigcap_{n=1}^{\infty} G^{p^n}H = H$. Thus, if $g \notin H$, then there exists n such that $g \notin G^{p^n}H$. It follows that $g \notin G^{p^n}\langle x\rangle\langle y\rangle$. Hence we can assume $g \in H \setminus \langle x\rangle\langle y\rangle$. If H is of nilpotency class m then $g = zx^iy^j$ where $z \in Z_{m-1}(H)$. Since $\bigcap_{n=1}^{\infty} H^{p^n} = 1$, it follows that there exists n such that $z \notin H^{p^n}$. We shall show that $g = zx^iy^j \notin H^{p^n}\langle x\rangle\langle y\rangle$. Suppose $g \in H^{p^n}\langle x\rangle\langle y\rangle$. Then $g = hx^sy^t$, where $h \in H^{p^n}$. This implies $h = w_1^{p^n} \cdots w_r^{p^n}$ where $w_1, \ldots, w_r \in H$ for some r. Let $\bar{H} = H/Z_{m-1}(H)$. Then $\bar{h} = \bar{x}^{kp^n}\bar{y}^{\ell p^n}$. It follows that $\bar{g} = \bar{x}^i\bar{y}^j = \bar{x}^{kp^n+s}\bar{y}^{\ell p^n+t}$. Since $Z_{m-1}(H) \cap \langle x\rangle\langle y\rangle = 1$, we have $i = kp^n + s$ and $j = \ell p^n + t$. Thus $g = zx^iy^j = hx^sy^t$ implies $z = hx^sy^{t-j}x^{-i} = hx^{i-kp^n}y^{-\ell p^n}x^{-i} = hx^{-kp^n}(x^iy^{-\ell p^n}x^{-i})$. But this implies $z \in H^{p^n}$ contradicting the choice of n. Hence $g \notin H^{p^n}\langle x\rangle\langle y\rangle$. Since $G^{p^n} \cap H = H^{p^n}$ for $p > c$, it follows that $g \notin G^{p^n}\langle x\rangle\langle y\rangle$ for every prime $p > c$. ∎

DEFINITION 3.3. Let $G = G_1 *_H G_2$. Let X, Y be subgroups of G_1, G_2 respectively. Let $\mathcal{N} = \{(N_i, M_i)\ ;\ i \in I\}$ be a collection of pairs of normal subgroups of G_1 and G_2 satisfying the following conditions.

(1) $N_i \triangleleft G_1$, $M_i \triangleleft G_2$ such that $N_i \cap H = M_i \cap H$ for all $i \in I$,
(2) $N_i \cap XH = (N_i \cap X)(N_i \cap H)$ and $M_i \cap YH = (M_i \cap Y)(M_i \cap H)$ for all $i \in I$,
(3) $\big(\bigcap_{j=1}^n N_{\alpha_j}, \bigcap_{j=1}^n M_{\alpha_j}\big) \in \mathcal{N}$ for all $\alpha_1, \ldots, \alpha_n \in I$, where n is finite,
(4) $\bigcap_{i\in I} N_iX = X$, $\bigcap_{i\in I} N_iH = H$, $\bigcap_{i\in I} M_iY = Y$ and $\bigcap_{i\in I} M_iH = H$,
(5) $\bigcap_{i\in I} N_iXH = XH$ and $\bigcap_{i\in I} M_iYH = YH$.

Then $\mathcal{N}$ is called a *compatible filter* of G with respect to the subgroups X and Y.

LEMMA 3.4. *Let $G = G_1 *_H G_2$. Let X, Y be subgroups of G_1, G_2 respectively such that $X \cap H = Y \cap H = 1$. Let $\mathcal{N}$ be a compatible filter of G with respect to X and Y. Then, for each $g \in G \setminus (X * Y)$ with $\|g\| \geq 1$, there exists $(N, M) \in \mathcal{N}$ such that $\|g\pi\| = \|g\|$ and $g\pi \notin X\pi * Y\pi$ where π is the canonical homomorphism of G onto $\bar{G} = \bar{G}_1 *_{\bar{H}} \bar{G}_2$, and where $\bar{G}_1 = G_1/N$, $\bar{G}_2 = G_2/M$ and $\bar{H} = HN/N = HM/M$.*

PROOF. We shall only consider the case $g = u_1v_1 \cdots u_nv_n$, where $u_i \in G_1 \setminus H$, $v_i \in G_2 \setminus H$ (other cases being similar). We first note that $g = x_1y_1 \cdots x_ny_n$ where $x_i \in X$ and $y_i \in Y$ if and only if there exist $h_1, k_1, \ldots, h_{n-1}, k_{n-1}, h_n \in H$ such that $u_1 = x_1h_1$, $v_1 = h_1^{-1}y_1k_1$, $u_2 = k_1^{-1}x_2h_2, \ldots, u_n = k_{n-1}^{-1}x_nh_n$ and $v_n = h_n^{-1}y_n$. Since $g \notin X * Y$, there exist x_i, y_i, h_i, k_i such that the following is true:

(1) $u_1 \notin XH$, or

(1′) $u_1 = x_1h_1$ but $h_1v_1 \notin YH$, or

(2) $u_1 = x_1h_1$, $h_1v_1 = y_1k_1$, but $k_1u_2 \notin XH$, or

(2′) $u_1 = x_1h_1$, $h_1v_1 = y_1k_1$, $k_1u_2 = x_2h_2$, but $h_2v_2 \notin YH$, or

⋮

(n) $u_1 = x_1h_1$, $h_1v_1 = y_1k_1, \ldots, h_{n-1}v_{n-1} = y_{n-1}k_{n-1}$, but $k_{n-1}u_n \notin XH$, or

(n') $u_1 = x_1h_1$, $h_1v_1 = y_1k_1, \ldots, k_{n-1}u_n = x_nh_n$, but $h_nv_n \notin Y$.

Let i be the smallest integer such that (i) (or (i')) is true. Since $X \cap H = Y \cap H = 1$, x_j, y_j, h_j, k_j, are all uniquely determined, by properties (4) and (5) of $\mathcal{N}$, there exist (N_0, M_0) $(N_{\alpha_j}, M_{\alpha_j})$, $(N_{\beta_j}, M_{\beta_j}) \in \mathcal{N}$ such that $k_{i-1}u_i \notin N_0XH$, $u_j \notin N_{\alpha_j}H$ and $v_j \notin M_{\beta_j}H$ for each $j = 1, \ldots, n$. Let $N = N_0 \cap \left(\bigcap_{j=1}^n N_{\alpha_j}\right) \cap \left(\bigcap_{j=1}^n N_{\beta_j}\right)$ and $M = M_0 \cap \left(\bigcap_{j=1}^n M_{\alpha_j}\right) \cap \left(\bigcap_{j=1}^n M_{\beta_j}\right)$. Then, by property (3) of $\mathcal{N}$, $(N, M) \in \mathcal{N}$. Let $\pi: G \longrightarrow \bar{G}$. Clearly $\|g\pi\| = \|g\|$ and, by property (2) of $\mathcal{N}$, $(X * Y)\pi = \bar{X} * \bar{Y}$. If $g\pi \in \bar{X} * \bar{Y}$ then $g\pi = \bar{u}_1\bar{v}_1 \cdots \bar{u}_n\bar{v}_n = \bar{s}_1\bar{t}_1 \cdots \bar{s}_n\bar{t}_n$ where $\bar{s}_i \in \bar{X}$ and $\bar{t}_i \in \bar{Y}$. This implies $\bar{u}_1 = \bar{s}_1\bar{r}_1$, $\bar{v}_1 = \bar{r}_1^{-1}\bar{t}_1\bar{w}_1$, $\bar{u}_2 = \bar{w}_1^{-1}\bar{s}_2\bar{r}_2, \ldots, \bar{u}_i = \bar{w}_{i-1}^{-1}\bar{s}_i\bar{r}_i, \ldots, \bar{v}_n = \bar{r}_n^{-1}\bar{t}_n$ where $\bar{r}_i, \bar{w}_i \in \bar{H}$. Now $\bar{u}_1 = \bar{x}_1\bar{h}_1$, $\bar{h}_1\bar{v}_1 = \bar{y}_1\bar{k}_1, \ldots, \bar{h}_{i-1}\bar{v}_{i-1} = \bar{y}_{i-1}\bar{k}_{i-1}$. Since $\bar{X} \cap \bar{H} = \bar{Y} \cap \bar{H} = 1$, it follows that $\bar{s}_1 = \bar{x}_1$, $\bar{r}_1 = \bar{h}_1$, $\bar{t}_1 = \bar{y}_1, \ldots, \bar{r}_{i-1} = \bar{h}_{i-1}$, $\bar{t}_{i-1} = \bar{y}_{i-1}$ and $\bar{w}_{i-1} = \bar{k}_{i-1}$. This implies $\bar{u}_i = \bar{w}_{i-1}^{-1}\bar{s}_i\bar{r}_i = \bar{k}_{i-1}^{-1}\bar{s}_i\bar{r}_i$. Thus $\bar{k}_{i-1}\bar{u}_i \in \bar{X}\bar{H}$ whence $k_{i-1}u_i \in NXH \subset N_0XH$ contradicting the choice of N_0. Hence $g\pi \notin \bar{X} * \bar{Y}$ as required. ∎

LEMMA 3.5. *Let $E_0 = A_0 *_{\langle b \rangle} B_0$. If $\langle a \rangle$, $\langle b \rangle$ are maximal cyclic subgroups of A_0 and $\langle b \rangle$, $\langle c \rangle$ are maximal cyclic subgroups of B_0 such that $\langle a, b \rangle$ and $\langle b, c \rangle$ are isolated in A_0 and B_0 respectively, then, for p greater than the nilpotency classes of A_0 and B_0, $\mathcal{N}_p = \{(A_0^{p^m}, B_0^{p^m})\ ;\ m = 1, 2, \ldots\}$ is a compatible filter of $E_0 = A_0 *_{\langle b \rangle} B_0$ with respect to $\langle a \rangle$ and $\langle c \rangle$.*

PROOF. Since $\langle a \rangle$, $\langle b \rangle$ and $\langle b \rangle$, $\langle c \rangle$ maximal cyclic subgroups of A_0 and B_0 respectively and p is greater than the nilpotency classes of A_0 and B_0, it follows that conditions (1), (3) and (4) of Definition 3.3 are satisfied. Now $\langle a, b \rangle$ and $\langle b, c \rangle$ are isolated in A_0 and B_0, whence (2) is satisfied. Also, by Lemma 3.2, condition (5) of Definition 3.3 is satisfied. Hence $\mathcal{N}_p$ is a compatible filter of $E_0 = A_0 *_{\langle b \rangle} B_0$ with respect to $\langle a \rangle$ and $\langle c \rangle$. ∎

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THEOREM 3.6. *Let P_0 be the polygonal product of A_0, B_0, C_0, D_0 amalgamating the maximal cyclic subgroups $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ and $\langle a\rangle$. If the subgroups A, B, C, D are isolated in A_0, B_0, C_0, D_0 respectively, then P_0 is* RF.

PROOF. We note that A, B, C, D isolated in A_0, B_0, C_0, D_0 respectively actually implies $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, $\langle a\rangle$ are maximal cyclic subgroups of the groups containing them, whence each of these subgroups is isolated in the groups containing them. It follows that $A_0^{p^m}\cap\langle b\rangle = B_0^{p^m}\cap\langle b\rangle = \langle b^{p^m}\rangle$, $B_0^{p^m}\cap\langle c\rangle = C_0^{p^m}\cap\langle c\rangle = \langle c^{p^m}\rangle$, $C_0^{p^m}\cap\langle d\rangle = D_0^{p^m}\cap\langle d\rangle = \langle d^{p^m}\rangle$, and $D_0^{p^m}\cap\langle a\rangle = A_0^{p^m}\cap\langle a\rangle = \langle a^{p^m}\rangle$ for each prime p greater than the nilpotency classes of A_0, B_0, C_0, D_0. Thus we can form the polygonal product of $A_0/A_0^{p^m}$, $B_0/B_0^{p^m}$, $C_0/C_0^{p^m}$ and $D_0/D_0^{p^m}$ amalgamating the subgroups $\langle b\rangle/\langle b^{p^m}\rangle$, $\langle c\rangle/\langle c^{p^m}\rangle$, $\langle d\rangle/\langle d^{p^m}\rangle$ and $\langle a\rangle/\langle a^{p^m}\rangle$. Let ϕ_{p^m} be the canonical homomorphism of P_0 onto this polygonal product. By Lemma 3.1, $P_0\phi_{p^m}$ is RF. Thus to prove the theorem, we need only find ϕ_{p^m} such that, for a given $1\neq g\in P_0$, $g\phi_{p^m}\neq 1$.

Let $E_0 = A_0 *_{\langle b\rangle} B_0$, $F_0 = C_0 *_{\langle d\rangle} D_0$ and $H = \langle a\rangle * \langle c\rangle$. Then $P_0 = E_0 *_H F_0$. Let t be the maximum of the nilpotency classes of A_0, B_0, C_0, D_0.

CASE 1. $\|g\| = 0$. Then $g\in H$. We shall only consider the case $g = a^{\alpha_1}c^{\beta_1}\cdots a^{\alpha_n}c^{\beta_n}$ with the other cases being similar. Choose $p > \max\{|\alpha_i|, |\beta_i|, t\ ;\ i = 1,\ldots,n\}$. Since $g\phi_p$ has the same free product length in $\langle a\phi_p\rangle * \langle c\phi_p\rangle$ as the free product length of g in $\langle a\rangle * \langle c\rangle$, $g\phi_p\neq 1$.

CASE 2. $\|g\| = 1$. Without loss of generality, we can assume $g\in E_0\setminus H$. If $g = b^k$, then we choose $p > \max(|k|, t)$. Clearly $g\phi_p\neq 1$. Thus we can assume g to be of length ≥ 1 in $E_0 = A_0 *_{\langle b\rangle} B_0$. By Lemma 3.5, $\mathcal{N}_p = \{(A_0^{p^m}, B_0^{p^m})\ ;\ m = 1,2,\ldots\}$ is a compatible filter of E_0 with respect to $\langle a\rangle$ and $\langle c\rangle$ for $p > t$. Thus, by Lemma 3.4, there exists an integer m such that $g\phi_{p^m}\notin H\phi_{p^m}$, whence $g\phi_{p^m}\neq 1$.

CASE 3. $\|g\|\geq 2$. Again we shall only consider the case $g = e_1f_1\cdots e_nf_n$ where $e_i\in E_0\setminus H$ and $f_i\in F_0\setminus H$. As in Case 2, for each i, there exist integers k_i, ℓ_i, $i = 1,\ldots,n$, such that $e_i\phi_{p^{k_i}}\notin H\phi_{p^{k_i}}$ and $f_i\phi_{p^{\ell_i}}\notin H\phi_{p^{\ell_i}}$ for sufficiently large prime p. Let $m = \max\{k_i, \ell_i\ ;\ i = 1,\ldots,n\}$. Then it is clear that $g\phi_{p^m}\neq 1$.

This completes the proof. ∎

4. **Other results.**

In this section, we generalize a result of Allenby and Tang [2]. Applying this result, we prove that the polygonal product P_0 of Theorem 3.6 is RF if A, B, C, D have the same nilpotency classes as A_0, B_0, C_0, D_0 respectively.

LEMMA 4.1. *Let G be a f.g. nilpotent group. Let $x, h\in G$ such that x is of finite order m and h is of infinite order. Then there exists an integer α such that for any integer $t\geq 1$, we can find $N_t \triangleleft_f G$ such that $\langle x\rangle\langle h\rangle\cap N_t = \langle h^{\alpha t}\rangle$.*

PROOF. Since G is $\langle h\rangle$-separable, and since $\langle x\rangle$ is finite, there exists $N_1\triangleleft_f G$ such that $\langle x\rangle\cap N_1\langle h\rangle = 1$. Let $N_1\cap\langle h\rangle = \langle h^\alpha\rangle$. Then $\langle x\rangle\langle h\rangle\cap N_1 = \langle h^\alpha\rangle$. Now let $t\geq 1$. Since $\bar G = G/\tau(G)$ is potent, there exists $M_t\triangleleft_f G$ such that $\langle\bar h\rangle\cap\bar M_t = \langle\bar h^{\alpha t}\rangle$. Then $N_t = M_t\cap N_1$ is the required normal subgroup. ∎

Applying Lemma 4.1, we immediately have the following result:

LEMMA 4.2. *Let G_1, G_2 be f.g. nilpotent groups such that $G_1 \cap G_2 = \langle h \rangle$ where h is of infinite order. Let $G = G_1 *_{\langle h \rangle} G_2$. If $x \in G_1$, $y \in G_2$ are of finite orders, then there exists an integer $\alpha > 0$ such that, for every $t \geq 1$, we can find $N_t \lhd_f G$ and $M_t \lhd_f G$ with $\langle x \rangle \langle h \rangle \cap N_t = \langle y \rangle \langle h \rangle \cap M_t = \langle h^{\alpha t} \rangle$.*

LEMMA 4.3. *Let G_1, G_2 be f.g. nilpotent groups such that $G_1 \cap G_2 = \langle h \rangle$ where h is of infinite order. Let $X = \langle x \rangle$, $H = \langle h \rangle$ and $Y = \langle y \rangle$ where $x \in G_1$ and $y \in G_2$ are of finite orders. If $\mathcal{N}$ is the set of all pairs (N, M) such that $N \lhd_f G_1$, $M \lhd_f G_2$ and $N \cap XH = M \cap YH = \langle h^{\alpha t} \rangle$, where α is determined by Lemma 4.2 and t ranges over the set of all positive integers then $\mathcal{N}$ is a compatible filter of $G = G_1 *_{\langle h \rangle} G_2$ with respect to the subgroups X and Y.*

PROOF. It is not difficult to check that conditions (1), (2) and (3) of Definition 3.3 are satisfied by $\mathcal{N}$. Let x be of order m. Let $g \in G_1 \backslash XH$. Then $x^i g \notin H$ for $i = 0, 1, \ldots, m-1$. Now G_1 is a f.g. nilpotent group. This implies G_1 is subgroup separable. Thus there exists $N_1 \lhd_f G_1$, such that $x^i g \notin N_1 H$ for all i. Let $N_1 \cap H = \langle h^k \rangle$. By Lemma 4.2, there exists $(N', M') \in \mathcal{N}$ such that $XH \cap N' = YH \cap M' = \langle h^{\alpha k} \rangle$. Let $N = N_1 \cap N'$. Then $XH \cap N = \langle h^{\alpha k} \rangle$. This implies $(N, M') \in \mathcal{N}$. Since $N \subset N_1$, it follows that $g \notin NXH$. Hence $\bigcap_{(N,M) \in \mathcal{N}} NXH = XH$. In the same way $\bigcap_{(N,M) \in \mathcal{N}} MYH = YH$. Thus $\mathcal{N}$ satisfies condition (5) of Definition 3.3. By a similar argument we can show that $\mathcal{N}$ satisfies condition (4) of Definition 3.3. Hence $\mathcal{N}$ is a compatible filter of G with respect to X and Y. ∎

We now prove a theorem which generalizes a result of Allenby and Tang (Theorem 4.4 [2]).

THEOREM 4.4. *Let P_0 be the polygonal product of f.g. nilpotent groups A_0, B_0, C_0, D_0 amalgamating $\langle b \rangle$, $\langle c \rangle$, $\langle d \rangle$ and $\langle a \rangle$ where $A_0 \cap B_0 = \langle b \rangle$, $B_0 \cap C_0 = \langle c \rangle$, $C_0 \cap D_0 = \langle d \rangle$ and $D_0 \cap A_0 = \langle a \rangle$ and $\langle a \rangle \cap \langle b \rangle = \langle b \rangle \cap \langle c \rangle = \langle c \rangle \cap \langle d \rangle = \langle d \rangle \cap \langle a \rangle = 1$. If a and c are of prime orders p and q respectively then P_0 is* RF.

PROOF. CASE 1. A_0, B_0, C_0 and D_0 are finite. Let $A = \langle a, b \rangle$, $B = \langle b, c \rangle$, $C = \langle c, d \rangle$ and $D = \langle d, a \rangle$. Let $E = A *_{\langle b \rangle} B$. Since $\langle a \rangle \cap \langle b \rangle^A = \langle c \rangle \cap \langle b \rangle^B = 1$, it follows that $E = \langle b \rangle^E H$ where $H = \langle a \rangle * \langle c \rangle$ and $H \cap \langle b \rangle^E = 1$. Thus H is a retract of E. In the same way, if we let $F = D *_{\langle d \rangle} C$ then H is a retract of F. Since the polygonal product P of A, B, C, D is the same as $E *_H F$, by Theorem 2.1, P is RF. Now A, B, C, D are finite. It follows that $P_0 = \Big(\big((P *_A A_0) *_B B_0 \big) *_C C_0 \Big) *_D D_0 \Big)$ is RF.

CASE 2. $|b| = \infty$, $|d| = m$, where m is finite. Let $\mathcal{N}$ be the compatible filter of $E_0 = A_0 *_{\langle b \rangle} B_0$ with respect to $\langle a \rangle$ and $\langle c \rangle$ as determined by Lemma 4.3. Let $\bar{A}_0 = A_0 / N$ and $\bar{B}_0 = B_0 / M$ where $(N, M) \in \mathcal{N}$. Since $\langle a \rangle \cap N = \langle c \rangle \cap M = 1$, $|\bar{a}| = p$, $|\bar{c}| = q$. Moreover, $\langle b \rangle \cap N = \langle b \rangle \cap M = \langle b^{\alpha t} \rangle$ implies that $\bar{b}$ has the same order in $\bar{A}_0$ and $\bar{B}_0$. Furthermore, $N \cap \langle a \rangle \langle b \rangle = M \cap \langle c \rangle \langle b \rangle = \langle b^{\alpha t} \rangle$ implies $\langle \bar{a} \rangle \cap \langle \bar{b} \rangle = \langle \bar{c} \rangle \cap \langle \bar{b} \rangle = 1$. Since C_0 and D_0 are RF, there exist $L \lhd_f C_0$ and $K \lhd_f D_0$ such that $L \cap \langle c \rangle \langle d \rangle = K \cap \langle a \rangle \langle d \rangle = 1$. Let $\bar{C}_0 = C_0 / L$ and $\bar{D}_0 = D_0 / K$. Then $|\bar{a}| = p$, $|\bar{d}| = m$ in $\bar{D}_0$ and $|\bar{c}| = q$, $|\bar{d}| = m$ in $\bar{C}_0$. Moreover, $\langle \bar{a} \rangle \cap \langle \bar{d} \rangle = \langle \bar{c} \rangle \cap \langle \bar{d} \rangle = 1$. Thus we can form the polygonal product $\bar{P}_0$ of $\bar{A}_0$,

$\bar{B}_0$, $\bar{C}_0$ and $\bar{D}_0$ amalgamating $\langle\bar{b}\rangle$, $\langle\bar{c}\rangle$, $\langle\bar{d}\rangle$ and $\langle\bar{a}\rangle$. Let ϕ be the canonical homomorphism of P_0 onto $\bar{P}_0$. By Case 1, $\bar{P}_0$ is RF. Thus, if, for each $1 \neq g \in P_0$, there exists a ϕ such that $g\phi \neq 1$ then $P_0 = E_0 *_H F_0$ is RF where $F_0 = D_0 *_{\langle d\rangle} C_0$.

Subcase (i). $g \in H$. Since C_0, D_0 are RF and a, c, d are all of finite order, there exist $L \triangleleft_f C_0$ and $K \triangleleft_f D_0$ such that $L \cap \langle c\rangle\langle d\rangle = K \cap \langle a\rangle\langle d\rangle = 1$. Let $(N,M) \in \mathcal{N}$. Then clearly if we let $\bar{A}_0 = A_0/N$, $\bar{B}_0 = B_0/M$, $\bar{C}_0 = C_0/L$, $\bar{D}_0 = D_0/K$ and let ϕ be the canonical homomorphism described above, we have $g\phi \neq 1$.

Subcase (ii). $g \in E_0 \setminus H$. If $g = b^k$ then, by Lemma 4.2, there exists $(N,M) \in \mathcal{N}$ such that $N \cap \langle b\rangle\langle a\rangle = M \cap \langle b\rangle\langle c\rangle = \langle b^{\alpha t}\rangle$ where $\alpha t > |k|$. Let L, K and ϕ be as defined in Subcase (i), then $g\phi = \bar{b}^k \notin \bar{H}$, whence $g\phi \neq 1$. If g is of length ≥ 1 in $E_0 = A_0 *_{\langle b\rangle} B_0$ then, by Lemma 3.4, there exists $(N,M) \in \mathcal{N}$ such that $g\pi \notin H\pi$ where π is the canonical homomorphism of $E_0 = A_0 *_{\langle b\rangle} B_0$ onto $\bar{A}_0 *_{\langle\bar{b}\rangle} \bar{B}_0$ where $\bar{A}_0 = A_0/N$ and $\bar{B}_0 = B_0/M$. Let L, K be as in Subcase (i). Then N, M, L, K define the required ϕ of P_0 onto $\bar{P}_0$. Moreover $g\pi \notin H\pi$ implies $g\phi \notin H\phi$ whence $g\phi \neq 1$.

Subcase (iii). $g \in F_0 \setminus H$. Since C_0, D_0 are subgroup separable and $\langle d\rangle$ is finite, F_0 is subgroup separable [1]. Thus there exists $R \triangleleft_f F_0$ such that $g \notin$ RH. Let L, K be as in Subcase (i). Let $L_1 = R \cap L$ and $K_1 = R \cap K$. Then $L_1 \cap \langle c\rangle\langle d\rangle = K_1 \cap \langle a\rangle\langle d\rangle = 1$. Let $(N,M) \in \mathcal{N}$. Let ϕ be the canonical homomorphism of P_0 onto the polygonal product of A_0/N, B_0/M, C_0/L_1 and D_0/K_1. Since $\langle L_1, K_1\rangle^{F_0} \subseteq R$ and $g \notin$ RH, it follows that $g\phi \notin H\phi$, whence $g\phi \neq 1$.

Subcase (iv). $g \notin E_0 \cup F_0$. We shall only consider the case $g = e_1 f_1 \cdots e_n f_n$ where $e_i \in E_0 \setminus H$ and $f_i \in F_0 \setminus H$ (other cases being similar). As in Subcase (ii), for each e_i, there exists $(N_i, M_i) \in \mathcal{N}$ such that $e_i\pi_i \notin H_{\pi_i}$ where π_i is the canonical homomorphism of $A_0 *_{\langle b\rangle} B_0$ onto $\tilde{A}_0 *_{\langle\tilde{b}\rangle} \tilde{B}_0$ where $\tilde{A}_0 = A_0/N_i$, $\tilde{B}_0 = B_0/M_i$ and $\langle\tilde{b}\rangle = N_i\langle b\rangle/N_i = M_i\langle b\rangle/M_i$. As in Subcase (iii), for each f_i, there exist $L_i \triangleleft_f C_0$ and $K_i \triangleleft_f D_0$ such that $L_i \cap \langle c\rangle\langle d\rangle = K_i \cap \langle a\rangle\langle d\rangle = 1$ and $f_i\theta_i \notin H\theta_i$ where θ_i is the canonical homomorphism of $D_0 *_{\langle d\rangle} C_0$ onto $\tilde{D}_0 *_{\langle\tilde{d}\rangle} \tilde{C}_0$ where $\tilde{C}_0 = C_0/L_i$, $\tilde{D}_0 = D_0/K_i$ and $\langle\tilde{d}\rangle = L_i\langle d\rangle/L_i = K_i\langle d\rangle/K_i$. Let $L_0 = \bigcap_{i=1}^n L_i$ and $K_0 = \bigcap_{i=1}^n K_i$. Then $L_0 \triangleleft_f C_0$, $K_0 \triangleleft_f D_0$ and $L_0 \cap \langle c\rangle\langle d\rangle = K_0 \cap \langle a\rangle\langle d\rangle = 1$. Moreover if θ is the canonical homomorphism of $D_0 *_{\langle d\rangle} C_0$ onto $\bar{D}_0 *_{\langle\bar{d}\rangle} \bar{C}_0$ where $\bar{C}_0 = C_0/L_0$, $\bar{D}_0 = D_0/K_0$ and $\langle\bar{d}\rangle = K_0\langle d\rangle/K_0 = L_0\langle d\rangle/L_0$ then $f_i\theta \notin H\theta$ for all i. Let $N = \bigcap_{i=1}^n N_i$ and $M = \bigcap_{i=1}^n M_i$. Then $(N,M) \in \mathcal{N}$. Let ϕ be the canonical homomorphism of P_0 onto the polygonal product of A_0/N, B_0/M, C_0/L_0 and D_0/K_0. Then $\|g\phi\|$ in $\bar{E}_0 *_{\bar{H}} \bar{F}_0$ is the same as $\|g\|$ in $E_0 *_H F_0$, where $\bar{E}_0 = A_0/N *_{N\langle b\rangle/N} B_0/M$, $\bar{F}_0 = F_0\theta$ and $\bar{H} = H\theta$. Thus $g\phi \neq 1$.

The remaining cases are:

CASE 3. $|b| < \infty$, $|d| = \infty$.

CASE 4. $|b| < \infty$, $|d| < \infty$.

CASE 5. $|b| = |d| = \infty$.

By suitable modification of the proof of Case 2, we can show that for each of the Cases 3, 4, 5 we can construct the required ϕ such that $g\phi \neq 1$ for every $1 \neq g \in G$. This completes the proof. ∎

We need the following lemma to prove our next result.

LEMMA 4.5. *Let G be a f.g. torsion-free nilpotent group. Let $a, b \in G$ such that $\langle a\rangle \cap \langle b\rangle = 1$. If Δ is an infinite set of primes and if $H = \langle a, b\rangle$ has the same nilpotency class as G then we have:*

(1) $$\bigcap_{p\in\Delta} \langle x^p\rangle^G \langle x\rangle = \langle x\rangle \text{ for every } x \in G\,;$$

(2) $$\bigcap_{p\in\Delta} \langle a^p\rangle^G \langle a\rangle\langle b\rangle = \langle a\rangle\langle b\rangle\,;$$

(3) $$\bigcap_{p\in\Delta} \langle a^p\rangle^G \langle b\rangle = \langle b\rangle.$$

PROOF. We first note that, by [7], if p is a prime greater than the nilpotency class of G then for any $x, y \in G$, $x^p y^p = w^p$ for some $w \in G$. Moreover $\bigcap_{p\in\Delta} G^p = 1$.

If G is abelian, then the lemma is trivial. So we can assume G and H are of nilpotency class $c > 1$.

(1) If $x \in Z(G)$ then (1) is obviously true. Therefore, let $x \in Z_{i+1}(G)\backslash Z_i(G)$, $1 \le i < c$. If $y \in \langle x^p\rangle^G\langle x\rangle$ then $y = g_1^{-1}x^{k_1p}g_1\cdots g_n^{-1}x^{k_np}g_nx^{i_p}$, for some $g_\ell \in G$ and integers k_ℓ, i_p. Clearly $[g, x] \equiv 1 \bmod Z_i$ for $g \in G$. Let $\bar{G} = G/Z_i(G)$. Then $\bar{y} = \bar{x}^{m_pp+i_p}$ where $m_p = k_1 + \cdots + k_n$. Since $\bar{G}$ is a $f.g.$ torsion-free nilpotent group, $m_pp + i_p$ must be a fixed integer α for each p. This implies $y = zx^\alpha = zx^{m_pp+i_p}$ where $z \in Z_i(G)$. If $p > c$ then $y = g_1^{-1}x^{k_1p}g_1\cdots g_n^{-1}x^{k_np}g_nx^{i_p} = w^px^{i_p}$ for some $w \in G$. Thus $z = w^px^{-m_pp} = u^p$ for some $u \in G$. It follows that if $y \in \langle x^p\rangle^G\langle x\rangle$ then $yx^{-\alpha} = z = u^p \in G^p$ for each $p > c$. Therefore, if $y \in \bigcap_{p\in\Delta}\langle x^p\rangle^G\langle x\rangle$ then $y \in \bigcap_{p\in\Delta, p>c}\langle x^p\rangle^G\langle x\rangle$. This implies $yx^{-\alpha} \in \bigcap_{p\in\Delta, p>c} G^p$. Since $\bigcap_{p\in\Delta, p>c} G^p = 1$, it follows that $y = x^\alpha \in \langle x\rangle$. This proves (1).

(2) Suppose $y \in \langle a^p\rangle^G\langle a\rangle\langle b\rangle$ then $y = g_1^{-1}a^{k_1p}g_1\cdots g_n^{-1}a^{k_np}g_na^{i_p}b^{j_p}$ for some $g_\ell \in G$ and integers k_ℓ, i_p and j_p. Let $\bar{G} = G/Z_{c-1}(G)$. Then $\bar{y} = \bar{a}^{m_pp+i_p}\bar{b}^{j_p}$ where $m_p = k_1 + \cdots + k_n$. If $y \in \langle a^q\rangle^G\langle a\rangle\langle b\rangle$ then $\bar{y} = \bar{a}^{m_qq+i_q}\bar{b}^{j_q}$ for some integers m_q, i_q and j_q. This implies $a^{m_pp+i_p-m_qq-i_q}b^{j_p-j_q} \in Z_{c-1}(G)\cap H \subseteq Z_{c-1}(H)$. Since H is of nilpotency class c, $Z_{c-1}(H) \cap \langle a\rangle\langle b\rangle = 1$. This implies $m_pp + i_p = m_qq + i_q$ and $j_p = j_q$. Thus, for each p, if $y \in \langle a^p\rangle^G\langle a\rangle\langle b\rangle$, then j_p is a fixed integer, say, α. Therefore, if $y \in \bigcap_{p\in\Delta}\langle a^p\rangle^G\langle a\rangle\langle b\rangle$, by (1), $yb^{-\alpha} \in \bigcap_{p\in\Delta}\langle a^p\rangle^G\langle a\rangle = \langle a\rangle$. Hence $y \in \langle a\rangle\langle b\rangle$ proving (2).

The proof of (3) is similar to (2). ∎

We are now ready to prove the following theorem.

THEOREM 4.6. *Let P_0 be the polygonal product of the f.g. torsion-free nilpotent groups A_0, B_0, C_0, D_0 amalgamating $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ and $\langle a\rangle$ where $A_0\cap B_0 = \langle b\rangle$, $B_0\cap C_0 = \langle c\rangle$, $C_0\cap D_0 = \langle d\rangle$, $D_0\cap A_0 = \langle a\rangle$ and $\langle a\rangle\cap\langle b\rangle = \langle b\rangle\cap\langle c\rangle = \langle c\rangle\cap\langle d\rangle = \langle d\rangle\cap\langle a\rangle = 1$. If $A = \langle a, b\rangle$, $B = \langle b, c\rangle$, $C = \langle c, d\rangle$ and $D = \langle d, a\rangle$ have the same nilpotency class as A_0, B_0, C_0 and D_0 respectively then P_0 is* RF.

PROOF. Let $\bar{A}_0 = A_0/\langle a^p\rangle^{A_0}$, $\bar{B}_0 = B_0/\langle c^p\rangle^{B_0}$, $\bar{C}_0 = C_0/\langle c^p\rangle^{C_0}$ and $\bar{D}_0 = D_0/\langle a^p\rangle^{D_0}$. Since A_0, B_0, C_0 and D_0 are $f.g.$ torsion-free nilpotent groups, it follows that $|\bar{a}| = |\bar{c}| = p$ and $|\bar{b}| = |\bar{d}| = \infty$ for every prime p. Thus we can form the polygonal product $\bar{P}_0$ of $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$, and $\bar{D}_0$ amalgamating $\langle\bar{b}\rangle$, $\langle\bar{c}\rangle$, $\langle\bar{d}\rangle$ and $\langle\bar{a}\rangle$. Let ϕ_p be the canonical homomorphism of P_0 to $\bar{P}_0$. By Theorem 4.4, $\bar{P}_0$ is RF. Thus, if, for each $1 \neq g \in P_0$, we can find a prime p such that $g\phi_p \neq 1$ then we have proved P_0 is RF. As before, we let $E_0 = A_0 *_{\langle b\rangle} B_0$, $F_0 = D_0 *_{\langle d\rangle} C_0$ and $H = \langle a\rangle * \langle c\rangle$. Then $P_0 = E_0 *_H F_0$. Let $1 \neq g \in P_0 = E_0 *_H F_0$.

CASE 1. $\|g\| = 0$. This implies $g \in H$. We shall only consider the case $g = a^{\alpha_1}c^{\beta_1}\cdots a^{\alpha_n}c^{\beta_n}$, other cases being similar. Let p be a prime such that $p > |\alpha_i|, |\beta_i|$, $i = 1,\ldots,n$. Then $g\phi_p$ has the same free product length in $\langle\bar{a}\rangle * \langle\bar{c}\rangle$ as g in $\langle a\rangle * \langle c\rangle$. This implies $g\phi_p \neq 1$ as required.

CASE 2. $\|g\| = 1$. Without loss of generality we can assume $g \in E_0 \setminus H$. If $g = b^k$ then $g\phi_p = \bar{b}^k \notin H\phi_p$ for all primes p, whence $g\phi_p \neq 1$ as required. We need only consider the case $g = u_1v_1\cdots u_nv_n$ where $u_i \in A_0 \setminus \langle b\rangle$ and $v_i \in B_0 \setminus \langle b\rangle$, other cases being similar. Since $g \notin H$, as in the proof of Lemma 3.4, there exists a^{r_i}, b^{s_i}, c^{ℓ_i}, b^{t_i} such that one of the following is true:

(1) $u_1 \notin \langle a\rangle\langle b\rangle$, or

(1′) $u_1 = a^{r_1}b^{s_1}$, but $b^{s_1}v_1 \notin \langle c\rangle\langle b\rangle$, or

(2) $u_1 = a^{r_1}b^{s_1}$, $b^{s_1}v_1 = c^{\ell_1}b^{t_1}$, but $b^{t_1}u_2 \notin \langle a\rangle\langle b\rangle$, or

⋮

(n) $u_1 = a^{r_1}b^{s_1}$, $b^{s_1}v_1 = c^{\ell_1}b^{t_1},\ldots, b^{s_{n-1}}v_{n-1} = c^{\ell_{n-1}}b^{t_{n-1}}$, but $b^{t_{n-1}}u_n \notin \langle a\rangle\langle b\rangle$, or

(n') $u_1 = a^{r_1}b^{s_1}$, $b^{s_1}v_1 = c^{\ell_1}b^{t_1},\ldots, b^{t_{n-1}}u_n = a^{r_n}b^{s_n}$, but $b^{s_n}v_n \notin \langle c\rangle$.

Let i be the smallest integer such that (i) (or (i')) is true. Then, by Lemma 4.5, for almost all primes p, $b^{t_{i-1}}u_i \notin \langle a^p\rangle^{A_0}\langle a\rangle\langle b\rangle$, $u_j \notin \langle a^p\rangle^{A_0}\langle b\rangle$ and $v_j \notin \langle c^p\rangle^{B_0}\langle b\rangle$ for $j = 1,\ldots,n$. Since $\langle\bar{a}\rangle\cap\langle\bar{b}\rangle = \langle\bar{b}\rangle\cap\langle\bar{c}\rangle = 1$, as in the proof of Lemma 3.4, we can show that, for almost all p, $g\phi_p \notin H\phi_p$, whence $g\phi_g \neq 1$ as required.

CASE 3. $\|g\| \geq 2$. Again we only consider the case $g = e_1f_1\cdots e_nf_n$ where $e_i \in E_0 \setminus H$ and $f_i \in F_0 \setminus H$, other cases being similar. By Case 2, we find a sufficiently large prime p such that $e_i\phi_p \notin H\phi_p$ and $f_i\phi_p \notin H\phi_p$ for $i = 1,\ldots,n$. This implies $g\phi_p \neq 1$ as required.

This completes the proof. ∎

REMARK. Theorem 3.4 [2] follows immediately from Theorem 4.6.

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