ON EXISTENCE THEOREMS FOR
DIFFERENTIAL EQUATIONS IN BANACH SPACES

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In this paper we show that a number of existence theorems for the
Cauchy problem of ordinary differential equations in Banach
spaces are only apparent generalizations of the previous ones.

1. Introduction

In the last years there have appeared a lot of papers containing
theorems on the existence of solutions of ordinary differential equations
in Banach spaces (see, for example, [7], [10], [11], [13]). In most of
those theorems it has been assumed that the right hand side of a
differential equation

(1) $$x' = f(t, x) \ , \quad x(0) = x_0 \ ,$$

is a uniformly continuous function in a Banach space and satisfies a
comparison condition of Kamke type translated in terms of a so-called
measure of noncompactness. Such conditions have the form

(2) $$\mu\big(f(t, X)\big) \le \omega\big(t, \mu(X)\big)$$

being a natural translation of the comparison condition of the type

$$\|f(t, x)-f(t, y)\| \le \omega(t, \|x-y\|) \ .$$

The first theorem of such type (involving the function $\omega(t, u) = ku$) has been proved by Ambrosetti [1]. Its later generalization has been given by Geobel and Rzymowski [8], Sadovskii [13], Szufla [15], Deimling [6, 7] and so on. It is worth while to note that the mentioned generalizations depend mostly on the assumption of a more general Kamke function $\omega(t, u)$. For instance, Sadovskii has assumed a comparison function of Perron type [13] and Szufla that of Coddington and Levinson type [15]. Conditions guaranteeing the existence of only trivial solutions both of a differential equation $u' = \omega(t, u)$ and of an integral inequality

$$u(t) \leq \int_0^t \omega(s, u(s))ds$$ have been considered. Also some more general

classes of comparison functions have been investigated [6, 7].

The main aim of this paper is to show that the generalizations of existence theorems for (1) in the direction of assuming more general comparison functions are only apparent generalizations, because all such theorems are equivalent to that one involving a comparison function of Perron and even Bompiani type (see [16]). This situation is similar to the result of Olech [12] who proved that if the right hand side of the equation (1) is continuous then the comparison conditions of Kamke or Coddington and Levinson type are only apparently more general than that of Perron type.

2. Notation and definitions

Let E be a real Banach space with the norm $\|\cdot\|$ and the zero element Θ . In this paper we will use the notation from the paper [2]. For example, the symbol $K(x, r)$ will denote the closed ball centered at x and radius r and by $K(X, r)$ we will denote the ball centered at a set X and with radius r ; that is

$$K(X, r) = \bigcup_{x \in X} K(x, r) .$$

The closure of a set X , its diameter and the distance between a point x and X are denoted by $\overline{X}$, diam X , dist(x, X) , respectively. The symbol Conv X denotes the closed convex hull of X . For a bounded set X we denote by $\|X\|$ its norm, $\|X\| = \sup[\|x\| : x \in X]$. For subsets of the space E we define the following algebraic operations

$$X + Y = [x+y : x \in X, y \in Y] ,$$

$$aX = [ax : x \in X] \ , \quad a \in R \ .$$

Further, denote by M_E (or shortly M) the family of all nonempty and bounded subsets of E and by N its subfamily consisting of all relatively compact sets. For $X, Y \in M$ let us put

$$d(X, Y) = \inf[r : X \subset K(Y, r)] \ ,$$

$$D(X, Y) = \max[d(X, Y), d(Y, X)] \ .$$

The number $D(X, Y)$ is called the Hausdorff distance between X and Y. For its properties we refer to [9].

In what follows let $X \in M_E$. By a measure of noncompactness of a set X we will understand the number $\mu(X)$ defined in the following way:

$$\mu(X) = \inf[r > 0 : X \text{ has a finite } r\text{-net in } E] \ .$$

Let us notice that $\mu : M_E \to R_+ \ \big(R_+ = \langle 0, +\infty)\big)$ and is the so-called Hausdorff or ball measure of noncompactness [2]. There exist various definitions of a measure of noncompactness [2], [13], but the above seems to be the most convenient for calculations and is very often used.

Now we quote a few properties of μ which will be used in the sequel:

1° $\mu(X) = 0 \iff X \in N_E$,

2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,

3° $\mu(aX) = |a|\mu(X) \ , \quad a \in R$,

4° $\mu(X+Y) \leq \mu(X) + \mu(Y)$,

5° $\mu(\text{Conv } X) = \mu(X)$,

6° $\mu(X \cup Y) = \max[\mu(X), \mu(Y)]$,

7° $\mu\big(K(x, r)\big) = r$,

8° $|\mu(X)-\mu(Y)| \leq D(X, Y)$.

The proofs may be found in [2], [13], for example.

3. Comparison functions of Kamke type

In the literature one can meet a lot of conditions guaranteeing the uniqueness of solutions of the Cauchy problem (1). Most of these

conditions have the "comparison" form [16], that is, one requiring a comparison problem with respect to (1) having a solution with specific properties. In the sequel we will discuss such conditions.

Let us denote by T a fixed positive real number and let $J = \langle 0, T \rangle$, $J_0 = (0, T)$. Denote by Ω an arbitrary open subset of a given Banach space E and fix $x_0 \in \Omega$. Let $f : J \times \Omega \to E$ be an arbitrary function.

DEFINITION. A function $\omega : J \times R_+ \to R_+$ $\big($ or $\omega : J_0 \times R_+ \to R_+ \big)$ will be called a Kamke comparison function provided the inequality

$$\|f(t, x)-f(t, y)\| \le \omega(t, \|x-y\|) , \quad x, y \in \Omega , \quad t \in J \quad \big(\text{or} \quad t \in J_0 \big),$$

together with some additional assumptions concerning the function ω , guarantees that the problem (1) has at most one solution.

Further we quote some frequently used classes of Kamke comparison functions.

Class A . This class contains all functions $\omega(t, u) = \omega : J \times R_+ \to R_+$ continuous and nondecreasing with respect to u , having the property $\omega(t, 0) \equiv 0$ and such that $u(t) \equiv 0$ is the only continuous function on J satisfying the inequality

$$u(t) \le \int_0^t \omega\big(s, u(s)\big)ds , \quad u(0) = 0 .$$

Class B . A continuous function $\omega : J \times R_+ \to R_+$ belongs to the class B provided it is nondecreasing with respect to the second variable, $\omega(t, 0) \equiv 0$ and the only differentiable function on J which satisfies on J the equation $u' = \omega(t, u)$ and the condition $u(0) = 0$, is $u(t) \equiv 0$.

Class C . The function $\omega : J_0 \times R_+ \to R_+$ belongs to the class C if for any $\varepsilon > 0$ there exists $\delta > 0$ and a sequence $t_i \to 0+$, $t_i > 0$ and a sequence of functions $\rho_i : \langle t_i, T \rangle \to R_+$ which are differentiable on $\langle t_i, T \rangle$ and satisfy the inequalities $\rho_i' > \omega\big(t, \rho_i\big)$, $\rho_i(t) \ge \delta t_i$, $0 < \rho_i(t) \le \varepsilon$ for $t \in \langle t_i, T \rangle$.

Let us notice that class B was first investigated by Bompiani [4]

and its generalization was given by Perron. A more general class than
those given by Bompiani and Perron was given in 1930 by Kamke. Next,
Coddington and Levinson have given their class of comparison functions [5]
being more general than the previous ones. It is worth while to mention
that the most general class of Kamke comparison functions considered in the
mathematical literature is the class C [16]. For other details and
references concerning Kamke classes we refer to the monograph [16] (see
also [3]).

In what follows we give two theorems which will be used in the sequel.

THEOREM 1. $A = B$.

We omit the easy proof (see [16]).

In view of the above theorem we can use classes A and B inter-
changeably.

THEOREM 2. *Let us assume that* $\omega_1 : J \times R_+ \to R_+$ *is continuous,
nondecreasing with respect to the second variable and* $\omega_1(t, 0) \equiv 0$.
Further let there exist a function $\omega \in C$ *such that* $\omega_1(t, u) \leq \omega(t, u)$,
$(t, u) \in J_0 \times R_+$. *Then* $\omega_1 \in A$.

Proof. Suppose, on the contrary, that $\omega_1 \notin A$ which means, in view
of Theorem 1, that there exists a solution of the equation $u' = \omega_1(t, u)$,
$u(0) = 0$ and such that $u(T) = \varepsilon > 0$. Actually this solution must
satisfy the condition $u(t) = o(t)$ as $t \to 0+$. On the other hand there
exists $\delta > 0$, a sequence $t_i \to 0$, $t_i > 0$ and a sequence of functions
$\rho_i : \langle t_i, T \rangle \to R_+$ which are differentiable on $\langle t_i, T \rangle$ and such that
$\rho_i' > \omega(t, \rho_i)$, $0 < \rho_i(t) \leq \varepsilon$, $\rho_i(t_i) \geq \delta t_i$ for $t \in \langle t_i, T \rangle$. Hence
and with respect to the appropriate theorem on differential inequalities
(see, for example, [14], Lemma 6.3, p. 18) it follows that $\rho_i(t) \leq u(t)$
for $t \in \langle t_i, T \rangle$, so that $\rho_i(t_i) = o(t_i)$ as i tends to infinity. This
contradicts the assumption about $\rho_i(t)$ and completes the proof.

4.   Relations among some existence theorems

For a given Banach space E denote by $C = C(J, E)$ $(J = \langle 0, T \rangle)$ the space of all continuous functions mapping the interval J into the space E with the usual maximum norm. Further, let μ be a measure of non-compactness in the space E . For $X \in M_C$ we will write

$$X(t) = [x(t) : x \in X] .$$

We recall the following lemma [2].

LEMMA 1. $D\big(X(t), X(s)\big) \leq \delta(X, |t-s|)$ *for any* $t, s \in J$.

Here $\delta(X, \varepsilon)$ denotes the modulus of continuity of the set X .

In what follows we consider the differential equation

$$(3) \qquad\qquad\qquad x' = f(t, x)$$

with the initial condition

$$(4) \qquad\qquad\qquad x(0) = x_0 .$$

We will assume that f is defined and uniformly continuous on the set $J \times K\big(x_0, r\big)$ and take values in E . Furthermore, let $\omega(t, u)$ be a given Kamke comparison function belonging to C . We assume that f satisfies the following comparison condition

$$(5) \qquad\qquad\qquad \mu\big(f(t, x)\big) \leq \omega\big(t, \mu(X)\big)$$

for every $X \in \dot{M}_E$ and for almost all $t \in J$.

Consider the function $\bar{\omega} : J \times R_+ \to R_+$ defined by the formula

$$\bar{\omega}(t, u) = \sup\big[\mu\big(f(t, X)\big) : \mu(X) = u, X \subset K\big(x_0, r\big)\big] .$$

Observe that

$$(6) \qquad\qquad\qquad \bar{\omega}(t, u) \leq \omega(t, u)$$

which follows directly from the definition of $\bar{\omega}(t, u)$ and (5). Next notice that monotonicity of μ and its continuity with respect to the Hausdorff distance (see Section 2) imply that $\bar{\omega}(t, u)$ is nondecreasing with respect to u .

Further, let us take $u, u_0 \in R_+$ such that $0 < u_0 < u \leq r = \mu\big(K(x_0, r)\big)$. Assume that $t \in J$ is fixed. Take an

arbitrarily small number $\varepsilon > 0$ and let $K \subset K(x_0, r)$ be convex and such that $\mu(X) = u$ and $\bar{\omega}(t, u) \leq \mu(f(t, X)) + \varepsilon$. Without loss of generality we may assume that $\Theta \in X$. Consider the set $Y = (u_0/u)X$. Actually Y is convex, $Y \subset X$ and $\mu(Y) = u_0$. Moreover we have

$$D(X, Y) \leq \frac{u-u_0}{u} \|X\| .$$

Hence we have

$$\bar{\omega}(t, u) - \bar{\omega}(t, u_0) \leq \mu(f(t, X)) - \mu(f(t, Y)) + \varepsilon \leq D(f(t, X), f(t, Y)) + \varepsilon$$

which in view of Lemma 1 gives

$$\bar{\omega}(t, u) - \bar{\omega}(t, u_0) \leq \delta\left(f\left(t, \frac{u-u_0}{u} \|X\|\right)\right) + \varepsilon ,$$

where we have denoted by $\delta(f(t, m))$ the modulus of continuity of the function $f(t, x)$ with respect to x . Finally, since ε was chosen arbitrarily we have

$$\bar{\omega}(t, u) - \bar{\omega}(t, u_0) \leq \delta\left(f\left(t, r\left(1 - \frac{u_0}{u}\right)\right)\right) .$$

In the same way we may carry over the proof of the following inequality

$$|\bar{\omega}(t_1, u) - \bar{\omega}(t_2, u)| \leq \delta(|t_1 - t_2|)$$

where $\delta(m)$ denotes the modulus of continuity of the function $f(t, x)$ on the set $J \times K(x_0, r)$.

Now we may deduce the following theorem.

THEOREM 3. *The function* $\bar{\omega}(t, u)$ *belongs to the class* A .

Proof. Up to now we have shown that $\bar{\omega}$ is nondecreasing with respect to the second variable and continuous on the set $J \times (0, +\infty)$. In virtue of (6) and Theorem 2 it suffices to show that $\bar{\omega}(t, 0) = 0$ for $t \in J$ and $\bar{\omega}(t, u)$ is continuous for $u = 0$. The first fact follows from the definition of $\bar{\omega}(t, u)$ and the property saying that the image of a compact set under a uniformly continuous function is a compact set. In order to prove the second assertion let us take $u > 0$, $\varepsilon > 0$ and a set X such that $\mu(X) = u$ and $\bar{\omega}(t, u) \leq \mu(f(t, X)) + \varepsilon$. In view of the definition

of a measure μ we can find a finite set $\{x_1,\ x_2,\ \ldots,\ x_n\} \subset E$ such that

$X \subset \bigcup\limits_{i=1}^{n} K(x_i,\ u+\varepsilon)$. Thus, since f is uniformly continuous we obtain

$$\bar{\omega}(t,\ u) \leq \mu\left\{f\left[t,\ \bigcup\limits_{i=1}^{n} K(x_i,\ u+\varepsilon)\right]\right\} + \varepsilon \leq \mu\left(\bigcup\limits_{i=1}^{n} K(f(t,\ x_i),\ \delta(f(t,\ u+\varepsilon)))+\varepsilon\right)$$

$$\leq \delta(f(t,\ u+\varepsilon)) + \varepsilon$$

which implies finally

$$\bar{\omega}(t,\ u) \leq \delta(f(t,\ u))\ .$$

Obviously the desired continuity of the function $\bar{\omega}(t,\ u)$ for $u = 0$
follows from the above inequality which completes the proof.

The theorem proved above shows that a lot of existence theorems for
the Cauchy problem (3) and (4) in a Banach space are no more general than
that given by Sadovskii [13], for instance. Moreover we are able to
formulate an existence theorem being formally stronger than all those given
up to now and involving an assumption of type [5].

THEOREM 4. *Let* $f : J \times K(x_0,\ r) \to E$ *be a uniformly continuous
function, bounded by a constant A and let the condition* (5) *be satisfied,
where* $\omega(t,\ u)$ *is a Kamke comparison function from the class C . If
AT ≤ r then there exists at least one solution of equation* (3) *satisfying
condition* (4) *defined for* $t \in J$ *.*

The proof follows immediately from Theorem 3 and the earlier proved
existence theorems.

Finally let us remark that Theorems 3 and 4 are also true in a more
general situation when we consider measures of noncompactness defined in an
axiomatic way. The detailed considerations will appear elsewhere.

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