THE PERIOD OF THE CHANDLER WOBBLE

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Realistic models of the Earth are known to possess a solid anelastic inner core, mantle and crust, and a fluid core and oceans. How might we go about calculating the theoretical free period of the Chandler wobble of such an Earth model? Let x_i be a set of Cartesian axes with an origin at the center of mass, and let ω_i be the instantaneous angular velocity of rotation of these axes with respect to inertial space. The net angular momentum is then $C_{ij}\omega_j + h_i$, where C_{ij} is the inertia tensor, and h_i is the relative angular momentum. Let us affix the axes x_i in the mantle and crust by stipulating that the relative angular momentum is that of the core and oceans alone, i.e., h_i (mantle and crust) = 0; $h_i = h_i$ (core and oceans). For an infinitesimal free oscillation of angular frequency σ, we can write $\omega_i = \Omega(\delta_{i3} + m_i\, e^{i\sigma t})$, $C_{ij} = A(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) + C\delta_{i3}\delta_{j3} + c_{ij}\, e^{i\sigma t}$, and $h_i = h_i\, e^{i\sigma t}$, where Ω is the mean rate of rotation and A and C are the mean equatorial and polar moments of inertia. Correct to first order, any such oscillation is governed by the well-known Liouville equations

$$
\begin{aligned}
i\sigma m_1 + A^{-1}(C-A)\Omega m_2 &= -A^{-1}[i\sigma(c_{13} + \Omega^{-1}h_1) - \Omega(c_{23} + \Omega^{-1}h_2)] \quad , \\
i\sigma m_2 - A^{-1}(C-A)\Omega m_1 &= -A^{-1}[i\sigma(c_{23} + \Omega^{-1}h_2) + \Omega(c_{13} + \Omega^{-1}h_1)] \quad , \\
i\sigma m_3 &= -C^{-1}[i\sigma(c_{33} + \Omega^{-1}h_3)] \quad . \qquad (1)
\end{aligned}
$$

Provided only that the anelastic rheology of the Earth model is everywhere linear, the quantities c_{ij} and h_i can be related to m_i by

$$c_{ij} = D_{ijk}m_k \quad , \quad h_i = E_{ij}m_j \quad , \qquad (2)$$

where D_{ijk} and E_{ij} are frequency-dependent tensors. The substitution (2) reduces the Liouville equations (1) to a set of three homogeneous linear algebraic equations for the components m_i. The determinant of

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these equations will vanish only for selected values of σ, namely whenever σ is an eigenfrequency of an oscillation with $m_i \neq 0$. Since the Chandler wobble is one such oscillation, we can calculate its period precisely if we can determine the two tensors D_{ijk} and E_{ij}, for periods $T = 2\pi/\sigma \sim 14$ months.

This is however a formidable task. Under some circumstances, a quasi-static approximation may be sufficiently accurate. In this approximation, E_{ij} is replaced by zero, and D_{ijk} by its static, or zero-frequency, value; in that case D_{ijk} satisfies $D_{i3k} = D_{k3i}$, and the secular equation is a cubic polynomial with one root $\sigma = 0$ and two of the form $\sigma = \pm\sigma_w$, where

$$\sigma_w = \left[\frac{(C-A-D_{131})(C-A-D_{232}) - D_{132}^2}{(A + D_{131})(A + D_{232}) - D_{132}^2}\right]^{1/2} \Omega \quad . \tag{3}$$

The zero root is associated with the axial spin mode, and σ_w with the Chandler wobble. In writing (3), the coupling of m_1 and m_2 to m_3 (produced by the terms D_{133} and D_{233}) has been neglected; this coupling has been investigated by Dahlen (1976), and found to be thoroughly negligible.

It is known that a quasi-static approximation can be justified for an everywhere solid Earth model, but not for one with a fluid core. In the first case, E_{ij} is zero by definition, and $D_{ijk}(T \sim 14 \text{ months}) \sim D_{ijk}(\text{static})$ is guaranteed by the isolation of the Chandler wobble in the eigenfrequency spectrum (for an everywhere solid Earth model, the next gravest mode is ${}_0S_2$, with $T \sim 1$ hour). This is the basis of the classical result due to Love and Larmor

$$\sigma_w = \frac{C-A-ka^5\Omega^2/3G}{A}\,\Omega \quad , \tag{4}$$

which shows that elasticity acts to lengthen the period (k is the tidal Love number, a is the radius of the Earth model, and G is Newton's constant). In the second case, $E_{ij} \sim 0$ is not a good approximation, since Hough has shown that a fluid core will not participate in the wobble.

A quasi-static theory can be used for the oceans only if they, unlike the core, do participate in the wobble. In the language of tidal theory, we must ask whether the global pole tide is an equilibrium tide. This question has been addressed both theoretically and observationally. Proudman (1960) has shown that an 18.6 year tide would be rendered equilibrium by the action of turbulent bottom friction, but that a fortnightly tide would not; at $T \sim 14$ months, he was unable to draw any firm conclusion. The most thorough observational study to date, that of Miller and Wunsch (1973), could conclude only that, "ports with lengthy records are too few, and too far apart, to allow one to say anything about the global structure of the pole tide ... There is no evidence in the data either to confirm or refute the equilibrium hypothesis."

There being no evidence to the contrary, let us adopt an equilibrium law. A self-consistent equilibrium calculation, which takes oceanic self-attraction and tidal loading into account, has been performed by Dahlen (1976). The resulting values of $D_{i3k} = (a^5\Omega^2/3G)d_{i3k}$, together with the Love number k, for two recent Earth models (Gilbert and Dziewonski, 1975), are shown below:

	1066A	1066B
k	0.30088	0.30097
d_{131}	0.35092	0.35102
d_{232}	0.34051	0.34061
$d_{132} = d_{231}$	-0.00109	-0.00110

The corresponding values of $T_w = 2\pi/\sigma_w$ are insignificantly different for the two models; we obtain $T_w = 481.2$ (sidereal) days with oceans from (3), and $T_w = 447.5$ days without oceans from (4).

A summary of recent determinations of the observed period T_o is given below:

Investigator	Method	T_o (sidereal days)
Jeffreys (1968)	maximum likelihood	434.3 ± 2.2
Currie (1974)	maximum entropy	434.1 ± 1.0
Wilson and Haubrich (1976)	narrow band maximum likelihood, Monte Carlo	435.2 ± 2.6

The best estimate is probably that of Wilson and Haubrich, as they have conducted the most careful investigation of the effects of noise.

The discrepancy between T_w (with oceans) and T_o is, of course, due to the core. An alternative, and better, procedure is to employ the equilibrium tidal calculation to correct the observed period T_o for the effect of the oceans, i.e., to determine $T_e = 2\pi/\sigma_e$, the period

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of the real Earth's wobble if the oceans were absent. To do this, we use

$$
\begin{aligned}
i\sigma_o m_1 + \sigma_e m_2 &= -A^{-1}(i\sigma_o c'_{13} - \Omega c'_{23}) \quad , \\
i\sigma_o m_2 - \sigma_e m_1 &= -A^{-1}(i\sigma_o c'_{23} + \Omega c'_{13}) \quad ,
\end{aligned}
\tag{5}
$$

where $c'_{13} = (D_{131} - ka^5\Omega^2/3G)m_1 + D_{132}m_2$ and $c'_{23} = D_{231}m_1 + (D_{232} - ka^5\Omega^2/3G)m_2$ are the inertia tensor perturbations due only to the effects of the oceans. This leads (with the neglect of some demonstrably small terms) to

$$
\sigma_e^2 - \sigma_e\Omega A^{-1}(D_{131} + D_{232} - 2ka^5\Omega^2/3G) - \sigma_o^2
- \Omega^2 A^{-2}[D_{132}^2 - (D_{131} - ka^5\Omega^2/3G)(D_{232} - ka^5\Omega^2/3G)] = 0 \quad .
\tag{6}
$$

With T_o = 435.2 days, we find from (6) that T_e = 407.6 days, i.e., the effect of an equilibrium pole tide is to decrease the period by 27.6 days.

The period T_e can also be calculated directly, by solving the dynamical elastic-gravitational equations for a rotating, ellipsoidal Earth model, without oceans, but with a fluid outer core. Smith (1977; also see this volume) has performed such calculations for several models, all of which have a constant Brunt-Väisälä frequency N in the outer core. For a model with a neutrally stable core ($N = 0$), he finds T_e = 403.6 days, and for a highly stable core ($2\pi/N \sim 3$ hours), he finds T_e = 405.2 days.

An important recent development in seismology is the appreciation that anelastic attenuation in the mantle and crust must be accompanied by physical dispersion, i.e., the elastic parameters must be frequency-dependent. Akopyan, Zharkov and Lyubimov (1975,1976) and Liu, Anderson and Kanamori (1976) have shown independently that the long-standing discrepancy between Earth models derived from free oscillations and travel times can be resolved by taking the frequency dependence of the shear modulus μ into account. For a wide class of linear dissipative mechanisms, if Q_μ is constant between two frequencies σ_1 and σ_2, then μ varies like $\mu_2/\mu_1 = 1 + 2(\pi Q_\mu)^{-1} \ln(\sigma_2/\sigma_1)$. In a recent study of the attenuation of the free oscillations of the Earth, Sailor and Dziewonski (1977) found that within the normal mode band (centered around $T \sim 200$ sec), $Q_\mu \sim 110$ in the crust and upper mantle and $Q_\mu \sim 350$ in the lower mantle (below 680 km).

Let us explore the implications of the hypothesis that these values prevail from the normal mode band down to $T \sim 14$ months. For a rough estimate of the effect on T_e, we shall adopt $Q_\mu = 300$ as a single mean value for the crust and mantle as a whole. The model of the crust and mantle used in Smith's dynamical calculations has been derived from free oscillations, so the shear modulus μ is that appropriate to $T \sim 200$ sec. If the above hypothesis is correct, the shear modulus at $T \sim 14$ months will be $\mu + \delta\mu$, where $\delta\mu/\mu \sim 2(\pi Q_\mu)^{-1} \ln(200 \text{ sec}/14 \text{ months}) = -0.026$. A simple approximate formula for σ_e, which accounts both for the elasticity of the crust and mantle and for the lack of participation by the core, is

$$\sigma_e \sim \frac{C - A - ka^5\Omega^2/3G}{A_m}\,\Omega \quad , \tag{7}$$

where A_m is the moment of inertia of the crust and mantle alone. A change in k by an amount δk is seen to produce a change in σ_e of an amount

$$\delta\sigma_e \sim -\delta k \ (a^5\Omega^2/3GA_m)\Omega \quad . \tag{8}$$

For a rough estimate of δk, let us make use of Kelvin's formula

$$k = \frac{3/2}{1 + (19/2)(\mu/\rho g a)} \quad , \tag{9}$$

for a homogeneous incompressible Earth model of density ρ and surface gravity g. The not unreasonable choice $\mu/\rho g a = 8/19$ makes $k = 0.3$, in which case $\delta k/k \sim -4\delta\mu/5\mu = 0.021$. From (8) and $\delta T_e/T_e = -\delta\sigma_e/\sigma_e$, we obtain the value $\delta T_e \sim 4.6$ days, which must be added to Smith's results; a final comparison is shown below:

Method	T_e (sidereal days)
T_o, corrected for oceans	407.6 ± 2.6
Smith ($N = 0$), corrected for dispersion	408.2
Smith ($2\pi/N \sim 3$ hours), corrected for dispersion	409.8

Consider now the following three statements about the Earth: (1) Globally, the pole tide is an equilibrium tide. (2) The fluid outer core is neutrally stratified. (3) Q_μ in the mantle and crust is independent of frequency from $T \sim 200$ sec down to $T \sim 14$ months. The upshot of the above comparison is that these three statements, considered simultaneously, are consistent with the measured period of the Chandler wobble at the level of a single standard deviation. This, of course, does not imply that any of the statements is necessarily true, but only that none of them may be eliminated for failing to predict correctly the single datum T_o.

Within the next few decades, the newly developed techniques for observing polar motion will probably succeed in reducing substantially the uncertainty in the measurement of T_o. Can such an improvement be of help in seeking to prove or disprove any of the above statements? Any contention that improved observations of polar motion might help us to understand the complex dynamics of the oceans at $T \sim 14$ months would no doubt strike an oceanographer as absurd. If, on the other hand, future oceanographic observations do discover a small global deviation from an equilibrium pole tide, its effect on T_o can easily be determined by inserting for c'_{13} and c'_{23} in (5) the correct values instead of those inferred by assuming an equilibrium law. The Chandler wobble is only one of an infinitude of free oscillations of the Earth, and the period of every mode depends at least partially upon the density stratification in the outer core. Seismological observations in the normal mode band should constrain N definitively in the near future, so that its influence on T_o can also be determined. Clearly, it is the final statement which can best be assessed by an improved measurement of T_o. Thus far, the Chandler wobble is the only unambiguously observed free oscillation of the Earth with a period graver than one hour. It therefore provides a unique opportunity to investigate the very long-period anelastic properties of the mantle and crust. The rough estimate of δT_o due to anelasticity derived here easily can, and should, be improved; the radial variation of Q_μ and the possibility of bulk dissipation Q_κ should be taken into account. Ultimately, of course, any statement about the Earth's very long-period anelasticity which is based upon calculating T_o must rest upon the oceanographic and seismological assessment of the first two statements.

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