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AUTOMATIC CONTINUITY OF n-HOMOMORPHISMS BETWEEN TOPOLOGICAL ALGEBRAS

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Abstract

A map $\theta : A \to B$ between algebras A and B is called n-multiplicative if $\theta(a_1 a_2 \cdots a_n) = \theta(a_1)\theta(a_2) \cdots \theta(a_n)$ for all elements $a_1, a_2, \ldots, a_n \in A$. If θ is also linear then it is called an n-homomorphism. This notion is an extension of a homomorphism. We obtain some results on automatic continuity of n-homomorphisms between certain topological algebras, as well as Banach algebras. The main results are extensions of Johnson's theorem to surjective n-homomorphisms on topological algebras, a theorem due to C. E. Rickart in 1950 to dense range n-homomorphisms on topological algebras and two theorems due to E. Park and J. Trout in 2009 to $*$-preserving n-homomorphisms on *lmc* $*$-algebras.

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1. Introduction

Let A and B be algebras and $n \geq 2$ be an integer. A mapping $\theta : A \to B$ is called n-multiplicative [anti-n-multiplicative] if

$$\theta(a_1 a_2 \cdots a_n) = \theta(a_1)\theta(a_2) \cdots \theta(a_n)[= \theta(a_n)\theta(a_2) \cdots \theta(a_1)]$$

for all elements $a_1, a_2, \ldots, a_n \in A$. If θ is also linear then it is called an n-homomorphism [anti-n-homomorphism]. Obviously, each homomorphism is an n-homomorphism for every $n \geq 2$, but the converse is not true, in general. For example, if φ is a homomorphism then $\theta = -\varphi$ is a 3-homomorphism, which is not a homomorphism. For certain properties of 3-homomorphisms one may refer to [1]. If A is unital with the unit element e_A and $\theta : A \to B$ is an n-homomorphism then by [7, Proposition 2.2], there exists a homomorphism $\varphi : A \to B$ such that $\theta(a) = \theta(e_A)\varphi(a)$ for all $a \in A$. Furthermore, a 2-homomorphism is then just a homomorphism, in the usual sense. Thus we may assume in the following that $n \geq 3$. The concept of n-homomorphism was studied for complex algebras by Hejazian *et al.* in [7]. Fragoulopoulou [4, 5] in 1991 and 1993, and then Honary and Najafi [8] in 2008, obtained some results on the automatic continuity of homomorphisms between topological Q-algebras. We extend some of these results to n-homomorphisms.

We now provide some notation and state some definitions and known results. For further details one can refer, for example, to [2, 6, 9]. If A is a unital complex algebra with the unit e_A then the spectrum of $a \in A$ is $\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} : \lambda e_A - a \notin \mathrm{Inv}\, A\}$, where Inv A is the set of invertible elements of A. If A is a nonunital complex algebra, then the spectrum of a is

$$\mathrm{sp}_A(a) = \{0\} \cup \left\{\lambda \in \mathbb{C} \setminus \{0\} : \frac{1}{\lambda} a \notin q\text{-}\mathrm{Inv}\, A\right\},$$

where q-Inv A is the set of quasi-invertible elements of A. If A^+ is the unitization of A, then $\mathrm{sp}_A(a) = \mathrm{sp}_{A^+}((a, 0))$ and so $\nu_A(a) = \nu_{A^+}((a, 0))$ for all $a \in A$, where $\nu_A(a)$ is the spectral radius of a with respect to the algebra A.

A left ideal I of an algebra A is a modular left ideal if there exists $u \in A$ such that $A(e_A - u) \subseteq I$, where $A(e_A - u) = \{x - xu : x \in A\}$. The Jacobson radical Rad(A) of A is the intersection of all maximal modular left ideals of A. The strong radical $\mathfrak{R}(A)$ of A is the intersection of all maximal modular (two-sided) ideals of A. An algebra A is called simple if $A^2 \neq 0$ and if 0 and A are the only ideals in A. An algebra A is called semisimple whenever its Jacobson radical Rad(A) is trivial and it is called strongly semisimple if $\mathfrak{R}(A)$ is trivial.

A locally multiplicatively convex (*lmc*) algebra is a topological algebra whose topology is defined by a separating family $\mathcal{P} = (p_\alpha)$ of submultiplicative seminorms. A complete metrizable *lmc* algebra is a Fréchet algebra. An F-algebra is a topological algebra whose underlying topological linear space is an F-space; in other words, the topology of an F-algebra is defined by a complete invariant metric. A Fréchet algebra is an F-algebra which is also an *lmc* algebra. The topology of a Fréchet algebra A can be generated by a sequence $(p_n)_{n\in\mathbb{N}}$ of separating submultiplicative seminorms, that is, $p_n(xy) \le p_n(x)p_n(y)$ for all $n \in \mathbb{N}$ and $x, y \in A$, such that $p_n(x) \le p_{n+1}(x)$ for all $x \in A$ and $n \in \mathbb{N}$.

An algebra A equipped with an involution is called an involutive algebra, or a $*$-algebra. A topological $*$-algebra is a topological algebra with a continuous involution. If $(A, (p_\alpha))$ is an involutive topological algebra with a family of seminorms (p_α) such that $p_\alpha(x^*) = p_\alpha(x)$ for all $x \in A$ and for every α, then A is clearly a topological $*$-algebra. If A is an involutive algebra and p is a seminorm on A, which satisfies the property $p(x^*x) = p(x)^2$ for all $x \in A$, then p is called a C*-seminorm. The completion of an involutive topological algebra, whose topology is defined by a family of C*-seminorms, is called a locally C*-algebra. An *lmc* $*$-algebra is an involutive *lmc* algebra with a family of seminorms $\mathcal{P} = (p_\alpha)$ such that $p_\alpha(a^*) = p_\alpha(a)$ for every α and all $a \in A$. If, moreover, p_α is a C*-seminorm for every α, it is called an *lmc* C*-algebra.

For a topological algebra $(A, (p_\alpha))$, with a family of submultiplicative seminorms $\mathcal{P} = (p_\alpha)$, the completion of $A/\ker p_\alpha$ with respect to the norm $p'_\alpha([x]_\alpha) = p_\alpha(x)$ is denoted by A_α, where $[x]_\alpha = x + \ker p_\alpha$. Clearly, A_α is a Banach algebra. If A is an *lmc* C*-algebra, then A_α is a (Banach) C*-algebra for every α.

If A and B are involutive algebras, then an n-homomorphism $\theta : A \to B$ is called a $*$-preserving n-homomorphism if $\theta(x^*) = \theta(x)^*$ for all $x \in A$.

A topological algebra A is advertibly complete if a Cauchy net (a_α) in A converges in A whenever, for some $b \in A$, both $a_\alpha + b - a_\alpha \cdot b$ and $a_\alpha + b - b \cdot a_\alpha$ converge to zero.

A topological algebra A is a Q-algebra if the set of its quasi-invertible elements (q-Inv A) is open in A.

It is interesting to note that every topological Q-algebra is advertibly complete [9, Theorem I.6.4]. Moreover, every complete topological algebra is also advertibly complete [9, p. 45].

PROPOSITION 1.1 [6, Theorem 4.6]. *If $(B, (p_\alpha))$ is an advertibly complete lmc algebra, then, for every $x \in B$,*

$$\mathrm{sp}_B(x) = \bigcup_\alpha \mathrm{sp}_{B_\alpha}([x]_\alpha),$$

$$\nu_B(x) = \sup_\alpha \nu_{B_\alpha}([x]_\alpha) = \sup_\alpha \lim_{m\to\infty} (p_\alpha(x^m))^{1/m}.$$

Let A and B be topological linear spaces, and let $\theta : A \to B$ be a linear mapping. The separating space of θ is defined by

$$\mathfrak{S}(\theta) = \{b \in B : \exists \text{ net } (a_\delta) \text{ in } A \text{ such that } a_\delta \to 0 \text{ and } \theta(a_\delta) \to b\}.$$

The separating space $\mathfrak{S}(\theta)$ is a closed linear subspace of B; moreover, if A and B are F-spaces, then, by the closed graph theorem, θ is continuous if and only if $\mathfrak{S}(\theta) = \{0\}$ [2, Proposition 5.1.2].

The following lemma has been proved by Ransford in [11], for unital Banach algebras, but it is also valid for nonunital algebras.

LEMMA 1.2. *Let B be an algebra, let $y \in B$, and suppose that $\nu_B(y'y) = 0$ for all $y' \in B$. Then $y \in \mathrm{Rad}(B)$.*

The following lemma, which will be used later, is also due to Ransford [11].

LEMMA 1.3. *Let B be a Banach algebra, let $p(z)$ be a polynomial with coefficients in B, and let $R > 0$. Then*

$$\nu_B(p(1))^2 \leq \sup_{|z|=R} \nu_B(p(z)) \sup_{|z|=1/R} \nu_B(p(z)).$$

2. Extensions of Johnson's theorem for n-homomorphisms on topological algebras

We first state the following theorem, which appeared in [7, Proposition 2.2], and then deduce two useful results.

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THEOREM 2.1. *Let A be a unital algebra with the identity e_A, let B be an algebra and $\theta : A \to B$ be an n-homomorphism. If $\psi : A \to B$ is defined by $\psi(x) = \theta(e_A)^{n-2}\theta(x)$ then ψ is a homomorphism and $\theta(x) = \theta(e_A)\psi(x)$.*

COROLLARY 2.2. *With the same hypotheses as in the theorem, if θ is surjective then ψ is also surjective.*

PROOF. Clearly, $\theta(e_A)^{n-1}\theta(x) = \theta(x)$ for all $x \in A$. For every $y \in B$ there exists $x \in A$ such that $\theta(x) = y$. Moreover, there exists $t \in A$ such that $\theta(t) = \theta(e_A)\theta(x)$. Hence $\theta(e_A)^{n-2}\theta(t) = \theta(e_A)^{n-1}\theta(x) = \theta(x)$ and so $\psi(t) = \theta(x) = y$. Therefore, ψ is surjective. □

COROLLARY 2.3. *Let A and B be topological algebras, where A is unital. If $\theta : A \to B$ is a dense range n-homomorphism, then ψ is a dense range homomorphism.*

PROOF. It is clear that B is also unital and $e_B = \theta(e_A)^{n-1}$. Let $y \in B$. For $z = \theta(e_A)y$ there is a net (x_α) in A such that $\theta(x_\alpha) \to z$. Hence

$$\psi(x_\alpha) = \theta(e_A)^{n-2}\theta(x_\alpha) \to \theta(e_A)^{n-2}z = y.$$

Since $y \in B$ is arbitrary, it follows that $\overline{\psi(A)} = B$. □

LEMMA 2.4. *Let A be an algebra, $\lambda \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$. If $a, d \in A$ and $\lambda \notin \mathrm{sp}_A(a^k)$ then there exists an element $c \in A$ such that $c(\lambda e_{A^+} - a^k) = d$.*

PROOF. If $\lambda \notin \mathrm{sp}_A(a^k)$, then $c = d(\lambda e_{A^+} - a^k)^{-1} \in A^+$ satisfies

$$c(\lambda e_{A^+} - a^k) = d(\lambda e_{A^+} - a^k)^{-1}(\lambda e_{A^+} - a^k) = d.$$

Since $c \in A^+$, there exist $\alpha \in \mathbb{C}$ and $b \in A$ such that $c = (b, \alpha)$ and so

$$(d, 0) = (b, \alpha)[(0, \lambda) - (a^k, 0)] = (\lambda b, \lambda\alpha) - (ba^k + \alpha a^k, 0).$$

Thus $\lambda\alpha = 0$ and hence $\alpha = 0$, which shows that $c \in A$. □

LEMMA 2.5. *Let $(B, (p_\alpha)_{\alpha\in I})$ be an lmc algebra, $\lambda \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$. If, for $b \in B$, there exists an element $c \in B$ such that $c(\lambda e_{B^+} - b^k) = b$, then $\lambda \notin \mathrm{bd}(\mathrm{sp}_{B^+_\alpha}[b^k]_\alpha)$ for all $\alpha \in I$, where* bd *denotes the boundary (of a set) in the complex plane.*

PROOF. If $\lambda \in \mathrm{bd}(\mathrm{sp}_{B^+_\alpha}[b^k]_\alpha)$ for some $\alpha \in I$, then, by [2, Theorem 2.3.21(ii)], there exists a sequence $c_n \in \mathrm{Inv}\, B^+_\alpha$ such that $\|c_n\|_\alpha = 1$, where $\|\cdot\|_\alpha$ is the norm on B^+_α, and

$$([\lambda e_{B^+}]_\alpha - [b^k]_\alpha)c_n \to 0.$$

Then by the hypothesis we have

$$[c]_\alpha([\lambda e_{B^+}]_\alpha - [b^k]_\alpha)c_n = [b]_\alpha c_n \to 0.$$

Since $\lambda \neq 0$ and $\lambda \in \mathrm{bd}(\mathrm{sp}_{B^+_\alpha}[b^k]_\alpha)$, it follows that $b \notin \ker p_\alpha$. Hence $\lim_{n\to\infty} c_n = 0$, which is a contradiction. □

LEMMA 2.6. *Let A be an lmc algebra and $(B, (p_\alpha)_{\alpha\in I})$ be an advertibly complete lmc algebra. If $\theta : A \to B$ is an n-homomorphism and $a \in A$, then*

$$\mathrm{bd}(\mathrm{sp}_{B_\alpha^+}([\theta(a)^{n-1}]_\alpha)) \subseteq \mathrm{sp}_A(a^{n-1}) \cup \{0\},$$

for all $\alpha \in I$. Moreover, $\nu_{B_\alpha}([\theta(a)^{n-1}]_\alpha) \le \nu_A(a^{n-1})$ for all $\alpha \in I$ and hence

$$\nu_B(\theta(a)^{n-1}) \le \nu_A(a^{n-1}).$$

PROOF. Since A is an *lmc* algebra and $\mathrm{sp}_{A^+}((a, 0)) = \mathrm{sp}_A(a)$ for every $a \in A$, by [9, Corollary II.4.1], $\mathrm{sp}_A(a) \ne \emptyset$ for every $a \in A$. Suppose that $\lambda \ne 0$ such that $\lambda \notin \mathrm{sp}_A(a^{n-1})$. By Lemma 2.4, for $d = a$, there exists an element $c \in A$ such that $a = c(\lambda e_{A^+} - a^{n-1}) = \lambda c - ca^{n-1}$. Hence

$$\theta(a) = \theta(\lambda c - ca^{n-1}) = \theta(c)(\lambda e_{B^+} - \theta(a)^{n-1}).$$

From Lemma 2.5, it follows that $\lambda \notin \mathrm{bd}(\mathrm{sp}_{B_\alpha^+}([\theta(a)^{n-1}]_\alpha))$ for all $\alpha \in I$ and hence that

$$\mathrm{bd}(\mathrm{sp}_{B_\alpha^+}([\theta(a)^{n-1}]_\alpha)) \subseteq \mathrm{sp}_A(a^{n-1}) \cup \{0\},$$

for all $\alpha \in I$. It is now clear from Proposition 1.1 that $\nu_B(\theta(a)^{n-1}) \le \nu_A(a^{n-1})$. □

It is interesting to note that the above lemma is also valid if θ is an anti-n-homomorphism.

THEOREM 2.7. *Let A be a unital topological Q-algebra and let B be an advertibly complete semisimple lmc algebra. If $\theta : A \to B$ is a surjective n-homomorphism then θ has a closed graph.*

PROOF. By Corollary 2.2 we have $\psi(A) = B$, where $\psi(x) = \theta(e_A)^{n-2}\theta(x)$. By [8, Theorem 2.3], ψ has a closed graph and hence $\theta(x) = \theta(e_A)\psi(x)$ also has a closed graph. □

COROLLARY 2.8. *Let A be a unital F-algebra which is also a Q-algebra and let B be a semisimple Fréchet algebra. Then every surjective n-homomorphism $\theta : A \to B$ is automatically continuous.*

An algebra A is called factorizable if, for each $a \in A$, there exist $b, c \in A$ such that $a = bc$. If A is not unital in the above theorem then we have the following result.

THEOREM 2.9. *Let A be an lmc Q-algebra and B be a factorizable advertibly complete lmc semisimple algebra. Then every surjective n-homomorphism $\theta : A \to B$ has a closed graph.*

PROOF. Let (p_β) be a family of seminorms on A, and (q_α) be a family of seminorms on B. Denote by B_α the Banach algebra obtained by the completion of $B/\ker q_\alpha$ with respect to the norm $q'_\alpha([b]_\alpha) = q_\alpha(b)$, for $b \in B$. It is enough to show that, for any net (x_δ) in A, if $x_\delta \to 0$ in A and $\theta(x_\delta) \to y$ in B, then $y = 0$.

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By the surjectivity of θ, there exists $x \in A$ such that $\theta(x) = y$. We define a polynomial with coefficients in B by $P_\delta(z) = z\theta(x_\delta) + \theta(x - x_\delta)$. Since B_α is a Banach algebra,

$$\nu_{B_\alpha}([P_\delta(z)]_\alpha) \le q_\alpha(P_\delta(z)) \le |z| q_\alpha(\theta(x_\delta)) + q_\alpha(\theta(x) - \theta(x_\delta)).$$

On the other hand, $[\theta]_\alpha = \theta + \ker q_\alpha$ is an n-homomorphism from A to B_α. By Lemma 2.6, for all $z \in \mathbb{C}$,

$$\nu_{B_\alpha}([P_\delta(z)]_\alpha^{n-1}) \le \nu_A((zx_\delta + (x - x_\delta))^{n-1}).$$

Since A is a Q-algebra, there exists p_β with $\nu_A \le p_\beta$ [6, Theorem 6.18]. Hence

$$\nu_{B_\alpha}([P_\delta(z)]_\alpha^{n-1}) \le p_\beta((zx_\delta + (x - x_\delta))^{n-1}) \le (|z| p_\beta(x_\delta) + p_\beta(x - x_\delta))^{n-1}.$$

If $\lambda \in \mathrm{sp}_{B_\alpha}([P_\delta(z)]_\alpha)$ then $\lambda^{n-1} \in \mathrm{sp}_{B_\alpha}([P_\delta(z)]_\alpha^{n-1})$ and so

$$|\lambda| \le (|z| p_\beta(x_\delta) + p_\beta(x - x_\delta)).$$

Therefore,

$$\nu_{B_\alpha}([P_\delta(z)]_\alpha) \le |z| p_\beta(x_\delta) + p_\beta(x - x_\delta).$$

Combining these estimates with Lemma 1.3, we deduce that, for all δ and all $R > 0$,

$$\begin{aligned} \nu_{B_\alpha}([y]_\alpha)^2 \le & \sup_{|z|=1/R} (|z| q_\alpha(\theta(x_\delta)) + q_\alpha(\theta(x) - \theta(x_\delta))) \\ & \times \sup_{|z|=R} (|z| p_\beta(x_\delta) + p_\beta(x - x_\delta)). \end{aligned}$$

Since $x_\delta \to 0$ and $\theta(x_\delta) \to \theta(x)$, we obtain

$$\nu_{B_\alpha}([y]_\alpha)^2 \le \frac{1}{R} q_\alpha(y) \cdot p_\beta(x).$$

By letting $R \to \infty$, it follows that $\nu_{B_\alpha}([y]_\alpha) = 0$. Therefore, by Proposition 1.1, $\nu_B(y) = 0$.

Now let $y' \in B$. Since B is a factorizable algebra there exist $y'_1, \ldots, y'_{n-1} \in B$ such that $y' = y'_1 \cdots y'_{n-1}$. Now we choose $x'_i \in A$, $i = 1, \ldots, n-1$, with $\theta(x'_i) = y'_i$, $i = 1, \ldots, n-1$. Then $x'_1 \cdots x'_{n-1} x_\delta \to 0$ in A and $\theta(x'_1 \cdots x'_{n-1} x_\delta) \to y'_1 \cdots y'_{n-1} y = y'y$ in B. Hence a repetition of the above argument shows that $\nu_B(y'y) = 0$. Since y' is arbitrary, by Lemma 1.2, it follows that $y \in \mathrm{Rad}(B)$ and hence $y = 0$, as desired. □

COROLLARY 2.10. *Let A and B be Fréchet algebras such that A is a Q-algebra and B is factorizable and semisimple. Then every surjective n-homomorphism $\theta : A \to B$ is automatically continuous.*

PROOF. This is immediate by the closed graph theorem. □

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Since any unital algebra is factorizable, we also conclude the following result.

COROLLARY 2.11. *Let A be an lmc Q-algebra and B be a unital advertibly complete semisimple lmc algebra. Then every surjective n-homomorphism $\theta : A \to B$ has a closed graph.*

Let $(A, (p_\alpha))$ be an *lmc* algebra. A uniformly bounded left (right) approximate identity for A is a net $\{e_\gamma\}_{\gamma\in\Lambda}$ such that:

(i) $\lim_\gamma e_\gamma a = a$ ($\lim_\gamma a e_\gamma = a$) for all $a \in A$;
(ii) $\sup_\alpha p_\alpha(e_\gamma) < \infty$ for all $\gamma \in \Lambda$.

REMARK 2.12. Many *lmc* algebras which do not have an identity do have uniformly bounded left or right approximate identities. For example, every locally C*-algebra has a uniformly bounded approximate identity [6, Theorem 11.5]. Moreover, every Fréchet algebra, with a uniformly bounded left approximate identity, is factorizable [3, Theorem 4.1].

Hence we have the following result.

COROLLARY 2.13. *Let A be an lmc Q-algebra and B be a semisimple Fréchet algebra with a uniformly bounded left approximate identity. Then every surjective n-homomorphism $\theta : A \to B$ has a closed graph. In particular, if A and B are Fréchet algebras such that A is a Q-algebra and B is a semisimple locally C*-algebra, then every surjective n-homomorphism $\theta : A \to B$ is continuous.*

PROPOSITION 2.14. *Every anti-n-homomorphism on an lmc Q-algebra A onto an advertibly complete factorizable semisimple lmc algebra B has a closed graph.*

PROOF. Since $\mathrm{sp}_B(y'y) \cup \{0\} = \mathrm{sp}_B(yy') \cup \{0\}$ for all $y, y' \in B$ we have $\nu_B(y'y) = \nu_B(yy')$ for all $y, y' \in B$. If we replace $y'y$ by yy' in Lemma 1.2 then the lemma is still true. Since Lemma 2.6 is also valid for anti-n-homomorphisms, by the same argument as in the proof of Theorem 2.9 the result follows. □

Let A and B be linear spaces over K ($\mathbb{R}$ or $\mathbb{C}$). A map $\theta : A \to B$ is conjugate-linear if

$$\theta(\lambda x + y) = \bar{\lambda}\theta(x) + \theta(y), \quad x, y \in A, \lambda \in K.$$

LEMMA 2.15. *Let A be an lmc algebra and $(B, (p_\alpha)_{\alpha\in I})$ be an advertibly complete lmc algebra. If $\theta : A \to B$ is a conjugate-linear and n-multiplicative (or anti-n-multiplicative) mapping, then for every $a \in A$ and for each $\alpha \in I$,*

$$\mathrm{bd}(\mathrm{sp}_{B_\alpha^+}([\theta(a)^{n-1}]_\alpha)) \subseteq \overline{\mathrm{sp}_A(a^{n-1})} \cup \{0\} \quad (\textit{conjugate of } \mathrm{sp}_A(a^{n-1})),$$

where bd *denotes the boundary (of a set) in the complex plane. Therefore,*

$$\nu_{B_\alpha}([\theta(a)^{n-1}]_\alpha) \le \nu_A(a^{n-1}),$$

for all $\alpha \in I$ and hence $\nu_B(\theta(a)^{n-1}) \le \nu_A(a^{n-1})$.

PROOF. By modifying the proof of Lemma 2.6, the result follows. □

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An n-involution on an algebra A over $\mathbb{C}$ is a map $* : A \to A$ satisfying:

(i) $(a+b)^* = a^* + b^*$;
(ii) $(a_1 a_2 \cdots a_n)^* = a_n^* a_{n-1}^* \cdots a_1^*$;
(iii) $(\lambda a)^* = \bar{\lambda} a^*$;
(iv) $\underbrace{(((a^*)^*)^* \cdots)^*}_{n} = a^{n*} = a$.

Note that every n-involution is conjugate-linear and anti-n-multiplicative. Hence we have the following result.

PROPOSITION 2.16. *Let A be a factorizable semisimple lmc Q-algebra. Then every n-involution on A has a closed graph. If, in addition, A is an F-algebra, then every n-involution on A is automatically continuous.*

PROOF. By Proposition 2.14, Lemma 2.15 and by modification of the proof of Theorem 2.9 the result follows. □

3. Extension of Rickart's theorem for dense range n-homomorphisms on topological algebras

We first extend [2, Proposition 5.1.3(i)] as follows.

PROPOSITION 3.1. *Let A and B be topological algebras and $\theta : A \to B$ be a dense range n-homomorphism such that $\theta(A)$ is factorizable. Then the separating space $\mathfrak{S}(\theta)$ is a closed (two-sided) ideal in B.*

PROOF. By [2, Proposition 5.1.2], the separating space $\mathfrak{S}(\theta)$ is a closed linear subspace of B. Let $b \in \mathfrak{S}(\theta)$ and $a \in A$. There exists a net $\{a_\delta\}$ in A such that $a_\delta \to 0$ and $\theta(a_\delta) \to b$. Since $\theta(A)$ is a factorizable algebra, there are $a'_1, \ldots, a'_{n-1} \in A$ such that $\theta(a) = \theta(a'_1) \cdots \theta(a'_{n-1})$. Since $a'_1 \cdots a'_{n-1} a_\delta \to 0$ and $\theta(a'_1 \cdots a'_{n-1} a_\delta) \to \theta(a'_1) \cdots \theta(a'_{n-1}) b = \theta(a) b$, it follows that $\theta(a) b \in \mathfrak{S}(\theta)$. Similarly, $b\theta(a) \in \mathfrak{S}(\theta)$.

If $b' \in B$ then there exists a net $\{a'_\beta\}$ in A such that $\theta(a'_\beta) \to b'$ and so $\theta(a'_\beta) b \to b'b$. Since $\theta(a'_\beta) b \in \mathfrak{S}(\theta)$ and $\mathfrak{S}(\theta)$ is closed, it follows that $b'b \in \mathfrak{S}(\theta)$. Similarly, $bb' \in \mathfrak{S}(\theta)$. Hence $\mathfrak{S}(\theta)$ is an ideal in B. □

We now state a result due to Rickart in 1950, see [12, Theorem 6.18] and then extend it to more general cases.

THEOREM 3.2. *Let A and B be Banach algebras such that A is unital and B is strongly semisimple. Then every dense range homomorphism $\theta : A \to B$ is automatically continuous.*

PROPOSITION 3.3. *Let A and B be Banach algebras such that A is unital and B is strongly semisimple. If $\theta : A \to B$ is a dense range n-homomorphism, then it is automatically continuous.*

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PROOF. By Corollary 2.3 we have $\overline{\psi(A)} = B$, where $\psi(x) = \theta^{n-2}(e_A)\theta(x)$, $x \in A$, is a homomorphism. By Theorem 3.2, ψ is continuous. Hence $\theta(x) = \theta(e_A)\psi(x)$ is also continuous. □

In a unital algebra every ideal is modular. Moreover, in a unital Q-algebra every maximal ideal is closed. We now extend the above result to certain topological algebras.

THEOREM 3.4. *Let A and B be lmc Q-algebras such that B is a unital, strongly semisimple algebra. If $\theta : A \to B$ is a dense range n-homomorphism such that $\theta(A)$ is factorizable, then θ has a closed graph.*

PROOF. It is enough to show that, for every net $x_\delta \in A$, if $x_\delta \to 0$ in A and $\theta(x_\delta) \to y$ in B, then $y = 0$. Let M be a maximal ideal of B. Since B is a unital Q-algebra, M is closed and so, by [6, 6.14(3)], B/M is a Q-algebra. Since ideals in B/M are in the form of J/M, where J is an ideal in B containing M, it is clear that the only ideals of B/M are zero (that is, M) and B/M. Hence B/M is simple. We now consider the n-homomorphism $\theta' : A \to B/M$, which is the composition of θ and the natural map of B onto B/M. By Proposition 3.1, $\mathfrak{S}(\theta')$ is an ideal of B/M. On the other hand, by Lemma 2.6 we have

$$\nu_{B/M}(\theta'(a)^{n-1}) \leq \nu_A(a^{n-1}) \quad (a \in A).$$

If $\lambda \in \mathrm{sp}_{B/M}(\theta'(a))$ then $\lambda^{n-1} \in \mathrm{sp}_{B/M}(\theta'(a)^{n-1})$ and so $\nu_{B/M}(\theta'(a)) \leq \nu_A(a)$.

If $e_{B/M} \in \mathfrak{S}(\theta')$ then there exists a net $\{a_\delta\}$ in A such that $a_\delta \to 0$ in A and $\theta'(a_\delta) \to e_{B/M}$ in B. Moreover,

$$\begin{aligned} 1 = \nu_{B/M}(e_{B/M}) &\leq \nu_{B/M}(\theta'(a_\delta)) + \nu_{B/M}(e_{B/M} - \theta'(a_\delta)) \\ &\leq \nu_A(a_\delta) + \nu_{B/M}(e_{B/M} - \theta'(a_\delta)). \end{aligned}$$

Since A and B/M are *lmc* Q-algebras, it follows from [9, Proposition III.6.2] that ν_A and $\nu_{B/M}$ are continuous at zero and so

$$\nu_A(a_\delta) + \nu_{B/M}(e_{B/M} - \theta'(a_\delta)) \to 0,$$

which is a contradiction. Hence $e_{B/M} \notin \mathfrak{S}(\theta')$. Since B/M is simple, it follows that $\mathfrak{S}(\theta') = M$, that is, θ' is continuous and hence $\theta'(x_\delta) \to 0$, which implies that $y \in M$. Since M is an arbitrary maximal (modular) ideal, we conclude that $y \in \mathfrak{R}(B)$. Since B is strongly semisimple, $y = 0$. Therefore, θ has a closed graph. □

COROLLARY 3.5. *Let A and B be Fréchet Q-algebras and let B be unital and strongly semisimple. Suppose that $\theta : A \to B$ is a dense range n-homomorphism such that $\theta(A)$ is factorizable. Then θ is automatically continuous.*

PROOF. By the closed graph theorem the result follows. □

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4. Automatic continuity of *n*-homomorphism on *lmc* C*-algebras

For certain results on the automatic continuity of n-homomorphisms on Banach C*-algebras one may refer to [10], and for the automatic continuity of homomorphisms on *lmc* C*-algebras one may refer to [4].

We now extend [10, Theorems 2.3 and 3.2], originally proved for Banach C*-algebras. For this purpose, we need the following useful lemmas.

LEMMA 4.1. *Let $(B, (p_\alpha)_{\alpha\in I})$ be an lmc $*$-algebra, $\lambda \in \mathbb{C}\setminus\{0\}$ and $k \in \mathbb{N}$. If, for $b \in B$, there exists an element $c \in B$ such that $c(\lambda e_{B^+} - (b^*b)^k) = b$, then $\lambda \notin$ $\mathrm{bd}(\mathrm{sp}_{B^+_\alpha}[(b^*b)^k]_\alpha)$ for all $\alpha \in I$.*

PROOF. The proof is similar to the proof of Lemma 2.5. □

LEMMA 4.2. *Let A be an lmc $*$-algebra and $(B, (p_\alpha)_{\alpha\in I})$ be an advertibly complete lmc $*$-algebra. If $\theta : A \to B$ is a $*$-preserving n-homomorphism, $n = 2k + 1$ and $a \in A$, then*

$$\mathrm{bd}(\mathrm{sp}_{B^+_\alpha}([(\theta(a)^*\theta(a))^k]_\alpha)) \subseteq \mathrm{sp}_A((a^*a)^k) \cup \{0\},$$

for all $\alpha \in I$. Moreover, $\nu_{B_\alpha}([(\theta(a)^*\theta(a))^k]_\alpha) \le \nu_A((a^*a)^k)$ for all $\alpha \in I$ and hence

$$\nu_B((\theta(a)^*\theta(a))^k) \le \nu_A((a^*a)^k).$$

PROOF. Let $\lambda \ne 0$ and $\lambda \notin \mathrm{sp}_A((a^*a)^k)$. By Lemma 2.4 there exists $c \in A$ such that $c(\lambda e_{A^+} - (a^*a)^k) = a$. By applying Lemmas 2.6 and 4.1 the result follows. □

THEOREM 4.3. *Let A be a topological Q-algebra, which is an lmc $*$-algebra with the family of seminorms $\mathcal{P} = (P_\alpha)$. Let B be an lmc C*-algebra with the family of seminorms $\mathcal{Q} = (q_\beta)$. If $\theta : a \to B$ is a $*$-preserving n-homomorphism, then for each q_β there exists a p_α such that $q_\beta(\theta(x)) \le p_\alpha(x)$ for all $x \in A$. Hence θ is continuous on A.*

PROOF. Since A is a Q-algebra, there is a p_α such that $\nu_A \le p_\alpha$ [6, Theorem 6.18]. Since B_β is a (Banach) C*-algebra, by [2, Proposition 3.2.3],

$$\nu_{B_\beta}([\theta(x)]^*_\beta[\theta(x)]_\beta) = q'_\beta([\theta(x)]^*_\beta[\theta(x)]_\beta) = q_\beta(\theta(x)^*\theta(x)) = q_\beta(\theta(x))^2. \quad (4.1)$$

Without loss of generality, we may assume that B is complete. By Proposition 1.1, for every β and for all $x \in A$, we have

$$\nu_{B_\beta}(([\theta(x)]^*_\beta[\theta(x)]_\beta)) \le \nu_B((\theta(x))^*\theta(x)). \quad (4.2)$$

We now consider two cases. First we assume that $n = 2k$. By Lemma 2.6, for every $x \in A$,

$$\begin{aligned}\nu_B(\theta((x^*x)^k)^{n-1}) &\le \nu_A((x^*x)^{k(n-1)}) \le p_\alpha((x^*x)^{k(n-1)}) \\ &\le (p_\alpha(x^*x))^{k(n-1)} \le p_\alpha(x)^{2k(n-1)}.\end{aligned}$$

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Also

$$\theta((x^*x)^k) = \theta(x^*x \cdots x^*x) = (\theta(x^*)\theta(x))^k = (\theta(x)^*\theta(x))^k \quad \text{for all } x \in A.$$

Hence $\lambda \in \mathrm{sp}_B((\theta(x))^*\theta(x))$ implies that $\lambda^{k(n-1)} \in \mathrm{sp}_B(\theta((x^*x)^k)^{n-1})$. Consequently,

$$\nu_B((\theta(x))^*\theta(x)) \le p_\alpha(x)^2.$$

From (4.1) and (4.2) we obtain $q_\beta(\theta(x)) \le p_\alpha(x)$.

Next we assume that $n = 2k + 1$. By Lemma 4.2 we have

$$\nu_B((\theta(x)^*\theta(x))^k) \le \nu_A((x^*x)^k) \le p_\alpha((x^*x)^k) \le p_\alpha(x)^{2k}.$$

If $\lambda \in \mathrm{sp}_B((\theta(x))^*\theta(x))$ then $\lambda^k \in \mathrm{sp}_B((\theta(x))^*\theta(x))^k$. Consequently,

$$\nu_B((\theta(x))^*\theta(x)) \le p_\alpha(x)^2.$$

Now by (4.1) and (4.2) we have

$$q_\beta(\theta(x))^2 \le \nu_B((\theta(x))^*\theta(x)) \le p_\alpha(x)^2.$$

Hence $q_\beta(\theta(x)) \le p_\alpha(x)$ for all $x \in A$. It is now clear that this inequality implies the continuity of θ at zero and hence θ is continuous on A. □

REMARK 4.4. It is clear that all results of this paper are valid for Banach algebras A and B if they have other properties required in each result.

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