



Remarks on the rigidity of CR-manifolds

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ABSTRACT

We propose a procedure to construct new smooth CR-manifolds whose local stability groups, equipped with their natural topologies, are subgroups of certain (finite-dimensional) Lie groups but not Lie groups themselves.

1. Introduction

Given a germ (M, p) of a real submanifold of \mathbb{C}^n , its basic invariant is the *local stability group* $\text{Aut}(M, p)$, i.e. the group of all germs at p of local biholomorphic maps of \mathbb{C}^n fixing p and preserving the germ (M, p) . By the work of several authors [CM74, BER97, Z97, ELZ03, LM05] it is known that this group is a (finite-dimensional) Lie group (in the natural inductive limit topology) for germs of *real-analytic* submanifolds satisfying certain nondegeneracy conditions, e.g. those having nondegenerate Levi form. On the other hand, in the absence of the nondegeneracy conditions, the group $\text{Aut}(M, p)$ can possibly be infinite-dimensional (in the sense that it contains Lie groups of arbitrarily large dimension). (For example, the local stability group of $(\mathbb{R}, 0)$ in \mathbb{C} consists of all convergent power series with real coefficients.) Furthermore, recent results in [BRWZ04] show that a similar principle also holds for *global CR-automorphisms*, both real-analytic and smooth.

One purpose of this paper is to show that, in contrast with the behaviour mentioned above, a similar alternative no longer holds for the *local stability group* of a *smooth* real submanifold. In particular, we show that, for any $n \geq 2$, there exists a germ (M, p) of a smooth strongly pseudoconvex hypersurface in \mathbb{C}^n with $\text{Aut}(M, p)$ being (topologically) isomorphic to a countable dense subgroup of the circle $S^1 \subset \mathbb{C}$ and hence not being a Lie group. In fact, $\text{Aut}(M, p)$ can be arranged to be isomorphic to any increasing countable union of finite subgroups of S^1 , for instance, to the subgroup

$$\{e^{2\pi i(l/2^m)} : l, m \in \mathbb{N}\} \subset S^1. \quad (1.1)$$

Furthermore, our construction also yields similar properties for the (generally larger) *local CR stability group* $\text{Aut}_{\text{CR}}(M, p)$, consisting of all germs at a point p of smooth CR-automorphisms of M fixing p . Recall that a germ of a smooth transformation $\varphi: (M, p) \rightarrow (M, p)$ is a *CR-automorphism* if it preserves the subbundle $T^c M$ and the restriction of its differential $d\varphi|_{T^c M}$ is \mathbb{C} -linear, where

$$T^c M := TM \cap iTM.$$

Another purpose of this paper is to provide a general construction of new smooth generic submanifolds with certain prescribed local CR stability groups (recall that a real submanifold $M \subset \mathbb{C}^n$ is *generic* if $T_q M + iT_q M = T_q \mathbb{C}^n$ for all $q \in M$). More precisely, we show the following.

THEOREM 1.1. *Let (M, p) be a germ of a smooth generic submanifold in \mathbb{C}^n of positive codimension and of finite type and assume that it is invariant under an increasing countable union G of finite*

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subgroups of $\text{Aut}(\mathbb{C}^n, p)$. Then there exists a G -invariant germ of another smooth generic submanifold (\widetilde{M}, p) of the same dimension as (M, p) , which is tangent to (M, p) of infinite order and has the following properties:

- (i) $\text{Aut}(\widetilde{M}, p) = G$;
- (ii) $\text{Aut}_{\text{CR}}(\widetilde{M}, p) = \{g|_M : g \in G\}$.

We use here the notion of finite type due to Kohn [Koh72] and Bloom–Graham [BG77]: a germ (M, p) is of *finite type* if all germs at p of smooth real vector fields on M tangent to T^cM span together with their iterated commutators the full tangent space T_pM .

We now illustrate Theorem 1.1 by an example, where it can be applied.

Example 1.2. Consider a real hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$, given in coordinates $(z_1, \dots, z_n, w) \in \mathbb{C}^{n+1}$ by

$$\text{Im } w = \varphi(|z_1|^2, z_2, \dots, z_n, \bar{z}_2, \dots, \bar{z}_n, \text{Re } w),$$

where φ is any smooth function such that $0 \in M$ and $(M, 0)$ is of finite type. Then $(M, 0)$ is clearly invariant under the rotation group consisting of all transformations $(z_1, z_2, \dots, z_n, w) \mapsto (e^{2\pi i\theta} z_1, z_2, \dots, z_n, w)$ for all real θ . Now we can take the subgroup G consisting of all of these transformations corresponding to $\theta = l/2^m$ with l, m being positive integers. Then G clearly satisfies the assumptions of Theorem 1.1. We then conclude that there exists a new real submanifold $\widetilde{M} \subset \mathbb{C}^{n+1}$ such that both $\text{Aut}(\widetilde{M}, p)$ and $\text{Aut}_{\text{CR}}(\widetilde{M}, p)$ are (topologically) isomorphic to G , which is a topological subgroup of S^1 but is not itself a Lie group. Similar examples can be obtained for other S^1 -actions or $S^1 \times S^1$ -actions or actions by more general compact groups leaving (M, p) invariant, where M can also be of any codimension.

As a remarkable consequence of Theorem 1.1 and the mentioned results [BER97, Z97], our construction provides germs of smooth generic submanifolds (even of strongly pseudoconvex hypersurfaces) that are not CR-equivalent to any germ of any real-analytic CR-manifold. Recall that (M, p) is called *finitely nondegenerate* if

$$\text{span}_{\mathbb{C}}\{L_1 \dots L_k \rho_Z^j(p) : k \geq 0, 1 \leq j \leq d\} = \mathbb{C}^n, \tag{1.2}$$

where $\rho = (\rho^1, \dots, \rho^d)$ is a defining function of M near p with $\partial\rho^1 \wedge \dots \wedge \partial\rho^d \neq 0$, ρ_Z^j denotes the complex gradient of ρ^j in \mathbb{C}^n and the span is taken over all collections of germs at p of smooth $(0, 1)$ vector fields $L_1 \dots L_k$ on M . We have:

COROLLARY 1.3. *For any germ (M, p) of a smooth generic submanifold in \mathbb{C}^n , which is finitely nondegenerate and of finite type, and which is invariant under the group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ acting on \mathbb{C}^n by multiplication, there exists a germ of another smooth generic submanifold (\widetilde{M}, p) , tangent to (M, p) of infinite order, which is not CR-equivalent to any germ of a real-analytic generic submanifold of \mathbb{C}^n .*

Indeed, since S^1 has many dense subgroups G satisfying the assumptions of Theorem 1.1 (e.g. the subgroup in (1.1)), Theorem 1.1 yields a germ (\widetilde{M}, p) whose local CR stability group is isomorphic to G and hence is not topologically isomorphic to any Lie group (and not locally compact). On the other hand, the local CR stability group of any real-analytic generic submanifold of \mathbb{C}^n , which is CR-equivalent to (\widetilde{M}, p) (and hence is also finitely nondegenerate and of finite type), is known to always be a Lie group (see [BER97] for hypersurfaces and [Z97] for higher codimension). Since the local CR stability group is a CR-invariant, (\widetilde{M}, p) cannot be CR-equivalent to any germ of a real-analytic generic submanifold of \mathbb{C}^n .

2. Jet spaces and jet groups

Here we recall the jet terminology and introduce the notation that will be used throughout the paper. Recall that, given two complex manifolds X and X' and an integer $k \geq 0$, a k -jet of a holomorphic map is an equivalence class of holomorphic maps from open neighborhoods of x in X into X' with fixed partial derivatives at x up to order k . Denote by $J_x^k(X, X')$ the set of all such k -jets. The union $J^k(X, X') := \bigcup_{x \in X} J_x^k(X, X')$ carries a natural fiber bundle structure over X . For a holomorphic map f from a neighborhood of x in X into X' , denote by $j_x^k f \in J_x^k(X, X')$ the corresponding k -jet. In local coordinates, $j_x^k f$ can be represented by the coordinates of the reference point x and all partial derivatives of f at x up to order k . If X and X' are smooth algebraic varieties, $J_x^k(X, X')$ and $J^k(X, X')$ are also of this type. We also denote by $J_{x,x'}^k(X, X')$ the space of all k -jets sending x into x' . The subset $G_x^k(X) \subset J_{x,x}^k(X, X)$ of all invertible k -jets forms an algebraic group with respect to composition. Completely analogously, k -jets of smooth maps between smooth real manifolds M and M' are defined, for which we shall use the same notation $J^k(M, M')$. The possible confusion will be eliminated by the convention that we write $X_{\mathbb{R}}$ whenever we consider a complex manifold X as a real manifold. Thus, if X and X' are complex manifolds, $J^k(X, X')$ is the space of all k -jets of holomorphic maps and $J^k(X_{\mathbb{R}}, X'_{\mathbb{R}})$ is the space of all k -jets of smooth maps.

Furthermore, we shall need k -jets of real submanifolds of fixed dimension of a smooth real manifold M . Let $\mathcal{C}_x^{k,m}(M)$ be the set of all germs at x of real C^k -smooth m -dimensional submanifolds of M through x . We say that two germs $V, V' \in \mathcal{C}_x^{k,m}(M)$ are k -equivalent, if, in a local coordinate neighborhood of x of the form $U_1 \times U_2$, V and V' can be given as graphs of smooth maps $\varphi, \varphi' : U_1 \rightarrow U_2$ such that $j_{x_1}^k \varphi = j_{x_1}^k \varphi'$, where $x = (x_1, x_2) \in U_1 \times U_2$. Denote by $J_x^{k,m}(M)$ the set of all k -equivalence classes of $\mathcal{C}_x^{k,m}(M)$ and by $J^{k,m}(M)$ the union $\bigcup_{x \in M} J_x^{k,m}(M)$ with the natural fiber bundle structure over M . Furthermore, for any real C^k -smooth m -dimensional submanifold $V \subset M$ through x , denote by $j_x^k(V) \in J_x^{k,m}(M)$ the corresponding k -jet. The space $J_x^{k,m}(M)$ carries a natural real (nonsingular) algebraic variety structure.

We now introduce the notions of equivalence and rigidity that will be crucial in the sequel.

DEFINITION 2.1.

- (1) Two k -jets of real submanifolds of the same dimension $\Lambda_j \in J_{p_j}^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$, $j = 1, 2$, are called *biholomorphically equivalent* if there exists a germ of a biholomorphic map $(\mathbb{C}^n, p_1) \rightarrow (\mathbb{C}^n, p_2)$ sending Λ_1 to Λ_2 .
- (2) A C^k -smooth generic submanifold $M \subset \mathbb{C}^n$ is called *totally rigid of order k* , if for any $p_1 \neq p_2 \in M$, the jets $j_{p_1}^k(M)$ and $j_{p_2}^k(M)$ are not biholomorphically equivalent in the sense of part (1).
- (3) A k -jet $\Lambda \in J_p^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$ is called *totally rigid* if any C^k -smooth submanifold passing through p and having Λ as its k -jet at p , contains a neighborhood of p that is totally rigid of order $k - 1$ in the sense of part (2).

Example 2.2. Any 0-jets of real submanifolds at p are obviously biholomorphically equivalent and 1-jets are equivalent if and only if their CR-dimensions at p are the same. Two 2-jets of generic submanifolds are equivalent if and only if their Levi forms at p are linearly equivalent (e.g. of the same rank and signature in the hypersurface case). Furthermore, it follows from the Chern–Moser theory [CM74] that two k -jets of Levi-nondegenerate hypersurfaces of the same signature are always biholomorphically equivalent for $k \leq 5$ in the case $n = 2$ and for $k \leq 3$ in the case $n > 2$, but may not be equivalent in general for larger k .

It is crucial for our method to consider the total rigidity of order $k - 1$ in part (3) (rather than, e.g., of order k) for the representing submanifolds M with $j_p^k(M) = \Lambda$. This allows us to achieve the total rigidity of M of order $k - 1$ near p by ensuring that the first-order derivatives of $j_q^{k-1}(M)$

at p as functions of $q \in M$ have suitable transversality property with respect to the submanifolds (orbits) of biholomorphically equivalent $(k - 1)$ -jets (see the proof of Proposition 2.3 below for more details). Thus, we need Λ to be of higher order than $k - 1$ to include the extra derivatives. More precisely, the existence of totally rigid jets is guaranteed by the following statement.

PROPOSITION 2.3. *For fixed integers $n < m < 2n$, a point $p \in \mathbb{C}^n$ and sufficiently large k (depending on n and m but not on p), the set of all totally rigid k -jets in $J_p^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$ contains an open dense subset.*

In fact we show that the number k in Proposition 2.3 can be chosen such that the following inequality holds:

$$(2n - m) \binom{k + m - 1}{m} - 1 - 2n \binom{k + n - 1}{n} - 1 \geq m. \tag{2.1}$$

The proof will be based on the following lemmas (of which the first is standard and provided with a short proof for the reader’s convenience). We write $\|\cdot\|_{C^l}$ for the standard C^l norm.

LEMMA 2.4. *Let $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a smooth map and V be an open neighborhood in \mathbb{R}^n of a point $a \in \mathbb{R}^n$. Then for any $\varepsilon > 0$ and any integers $0 \leq k \leq l$, there exists $\delta > 0$ such that, if $\Lambda \in J_a^k(\mathbb{R}^n, \mathbb{R}^m)$ is a k -jet with $\|\Lambda - j_a^k \varphi\| < \delta$, then there exists another smooth map $\tilde{\varphi}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $j_a^k \tilde{\varphi} = \Lambda$, $\tilde{\varphi}|_{\mathbb{R}^n \setminus V} = \varphi|_{\mathbb{R}^n \setminus V}$ and $\|\tilde{\varphi} - \varphi\|_{C^l} < \varepsilon$.*

Proof. Without loss of generality, V is bounded. We shall look for a map $\tilde{\varphi}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ of the form

$$\tilde{\varphi}(x) := \varphi(x) + \chi(x) \cdot (\psi(x) - \varphi(x)), \tag{2.2}$$

where $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a smooth map with $j_a^k \psi = \Lambda$ and $\chi: \mathbb{R}^n \rightarrow \mathbb{R}$ is a fixed smooth function, which is 1 in a neighborhood of a and 0 outside V . Then $j_a^k \tilde{\varphi} = \Lambda$ and $\tilde{\varphi}|_{\mathbb{R}^n \setminus V} = \varphi|_{\mathbb{R}^n \setminus V}$. Furthermore, there exists $C > 0$ depending on χ but not on ψ such that

$$\|\tilde{\varphi} - \varphi\|_{C^l} < C \|\psi - \varphi\|_{C^l}.$$

If δ is sufficiently small and Λ satisfies our assumption, we can always choose ψ with $\|\psi - \varphi\|_{C^l} < \varepsilon/C$ on the closure \bar{V} . Then the map $\tilde{\varphi}$ given by (2.2) satisfies the required properties. \square

LEMMA 2.5. *Let $k, l, s \geq 0$ and $0 \leq r \leq m$ be any integers, $U \subset J^l(\mathbb{R}^m, \mathbb{R}^s)$ be an open set and $F: U \rightarrow \mathbb{R}^r$ be a smooth map of constant rank r . Then, for any nonempty open set $B \subset \mathbb{R}^m$ and a smooth map $f: B \rightarrow \mathbb{R}^s$ satisfying $j_x^l f \in U$ for all $x \in B$, there exists another nonempty open subset $\hat{B} \subset B$ and another smooth map $\hat{f}: \hat{B} \rightarrow \mathbb{R}^s$ such that the following holds:*

- (1) *the C^k norm of $\hat{f} - (f|_{\hat{B}})$ can be chosen arbitrarily small;*
- (2) *$j_x^l \hat{f} \in U$ for all $x \in \hat{B}$;*
- (3) *the map $x \in \hat{B} \mapsto F(j_x^l \hat{f}) \in \mathbb{R}^r$ is also of constant rank r .*

Proof. Without loss of generality, the integer k (used only in property (1)) is at least l . We prove the lemma by induction on r . For $r = 0$ the statement is trivial. Suppose it holds for any $r < r_0 \leq m$ and we are given a map $F: U \rightarrow \mathbb{R}^{r_0}$ as in the lemma. Consider the standard splitting $\mathbb{R}^{r_0} = \mathbb{R}^{r_0-1} \times \mathbb{R}$ and the corresponding components $F_1: U \rightarrow \mathbb{R}^{r_0-1}$ and $F_2: U \rightarrow \mathbb{R}$ of F (so that $F = (F_1, F_2)$). Then the induction assumption for F_1 yields a map $\hat{f}: \hat{B} \rightarrow \mathbb{R}^s$ such that $\|\hat{f} - (f|_{\hat{B}})\|_{C^k}$ is arbitrarily small, $j_x^l \hat{f} \in U$ for all $x \in \hat{B}$ and the map $\Phi_1(x) := F_1(j_x^l \hat{f}) \in \mathbb{R}^{r_0-1}$ is of constant rank $r_0 - 1 < m$ for $x \in \hat{B}$. By shrinking \hat{B} if necessary, we may assume it is connected and of the form $\hat{B} = \hat{B}_1 \times \hat{B}_2 \subset \mathbb{R}^{m-r_0+1} \times \mathbb{R}^{r_0-1}$ and such that, for some $c \in \mathbb{R}^{r_0-1}$, the level set $C := \{x \in \hat{B} : \Phi_1(x) = c\}$ is a graph of a smooth map $\varphi: \hat{B}_1 \rightarrow \hat{B}_2$.

We now consider two points $x_1 \neq x_2 \in C$ and look for a small perturbation \tilde{f} of \hat{f} and \tilde{x}_2 of x_2 such that x_1 and \tilde{x}_2 still belong to the same level set \tilde{C} of $\tilde{\Phi}_1(x) := F_1(j_x^l \tilde{f})$ but $F_2(j_{x_1}^l \tilde{f}) \neq F_2(j_{\tilde{x}_2}^l \tilde{f})$. More precisely, suppose that $F(j_{x_1}^l \hat{f}) = F(j_{x_2}^l \hat{f})$ (otherwise no perturbation is needed). By the assumption of the lemma, the map $F = (F_1, F_2)$ has constant rank r_0 in U . Since $r_0 \leq m \leq \dim U$, we can find a point $\tilde{x}_2 \in \hat{B}$ and a jet $\Lambda \in U \cap J_{x_2}^l(\mathbb{R}^m, \mathbb{R}^s)$ arbitrarily close to $j_{x_2}^l \hat{f}$ such that $F_1(\Lambda) = F_1(j_{x_2}^l \hat{f})$ but $F_2(\Lambda) \neq F_2(j_{x_2}^l \hat{f})$. Since Λ can be chosen arbitrarily close to $j_{x_2}^l \hat{f}$, it can be represented by a smooth map \tilde{f} that is arbitrarily close to \hat{f} in the C^k norm and differs from it on a neighborhood of x_2 with compact support in $\hat{B} \setminus \{x_1\}$ in view of Lemma 2.4. By choosing the norm $\|\tilde{f} - \hat{f}\|_{C^k}$ sufficiently small, we shall preserve the properties that $j_x^l \tilde{f} \in U$ for all $x \in \hat{B}$, the rank of $\tilde{\Phi}_1(x) := F_1(j_x^l \tilde{f})$ is still $r_0 - 1$ and the level set $\tilde{C} := \{x \in \hat{B} : \tilde{\Phi}_1(x) = c\}$ is still a graph of a smooth map $\tilde{\varphi} : \hat{B}_1 \rightarrow \hat{B}_2$. Hence, \tilde{C} is a connected manifold containing two points $x_1, \tilde{x}_2 \in \tilde{C}$ such that $F_2(j_{x_1}^l \tilde{f}) \neq F_2(j_{\tilde{x}_2}^l \tilde{f})$. The latter fact implies that the function $\tilde{\Phi}_2(x) := F_2(j_x^l \tilde{f})$ is not constant on \tilde{C} and therefore its differential is somewhere nonzero. Putting this property together with the rank property of $\tilde{\Phi}_1$, we conclude that the rank of $\tilde{\Phi}(x) := F(j_x^l \tilde{f})$ is r_0 at some point $x_0 \in \hat{B}$. The required conclusion is obtained by replacing \hat{f} with \tilde{f} and \hat{B} with a sufficiently small open neighborhood of x_0 . \square

Proof of Proposition 2.3. We may assume $p = 0$. Consider the natural action of the group $G_0^{k-1}(\mathbb{C}^n)$ (consisting of all $(k - 1)$ -jets at 0 of local biholomorphic maps of \mathbb{C}^n) on the space $J_0^{k-1,m}(\mathbb{C}_{\mathbb{R}}^n)$ (consisting of all $(k - 1)$ -jets at 0 of real m -dimensional submanifolds of $\mathbb{C}_{\mathbb{R}}^n$ passing through 0). The dimensions of the jet spaces can be computed directly:

$$\dim_{\mathbb{R}} G_0^{k-1}(\mathbb{C}^n) = 2n \binom{k+n-1}{n} - 1, \quad \dim_{\mathbb{R}} J_0^{k-1,m}(\mathbb{C}_{\mathbb{R}}^n) = (2n - m) \binom{k+m-1}{m} - 1. \tag{2.3}$$

Hence, the inequality (2.1) is equivalent to

$$\dim_{\mathbb{R}} J_0^{k-1,m}(\mathbb{C}_{\mathbb{R}}^n) - \dim_{\mathbb{R}} G_0^{k-1}(\mathbb{C}^n) \geq m. \tag{2.4}$$

In particular, for any sufficiently large k , all orbits of $G_0^{k-1}(\mathbb{C}^n)$ in $J_0^{k-1,m}(\mathbb{C}_{\mathbb{R}}^n)$ have their (real) codimension at least m . In the rest of the proof we shall assume that (2.4) is satisfied.

It is easy to see that $G_0^{k-1}(\mathbb{C}^n)$ is an algebraic group acting rationally on $J_0^{k-1,m}(\mathbb{C}_{\mathbb{R}}^n)$ by calculating the group operation and the action in local coordinates. Hence, the orbits of $G_0^{k-1}(\mathbb{C}^n)$ form, on an open dense subset $\Omega \subset J_0^{k-1,m}(\mathbb{C}_{\mathbb{R}}^n)$, a foliation into real submanifolds of a fixed constant codimension $\geq m$. Consider any k -jet $\Lambda_0 \in J_0^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$ represented by the graph of a C^∞ -smooth map $\varphi_0 : \mathbb{R}^m \rightarrow \mathbb{R}^{2n-m}$ with $\varphi_0(0) = 0$, where we choose a suitable identification of $\mathbb{C}_{\mathbb{R}}^n$ with $\mathbb{R}^m \times \mathbb{R}^{2n-m}$ (after a possible permutation of the real coordinates). Thus, $\Lambda_0 = j_0^k((\text{id} \times \varphi_0)(\mathbb{R}^m))$. By the density of Ω , we can find another C^∞ -smooth map $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^{2n-m}$ with $\varphi(0) = 0$ and $\Theta := j_0^k \varphi$ arbitrarily close to $j_0^k \varphi_0$ such that the $(k - 1)$ -jet $\Lambda \in J_0^{k-1,m}(\mathbb{C}_{\mathbb{R}}^n)$ at 0 of the graph of φ is contained in Ω . (By this choice, Λ is also arbitrarily close to Λ_0 .) We can now find an open neighborhood U of $j_0^{k-1} \varphi$ in $J^{k-1}(\mathbb{R}^m, \mathbb{R}^{2n-m})$ and a smooth map $F : U \rightarrow \mathbb{R}^m$ of constant rank m and constant on the orbits (recall that m does not exceed the orbit codimension), such that two $(k - 1)$ -jets $\Lambda_j \in J_{(x_j, x'_j)}^{k-1,m}(\mathbb{C}_{\mathbb{R}}^n)$, $j = 1, 2$, near Λ_0 (where $(x_j, x'_j) \in \mathbb{R}^m \times \mathbb{R}^{2n-m}$), which are represented by graphs of some smooth maps $\varphi_1, \varphi_2 : \mathbb{R}^m \rightarrow \mathbb{R}^{2n-m}$, are biholomorphically inequivalent (in the sense of Definition 2.1) whenever $F(j_{x_1}^{k-1} \varphi_1) \neq F(j_{x_2}^{k-1} \varphi_2)$. Here F can be obtained by taking the first m coordinates in any real coordinate system $(x, y) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, for which the orbits are given by $x = \text{constant}$. Then Lemma 2.5 can be applied to $U, F, f := \varphi$ and an arbitrarily small neighborhood of $B \subset \mathbb{R}^m$ of 0. Let \hat{B} and $\hat{f} : \hat{B} \rightarrow \mathbb{R}^{2n-m}$ be given by the lemma.

We claim that, for any $x_0 \in \widehat{B}$, the k -jet $\Lambda(x_0) \in J^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$ of the graph of \widehat{f} at $(x_0, \widehat{f}(x_0))$ is totally rigid in the sense of Definition 2.1(3). Indeed, fix any $x_0 \in \widehat{B}$ and consider any C^k -smooth real m -dimensional submanifold $V \subset \mathbb{C}_{\mathbb{R}}^n$ passing through $(x_0, \widehat{f}(x_0))$ with $j_{(x_0, \widehat{f}(x_0))}^k(V) = \Lambda(x_0)$. By shrinking V , if necessary, we may assume that V is a graph of a smooth map $g: B(x_0) \rightarrow \mathbb{R}^{2n-m}$, where $B(x_0)$ is a suitable open neighborhood of x_0 . Then $j_{x_0}^k g = j_{x_0}^k \widehat{f}$ and, therefore, the ranks of the maps $x \mapsto F(j_x^{k-1}g)$ and $x \mapsto F(j_x^{k-1}\widehat{f})$ coincide at $x = x_0$. (Here is the step, where we use the different integers k and $k - 1$ for x_0 and points nearby respectively.) By property (3) in Lemma 2.5, the rank of the second map is m and, hence, so is the rank of the first map. However, the latter fact implies that $F(j_{x_1}^{k-1}g) \neq F(j_{x_2}^{k-1}g)$ for any $x_1 \neq x_2$ sufficiently close to x_0 . In view of the choice of F , it follows that the jets $j_{(x_1, g(x_1))}^{k-1}(V)$ and $j_{(x_2, g(x_2))}^{k-1}(V)$ are biholomorphically inequivalent for any such $x_1 \neq x_2$, which is precisely what is needed to show that $\Lambda(x_0)$ is totally rigid. It remains to observe that any translation of $\Lambda(x_0)$ is also totally rigid, hence we can also find totally rigid k -jets in $J_0^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$ arbitrarily close to the original jet Λ_0 . The proof is complete. \square

3. Realization of certain groups as CR stability groups

In the sequel we shall use the same letter for a germ and its representative unless there will be a danger of confusion. We begin with a standard lemma, whose proof is given here for the reader's convenience.

LEMMA 3.1. *Let $(G_k)_{k \geq 1}$ be an increasing sequence of finite groups of germs of local biholomorphic maps of \mathbb{C}^n in a neighborhood of a point p fixing that point. Let M be a smooth real submanifold of \mathbb{C}^n passing through p . Let $(D_k)_{k \geq 1}$ be a sequence of domains containing p and such that, for each k , the germs from G_k can be represented by biholomorphic self-maps of D_k . Then there exist a sequence of points $p_k \in M$, $k \geq 1$, converging to p and a sequence of mutually disjoint open neighborhoods V_k of p_k in \mathbb{C}^n such that $\overline{V}_k \subset D_k$ and, if $g(\overline{V}_k) \cap \overline{V}_l \neq \emptyset$ for some k, l and $g \in G_k$, then necessarily $k = l$ and $g \equiv \text{id}$.*

Note that the existence of domains D_k easily follows from the finiteness of each G_k . Indeed, if \widetilde{D}_k is any domain where all germs from G_k biholomorphically extend, it suffices to take $D_k := \bigcap_{g \in G_k} g(\widetilde{D}_k)$.

Proof. We shall construct $p_k \in M$ and $V_k \subset \mathbb{C}^n$ inductively. Let $k = 1$. Since G_1 is finite, the set of points $x \in D_1$, such that there are two elements $g_1 \neq g_2 \in G_1$ with $g_1(x) = g_2(x)$, is a complement of a proper analytic subset. Hence, we can choose $p_1 \in M$ and a neighborhood $V_1 \subset\subset D_1$ of p_1 in \mathbb{C}^n with $p \notin \overline{V}_1$ such that $g(\overline{V}_1) \cap \overline{V}_1 \neq \emptyset$ for $g \in G_1$ implies $g \equiv \text{id}$.

Now suppose that p_l and V_l with $p \notin \overline{V}_l$ have been chosen for all $l < k$. Since G_k is finite, we can choose a neighborhood U of p in D_k such that $g(U) \cap \overline{V}_l = \emptyset$ for all $l < k$ and $g \in G_k$. Using the same argument as before we can choose p_k arbitrarily close to p and V_k with $p \notin \overline{V}_k$ such that $p_k \in V_k \subset \overline{V}_k \subset U$ and, for any $g \neq \text{id} \in G_k$, $g(\overline{V}_k) \cap \overline{V}_k = \emptyset$. Since $\overline{V}_k \subset U$ and $g(U) \cap \overline{V}_l = \emptyset$ for all $l < k$ and $g \in G_k$, it follows that $g(\overline{V}_k) \cap \overline{V}_l \neq \emptyset$ can hold for some $l \leq k$ and $g \in G_k (\supset G_l)$ if and only if $k = l$ and $g \equiv \text{id}$. It is easy to see that the sequences (p_k) and (V_k) so constructed satisfy the required properties. \square

Proof of Theorem 1.1. Since the statement is local, we fix an identification $\mathbb{C}_{\mathbb{R}}^n \cong \mathbb{R}^m \times \mathbb{R}^{2n-m}$ near p such that M is represented by the graph of a smooth map $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^{2n-m}$ with $\|\varphi\|_{C^1}$ sufficiently small. We shall write $B_r(a)$ for the open ball with center a and radius r with respect to the product metric of the Euclidean metrics of \mathbb{R}^m and \mathbb{R}^{2n-m} . With this choice of the metric on $\mathbb{R}^m \times \mathbb{R}^{2n-m}$ and $\|\varphi\|_{C^1}$ sufficiently small, we have the property that, for any $a \in M$, the intersection $B_r(a) \cap M$

coincides with the graph of φ over the projection of $B_r(a)$ to \mathbb{R}^m (which is an Euclidean ball in \mathbb{R}^m). In the course of the proof we shall consider small perturbations of M obtained as graphs of small perturbations of φ . We shall always assume that the C^1 norms of these perturbations are still small, so that the mentioned relation between ball intersections with their graphs and graphs over balls still holds.

By the assumption, $G = \bigcup_k G_k$, where $G_k, k \geq 1$, is an increasing sequence of finite groups of local biholomorphic maps of \mathbb{C}^n in a neighborhood of p , fixing p and preserving the germ (M, p) . As indicated above, we can choose a decreasing sequence of open neighborhoods D_k of p in the unit ball $B_1(p)$ in \mathbb{C}^n centered at p , such that, for each k , all germs in G_k can be represented by biholomorphic self-maps of a neighborhood of \overline{D}_k . In the sequel we shall identify the germs from G_k with their biholomorphic representatives.

Let $p_k \in D_k \cap M$ and V_k be given by Lemma 3.1. Moreover, we can make the above choice of p_k, D_k, V_k inductively such that, in addition,

$$\max \left(\sup_{z \in D_{k+1}, g \in G_{k+1}} \|g'(z)\|, \sup_{z \in D_k, g \in G_k} \|g'(z)\| \right) \frac{\sup_{z \in V_{k+1}} |z - p|}{\inf_{z \in V_k} |z - p|} \rightarrow 0, \quad k \rightarrow \infty, \tag{3.1}$$

where g' is the Jacobian matrix of g .

For an l -jet $\Lambda \in J_x^{l,m}(\mathbb{C}^n_{\mathbb{R}})$, we denote by $\text{Orb}(\Lambda) \subset J^{l,m}(\mathbb{C}^n_{\mathbb{R}})$ the set of all l -jets $\tilde{\Lambda} \in J_y^{l,m}(\mathbb{C}^n_{\mathbb{R}})$ for all $y \in \mathbb{C}^n_{\mathbb{R}}$ that are biholomorphically equivalent to Λ (in the sense of Definition 2.1). Then it follows from (2.3) that, for l sufficiently large, the subset $\bigcup_{x \in M \cap D_1} \text{Orb}(j_x^l M) \subset J^{l,m}(\mathbb{C}^n_{\mathbb{R}})$ has Lebesgue measure zero and therefore its complement is dense in $J^{l,m}(\mathbb{C}^n_{\mathbb{R}})$. The same argument obviously applies to any other real submanifold of \mathbb{C}^n of the same dimension as M . We shall use this property to choose jets that are not in the certain unions of orbits. We shall consider l sufficiently large so that this choice is always possible.

We next consider a sequence $\varepsilon_k, 0 < \varepsilon_k < 1$, converging to 0 and such that

$$\varepsilon_k \left(\sup_{x \in D_k, g \in G_k} \|g'(x)\| \right) \rightarrow 0, \quad k \rightarrow \infty. \tag{3.2}$$

Then, as a consequence of Lemma 2.4 and Proposition 2.3, we can find a sequence of neighborhoods U_k of p_k in V_k with $\overline{U}_k \subset V_k$ and a sequence of graphs N_k of smooth maps $\varphi_k: \mathbb{R}^m \rightarrow \mathbb{R}^{2n-m}$ with

$$\|\varphi_k - \varphi\|_{C^k} < \varepsilon_k, \quad N_k \setminus \overline{U}_k = M \setminus \overline{U}_k, \tag{3.3}$$

and such that the following holds. There exist points $q_k \in N_k$ and real numbers $\delta_k > 0$ such that, for each k , the $2\delta_k$ -neighborhood of q_k in $N_k, B_{2\delta_k}(q_k) \cap N_k$, is contained in U_k , totally rigid (in the sense of Definition 2.1) and

$$j_{q_k}^l N_k \notin \bigcup_{x \in M \cap D_1} \text{Orb}(j_x^l M). \tag{3.4}$$

As the next step, we define $X_k \subset N_k \cap U_k$ to be the subset of all points $y \neq q_k$ for which there exists a CR-diffeomorphism from $B_{\varepsilon_k \delta_k}(y) \cap N_k$ into $B_{\delta_k}(q_k) \cap N_k$, sending y into q_k . Since the CR-manifold $B_{2\delta_k}(q_k) \cap N_k$ is totally rigid by our construction, it is clear that for any $y_1 \neq y_2 \in X_k$, the neighborhood $B_{\varepsilon_k \delta_k}(y_1) \cap N_k$ cannot contain y_2 . Hence, the neighborhoods $B_{\varepsilon_k \delta_k/2}(y) \cap N_k, y \in X_k$, do not intersect and so X_k must be a finite set. It is also clear that $X_k \cap B_{2\delta_k}(q_k) \cap N_k = \emptyset$ again by the total rigidity of $B_{2\delta_k}(q_k) \cap N_k$. Furthermore, we must have $X_k \subset N_k \cap U_k$ in view of (3.3) and (3.4).

We next choose a sequence $\eta_k, 0 < \eta_k < \varepsilon_k \delta_k$ and again apply Lemma 2.4 to obtain a sequence of graphs M_k of smooth maps $\psi_k: \mathbb{R}^m \rightarrow \mathbb{R}^{2n-m}$ with

$$\|\psi_k - \varphi_k\|_{C^k} < \varepsilon_k \tag{3.5}$$

and finite subsets $\tilde{X}_k \subset M_k$ with

$$j_y^l M_k \not\subset \bigcup_{x \in N_k \cap D_1} \text{Orb}(j_x^l N_k) \quad \forall y \in \tilde{X}_k, \tag{3.6}$$

and such that, for $W_k := \bigcup_{y \in \tilde{X}_k} B_{\eta_k}(y)$,

$$X_k \cap M_k = \emptyset, \quad M_k \setminus \overline{W}_k = N_k \setminus \overline{W}_k, \quad \overline{W}_k \cap B_{\delta_k}(q_k) = \emptyset. \tag{3.7}$$

We may in addition assume that η_k is sufficiently small so that $\overline{W}_k \subset V_k$.

Finally, we define the new generic submanifold $\tilde{M} \subset \mathbb{C}^n$ by replacing $g(M \cap V_k)$ with $g(M_k \cap V_k)$ for every sufficiently large k and every $g \in G_k$, i.e.

$$\tilde{M} := \left(M \setminus \bigcup_{k \geq k_0, g \in G_k} g(M \cap V_k) \right) \cup \left(\bigcup_{k \geq k_0, g \in G_k} g(M_k \cap V_k) \right), \tag{3.8}$$

where k_0 is sufficiently large. (Note that all neighborhoods $g(V_k)$, $g \in G_k$, $k \geq k_0$, are disjoint together with their closures since they are given by Lemma 3.1.) Then \tilde{M} is a smooth submanifold through p and, if ε_k have been chosen sufficiently rapidly converging to 0, \tilde{M} is tangent to M of infinite order at p in view of (3.3) and (3.5). Consequently \tilde{M} is also of finite type at p . Furthermore, the germ (\tilde{M}, p) is clearly invariant under the action of G , i.e. the group $\text{Aut}_{\text{CR}}(\tilde{M}, p)$ of germs at p of all local CR-automorphisms of M fixing p contains G .

We now claim that $\text{Aut}_{\text{CR}}(\tilde{M}, p) = G$. Indeed, fix any $f \in \text{Aut}_{\text{CR}}(\tilde{M}, p)$ and its representative defined in some neighborhood of p in \tilde{M} , denoted by the same letter. Then for k sufficiently large, f is defined in $D_k \cap \tilde{M}$ with $f(D_k \cap \tilde{M}) \subset D_1$. By the construction, each $q_k \in N_k \cap U_k$ is not contained in W_k , hence it is in $M_k \cap V_k$ and therefore in \tilde{M} so that we can evaluate $f(q_k)$. Then (3.4) implies that $f(q_k) \notin M$ and hence $f(q_k) \in g(U_s) \subset g(V_s)$ for some s and some $g \in G_s$. Thus, we have the estimates

$$\left(\sup_{D_s} \|(g^{-1})'\| \right)^{-1} \frac{\inf_{z \in V_s} |z - p|}{\sup_{z \in V_k} |z - p|} \leq \frac{|f(q_k) - p|}{|q_k - p|} \leq \sup_{D_s} \|g'\| \frac{\sup_{z \in V_s} |z - p|}{\inf_{z \in V_k} |z - p|}. \tag{3.9}$$

On the other hand, since f is a local diffeomorphism of \tilde{M} fixing p , there exist constants $0 < c < C$ such that

$$c \leq \frac{|f(z) - p|}{|z - p|} \leq C, \quad c \leq \|f'(z)\| \leq C \tag{3.10}$$

for $z \neq p \in \tilde{M}$ sufficiently close to p . Then if k is sufficiently large, we must have $s = k$ in view of (3.1). Hence, $f(q_k) \in g_k(U_k)$ for suitable $g_k \in G_k$. Setting $f_k := g_k^{-1} \circ f \in \text{Aut}_{\text{CR}}(M, p)$, we have $f_k(q_k) \in U_k \cap \tilde{M}$. In view of (3.8), this means $f_k(q_k) \in U_k \cap M_k$.

We now claim that, for k sufficiently large, we must have $B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap \tilde{M} \subset N_k$. Indeed, otherwise, in view of (3.7), we would have $W_k \cap B_{\varepsilon_k \delta_k}(f_k(q_k)) \neq \emptyset$ and, since $\eta_k < \varepsilon_k \delta_k$, it would imply $\tilde{X}_k \cap B_{2\varepsilon_k \delta_k}(f_k(q_k)) \neq \emptyset$. However, in view of (3.2) and (3.10), this would mean that, for k sufficiently large and some point $y \in \tilde{X}_k$, the inclusion $f_k^{-1}(y) \in B_{\delta_k}(q_k) \cap \tilde{M}$ would hold. By our construction, $B_{\delta_k}(q_k) \cap \tilde{M} \subset N_k$ and hence we would have a contradiction with (3.6). Hence, we have $B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap \tilde{M} \subset N_k$ as claimed. Thus, $B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap \tilde{M} = B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap N_k$. Again, using (3.2) and (3.10), we conclude that, for k sufficiently large, f_k^{-1} sends $B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap N_k$ into $B_{\delta_k}(q_k) \cap N_k$. By our construction of the set $X_k \subset N_k$, the latter conclusion means either $f_k(q_k) \in X_k$ or $f_k(q_k) = q_k$. The first case is impossible in view of the first condition in (3.7). Hence, we have $f_k(q_k) = q_k$ and, since a neighborhood of q_k in M_k is totally rigid, this means $f_k \equiv \text{id}$ in a neighborhood of q_k . Since (\tilde{M}, p) is of finite type, we have $f_k \equiv \text{id}$ as germs at p , and hence $f \equiv g_k \in G$ implying the desired conclusion. \square

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