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FINITE TYPE IMMERSIONS OF FLAT TORI INTO EUCLIDEAN SPACES

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We prove the existence of nontrivial k-type surfaces by constructing k-type immersions of flat tori in $\mathbb{E}^6$ which are *not* product immersions.

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Minimal submanifolds of a Euclidean space are contained in a much larger class of submanifolds, namely in the class of submanifolds of finite type. Submanifolds of finite type were introduced about a decade ago by B.-Y. Chen in [**4**]; the first results on this subject have been collected in the books [**4, 5**]; for recent surveys, see [**6**, **7**].

Let M^n be an n-dimensional, connected submanifold of the Euclidean space $\mathbb{E}^m$. Denote by Δ the Laplace operator on M^n, with respect to the Riemannian metric g on M^n, induced from the Euclidean metric of the ambient space $\mathbb{E}^m$. M^n is said to be of finite type if each component of the position vector field X of M^n in $\mathbb{E}^m$, can be written as a finite sum of eigenfunctions of the Laplacian operator, that is, if

$$X = c + \sum_{i=1}^{k} X_i \tag{1}$$

where c is a constant vector, and $X_1, \ldots, X_k$ are nonconstant maps satisfying $\Delta X_i = \lambda_i X_i$, for $i = 1, \ldots, k$.

The class of finite type submanifolds is very large. For instance, minimal submanifolds of a Euclidean space, and minimal submanifolds of a hypersphere are of 1-type, and compact, homogeneous submanifolds, equivariantly immersed, are of finite type [**4**]. Therefore the following problem seems to be very interesting:

Problem. Classify all finite type submanifolds of Euclidean spaces.

Far from being solved in general, there exist quite a lot of partial results which contribute to the solution of this problem. For example there are several results on

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finite type curves [13]; in particular, for finite type curves we have the following theorem (†): for all $k\in\mathbb{N}$, k-type curves exist in $\mathbb{E}^3$ [8]. In his list [6] of open problems and conjectures concerning submanifolds of finite type, B.-Y. Chen gives a survey of results on the classification of hypersurfaces, and the classification of hypersurfaces of hyperspheres. In [6] B.-Y. Chen gives also a good survey of what is known about the classification of 2-type (spherical) submanifolds with arbitrary codimension. [6] contains also an extensive bibliography on the subject of finite type submanifolds.

However, aside from finite type curves, all explicit examples of finite type submanifolds are of 1-type, 2-type, or 3-type, and/or product immersions. In particular, for higher dimensional submanifolds, an analogue to the theorem (†) for curves of finite type is missing. Therefore we can ask the following question:

Question. What is the lowest-dimensional Euclidean space $\mathbb{E}^p$ for which (nontrivial) k-type submanifolds M^n of dimension $n\geqq 2$ exist for any $k\in\mathbb{N}$?

This paper gives a partial answer to this question by showing that for surfaces ($n=2$) the dimension p of the ambient space is not higher than 6, and contributes to the solution of the problem by constructing explicit examples of k-type surfaces for arbitrary $k\in\mathbb{N}$. We prove the following theorem:

Theorem. *For all $k\in\mathbb{N}$, k-type surfaces exist in $\mathbb{E}^6$, which are* **not** *product immersions.*

In order to prove the theorem, we construct finite type isometrical immersions of flat tori into $\mathbb{E}^6$. For all $k\in\mathbb{N}$, we obtain compact k-type surfaces which lie fully in $\mathbb{E}^6$; the surfaces are nonspherical in general, i.e. not contained in $S^5\subseteq\mathbb{E}^6$, and are *not* product immersions.

We consider the flat torus $T^2=\mathbb{R}^2/\Lambda$ is a lattice

$$\Lambda=\{(2\pi ka, 2\pi lb)\,|\,k,l\in\mathbb{Z}\}$$

where a, and b are real numbers $a,b>0$. If (u,v) are Euclidean coordinates on $\mathbb{R}^2$, then the Laplacian Δ is given by

$$\Delta=-\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right).$$

The eigenfunctions of Δ are given by

$$\left\{\cos\frac{mu}{a}\cos\frac{nv}{b}, \cos\frac{mu}{a}\sin\frac{nv}{b}, \sin\frac{mu}{a}\cos\frac{nv}{b}, \sin\frac{mu}{a}\sin\frac{nv}{b}\,\middle|\, m,n\in\mathbb{Z}\right\}$$

and the spectrum of Δ is given by

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$$\mathrm{Spec}(T^2)=\left\{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2 \,\middle|\, m,n\in\mathbb{Z}\right\}.$$

We consider now an immersion ϕ of the torus T^2 into the Euclidean space $\mathbb{E}^p$; the position vector $\mathbf{r}(u,v)$ of ϕ in $\mathbb{E}^p$ can be decomposed in terms of the eigenfunctions of Δ

$$\begin{aligned}\mathbf{r}=\sum_{m,n}\mathbf{A}_{mn}\cos\frac{mu}{a}\cos\frac{nv}{b}+\mathbf{B}_{mn}\cos\frac{mu}{a}\sin\frac{nv}{b}\\ +\mathbf{C}_{mn}\sin\frac{mu}{a}\cos\frac{nv}{b}+\mathbf{D}_{mn}\sin\frac{mu}{a}\sin\frac{nv}{b}\end{aligned}\tag{2}$$

where $\mathbf{A}_{mn}, \mathbf{B}_{mn}, \mathbf{C}_{mn}$, and $\mathbf{D}_{mn}\in\mathbb{R}^{p\times 1}$.

We notice that if the decomposition (2) contains only a finite number of terms, and the immersion ϕ is isometrical, then the immersed torus $\mathcal{T}^2=\phi(T^2)$ will be of finite type. In order ϕ to be isometrical, we require

$$\begin{cases}\langle\mathbf{r}_u,\mathbf{r}_u\rangle=\langle\mathbf{r}_v,\mathbf{r}_v\rangle=1\\ \langle\mathbf{r}_u,\mathbf{r}_v\rangle=\langle\mathbf{r}_v,\mathbf{r}_u\rangle=0\end{cases}\tag{3}$$

We will show that it is possible to satisfy the equation for the expansion coefficients following from (3) with only a finite number of $\mathbf{A}_{mn}, \mathbf{B}_{mn}, \mathbf{C}_{mn}$, and $\mathbf{D}_{mn}$, belonging to precisely k different eigenvalues, for arbitrary $k\in\mathbb{N}$.

We will not write down explicitly the equations for the coefficients $\mathbf{A}_{mn}, \mathbf{B}_{mn}, \mathbf{C}_{mn}$, and $\mathbf{D}_{mn}$, following from (3) after substitution of (2). We only briefly list the contributions to the coefficients of the independent terms occurring in the inner products of (3) which descend from two groups in (2) belonging to the eigenvalues corresponding to $(m,n)=(p,q)$, and $(m,n)=(r,s)$ respectively.

In the generic cases, the terms in

$$\cos\frac{Au}{a}\cos\frac{Bv}{b},\ \cos\frac{Au}{a}\sin\frac{Bv}{b},\ \sin\frac{Au}{a}\cos\frac{Bv}{b},\ \text{and}\ \sin\frac{Au}{a}\sin\frac{Bv}{b},$$

following from $\langle\mathbf{r}_u,\mathbf{r}_u\rangle$ are:

$$\frac{pr}{2a^2}\cos\frac{(p-r)u}{a}\cos\frac{(q-s)v}{b}(\langle\mathbf{A}_{pq},\mathbf{A}_{rs}\rangle+\langle\mathbf{B}_{pq},\mathbf{B}_{rs}\rangle+\langle\mathbf{C}_{pq},\mathbf{C}_{rs}\rangle+\langle\mathbf{D}_{pq},\mathbf{D}_{rs}\rangle)\tag{4}$$

$$\frac{pr}{2a^2}\cos\frac{(p-r)u}{a}\sin\frac{(q-s)v}{b}(-\langle\mathbf{A}_{pq},\mathbf{B}_{rs}\rangle+\langle\mathbf{B}_{pq},\mathbf{A}_{rs}\rangle-\langle\mathbf{C}_{pq},\mathbf{D}_{rs}\rangle+\langle\mathbf{D}_{pq},\mathbf{C}_{rs}\rangle)\tag{5}$$

$$\frac{pr}{2a^2}\sin\frac{(p-r)u}{a}\cos\frac{(q-s)v}{b}(-\langle\mathbf{A}_{pq},\mathbf{C}_{rs}\rangle+\langle\mathbf{B}_{pq},\mathbf{D}_{rs}\rangle+\langle\mathbf{C}_{pq},\mathbf{A}_{rs}\rangle+\langle\mathbf{D}_{pq},\mathbf{B}_{rs}\rangle)\tag{6}$$

$$\frac{pr}{2a^2}\sin\frac{(p-r)u}{a}\sin\frac{(q-s)v}{b}(-\langle\mathbf{A}_{pq},\mathbf{D}_{rs}\rangle-\langle\mathbf{B}_{pq},\mathbf{C}_{rs}\rangle-\langle\mathbf{C}_{pq},\mathbf{B}_{rs}\rangle+\langle\mathbf{D}_{pq},\mathbf{A}_{rs}\rangle)\tag{7}$$

for $A=p-r, B=q-s$. For $A=p-r, B=q+s$, and $A=p+r, B=q-s$, and $A=p+r, B=q+s$, we get similar contributions with the factors following the sin's and cos's involving other combinations of different inner products. In addition, one has to take into account the contributions for the special cases $(p,q)=(r,s)$. These follow immediately, up to a factor 2, from (4)–(7) and their homologues, after specifying $(p,q)=(r,s):=(p,q)$, and $(p,q)=(r,s):=(r,s)$ respectively. In particular, the contribution to the constant term is:

$$\frac{p^2}{4a^2}(\langle \mathbf{A}_{pq},\mathbf{A}_{pq}\rangle+\langle \mathbf{B}_{pq},\mathbf{B}_{pq}\rangle+\langle \mathbf{C}_{pq},\mathbf{C}_{pq}\rangle+\langle \mathbf{D}_{pq},\mathbf{D}_{pq}\rangle)$$
$$+\frac{r^2}{4a^2}(\langle \mathbf{A}_{rs},\mathbf{A}_{rs}\rangle+\langle \mathbf{B}_{rs},\mathbf{B}_{rs}\rangle+\langle \mathbf{C}_{rs},\mathbf{C}_{rs}\rangle+\langle \mathbf{D}_{rs},\mathbf{D}_{rs}\rangle). \tag{8}$$

From $\langle \mathbf{r}_v,\mathbf{r}_v\rangle$ follow exactly the same expressions, up to over-all changes of signs, provided one replaces $a\to b$, $p\to q$, and $r\to s$ in front of the sin's and cos's; all factors following the sin's and cos's, involving inner products are kept unaltered.

In particular, the contribution to the constant term is:

$$\frac{q^2}{4b^2}(\langle \mathbf{A}_{pq},\mathbf{A}_{pq}\rangle+\langle \mathbf{B}_{pq},\mathbf{B}_{pq}\rangle+\langle \mathbf{C}_{pq},\mathbf{C}_{pq}\rangle+\langle \mathbf{D}_{pq},\mathbf{D}_{pq}\rangle)$$
$$+\frac{s^2}{4b^2}(\langle \mathbf{A}_{rs},\mathbf{A}_{rs}\rangle+\langle \mathbf{B}_{rs},\mathbf{B}_{rs}\rangle+\langle \mathbf{C}_{rs},\mathbf{C}_{rs}\rangle+\langle \mathbf{D}_{rs},\mathbf{D}_{rs}\rangle). \tag{9}$$

The contributions in the generic cases, following from $\langle \mathbf{r}_u,\mathbf{r}_v\rangle$ take a similar form to the expressions (4)–(7) and their homologues. The coefficients in front of the sin's and cos's have to be replaced by an expression of the form

$$\frac{pr}{2a^2}\to\frac{ps\pm rq}{4ab}.$$

The same factors following the sin's and cos's appear in a different order together with other combinations of sin's and cos's eventually with over-all sign changes. We do not list them all explicitly, but give the contribution to the constant term:

$$\frac{pq}{2ab}(-\langle \mathbf{A}_{pq},\mathbf{D}_{pq}\rangle+\langle \mathbf{B}_{pq},\mathbf{C}_{pq}\rangle)+\frac{rs}{2ab}(-\langle \mathbf{A}_{rs},\mathbf{D}_{rs}\rangle+\langle \mathbf{B}_{rs},\mathbf{C}_{rs}\rangle). \tag{10}$$

Looking for solutions of the equations, involving terms belonging to precisely k different eigenvalues, we restrict our attention to the special case where $a=b:=\rho$, and we consider k different eigenvalues corresponding to couples $(m,n):=(p_i,p_i)$ for $i=1,\dots,k$. We make the following Ansatz for the expansion coefficients $\mathbf{A}_{p_ip_i}:=\mathbf{A}_i, \mathbf{B}_{p_ip_i}:=\mathbf{B}_i, \mathbf{C}_{p_ip_i}:=\mathbf{C}_i$, and $\mathbf{D}_{p_ip_i}:=\mathbf{D}_i (i=1,\dots,k)$:

$$\mathbf{A}_1=\mathbf{o},\quad \mathbf{B}_1=\frac{c}{p_1}\mathbf{e}_5,\quad \mathbf{C}_1=\frac{c}{p_1}\mathbf{e}_6,\quad \mathbf{D}_1=\mathbf{o},$$

$$\mathbf{A}_i=\frac{u_i}{p_i}\mathbf{e}_1,\quad \mathbf{B}_i=\frac{u_i}{p_i}\mathbf{e}_2,\quad \mathbf{C}_i=\frac{u_i}{p_i}\mathbf{e}_3,\quad \mathbf{D}_i=\frac{u_i}{p_i}\mathbf{e}_4,$$

where $i=2,\dots,k$, and p has been put equal to 6; here $\mathbf{e}_1,\dots,\mathbf{e}_6$ are the standard orthonormal basis vectors and $\mathbf{o}$ is the zero vector in $\mathbb{E}^6$.

If we choose $p_1,\dots,p_k\in\mathbb{N}_0$, such that, for a number $d\in\mathbb{N}_0$,

$$2p_1=p_k-p_2, \tag{11}$$

$$(k-2)d=2p_1, \tag{12}$$

$$p_{i+1}=p_i+d, \tag{13}$$

then suitable $c,u_2,\dots,u_k,\rho\in\mathbb{R}_0$ remains to be determined. Taking into 'account the above choices, the equations for the unknowns $c,u_2,\dots,u_k,\rho$ following from (3) reduce to

$$\sum_{i=2}^{k}u_i^2+\frac{1}{2}c^2=\rho^2, \tag{14}$$

$$c^2+4u_2u_k=0, \tag{15}$$

$$\sum_{i=2}^{k-l}u_iu_{i+l}=0,\quad l=1,\dots,k-3. \tag{16}$$

Lemma 8.1 of [8] provides a way to find solutions for this system of equations. We can formulate the result in the following way:

For every $\mu\geqq 2$ ($\mu\in\mathbb{R}$), and defining $\alpha:=\sqrt[k-2]{\dfrac{\mu+\sqrt{\mu^2-4}}{2}}$, then

$$c-2\sqrt{\alpha^{k-4}}, \tag{17}$$

$$u_2=-\alpha^{k-2}, \tag{18}$$

$$u_i=(\alpha^2-1)\alpha^{k-2-i},\quad i=3,\dots,k-1 \tag{19}$$

$$u_k=1, \tag{20}$$

$$\rho=\sqrt{\alpha^{2k-g}+\alpha^{-2}+2\alpha^{k-4}} \tag{21}$$

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solve the equations (14)–(16). This proves the existence of k-type immersions of flat tori in $\mathbb{E}^6$, and finishes the proof of the theorem.

Concluding remarks

We constructed explicit examples of k-type surfaces lying fully in $\mathbb{E}^6$. They are nonspherical in general. The surfaces being spherical would imply that the u-line $(u,0)$, which is a geodesic, should be a finite type curve in $S^2 \subset \mathbb{E}^3$, hence a circle, which is of 1-type. So there can only be expansion coefficients $\mathbf{A}_i, \mathbf{C}_i$ corresponding to 1 eigenvalue. A similar argument for the v-line $(0,v)$, lets us conclude that in this case the surface can be of at most 2-type.

Applying the lemma of Moore [**12**], it can be seen that the constructed surfaces are not product immersions. Indeed, suppose they are product immersions, then, following [**12**],

$$h\left(\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right)=0$$

where h denotes the second fundamental form, or, there exist functions λ and μ so that

$$\frac{\partial^2 \mathbf{r}}{\partial u \partial v}=\lambda(u,v)\frac{\partial \mathbf{r}}{\partial u}+\mu(u,v)\frac{\partial \mathbf{r}}{\partial v} \tag{22}$$

By expressing (22) explicitly in components, one can deduce from this set of equations $\lambda(u,v)$ and $\mu(u,v)$ have to be equal to (with $\alpha := (p_1/\rho)$ and $\alpha_i := (p_i/\rho)$)

$$\lambda=-\alpha\frac{\sin\alpha v\cos\alpha v}{\cos^2\alpha u-\sin^2\alpha v}$$

$$\mu=-\alpha\frac{\sin\alpha u\cos\alpha u}{\cos^2\alpha u-\sin^2\alpha v}$$

Furthermore, the other equations then imply that for all (u,v)

$$0=\sum_{i=2}^{k} u_i(p_i-p_1)\cos(2\alpha+\alpha_i)u\sin\alpha_i v+u_i(p_i-p_1)\cos(2\alpha-\alpha_i)u\sin\alpha_i v$$
$$+u_i(p_i-p_1)\cos\alpha_i u\sin(2\alpha+\alpha_i)v+u_i(p_i-p_1)\cos\alpha_i u\sin(2\alpha-\alpha_i)v.$$

This has to hold in particular for all v with $u=0$. In view of the independence of the set

$$\left\{\sin\frac{mv}{\rho}\,\middle|\, m\in\mathbb{N}\right\} \text{ on } [0,\rho],$$

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we focus on the term with highest m. As the coefficient has to vanish, there follows that $u_k(p_k - p_1) = 0$. This is easily seen to be in contradiction with (11)–(13) and the solution (17)–(21).

Finally we remark that the main theorem of [**12**] implies that an isometric immersion of a flat torus in $\mathbb{E}^4$ is rigid, i.e. necessarily a product immersion of 2 circles. Consequently, the present construction of finite type immersions of flat tori as we performed in $\mathbb{E}^6$, is not possible in $\mathbb{E}^4$.

It is interesting to notice that the example of the 2-type flat torus on page 261 of [**4**] fits in our general construction scheme. The present paper also contributes to the classification of the finite type surfaces in $\mathbb{E}^6$. Available information on this subject concerned mainly 2-type spherical surfaces, i.e. lying in $S^5 \subset \mathbb{E}^6$. We mention the following results.

In [**1**] the 2-type mass-symmetrical integral surfaces in S^5 are classified; such a surface is the product of a plane circle and a helix of order 4 or the product of two circles. In [**2**] the surfaces in S^5 which are coordinate finite type and integral are determined. It is also noticed that these are precisely the integral surfaces in S^5 having C-parallel second fundamental form. In [**3**] stationary, mass-symmetric, 2-type surfaces of S^m are studied in detail. In particular, it is shown that such surfaces are in fact flat surfaces which lie fully in S^5 or S^7. [**9**] classifies spherical Chen surfaces which are mass-symmetric and of 2-type. It is proved that these surfaces are either pseudoumbilical or flat; moreover, in the latter case, they lie fully in a 3-sphere, a 5-sphere, or a 7-sphere. [**10**] studies mass-symmetric proper 2-type immersions of a topological 2-sphere into S^m. In particular, it is shown that such an immersion is a direct sum of two minimal immersions into spheres; moreover, for small dimension $m = 9$, it is proved that the 2-sphere is of constant curvature. In [**11**] the mass-symmetric 2-type immersions of surfaces of constant curvature into S^n are classified.

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