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A Basis of Bachmuth Type in the Commutator Subgroup of a Free Group

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Abstract. We show here that the commutator subgroup of a free group of finite rank poses a basis of Bachmuth's type.

1 Introduction

Let F_n denote a free group with an ordered basis $x_1, x_2, \dots, x_n$, F_n' the commutator subgroup and F_n'' the second commutator subgroup in F_n.

Let us consider the set of all non-trivial commutators:

$$\{[x_i, x_k]^{x_i^{d_i} \cdots x_n^{d_n}}, 1 \le i < k \le n, d_l \in \mathbb{Z}, i \le l \le n\}, \tag{1}$$

where all subscripts in the exponent are in ascending order.

In 1965 S. Bachmuth [1] proved:

Theorem *F_n' is freely generated modulo F_n'' by the set (1).*

The aim of this paper is to drop the restriction "modulo F_n''". We prove:

Theorem *F_n' is freely generated by the set (1).*

We use following notations: $[a, b] = a^{-1}b^{-1}ab$, $a^c = c^{-1}ac$, and commutator identities:

$$[ab, c] \equiv [a, c]^b [b, c], \quad [a, bc] \equiv [a, c][a, b]^c,$$

$$[a, b_1 b_2 \cdots b_n] \equiv [a, b_1][a, b_2]^{b_1}[a, b_3]^{b_1 b_2} \cdots [a, b_n]^{b_1 b_2 \cdots b_{n-1}}.$$

2 Sketch of the Proof

The proof consists of five technical steps, modifying consequently the basis in F_n'.

Step 1 F_n' is freely generated by the non-trivial elements of the set:

$$\mathfrak{P} = \{[x_1^{d_1} \cdots x_{i-1}^{d_{i-1}}, x_i^{d_i} \cdots x_n^{d_n}] \,;\, d_l \in \mathbb{Z}, 2 \le i \le n\}.$$

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Step 2 F'_n is freely generated by the non-trivial elements of the set:

$$\{[x_1^{d_1}\cdots x_j^{d_j}, x_k^{d_k}]^{x_{k+1}^{d_{k+1}}\cdots x_n^{d_n}}\ ;\ d_l\in\mathbb{Z},\ j<k\}$$

where all subscripts are in ascending order.

Since $x_k^d = x_k x_k^{d-1}$ we obtain:

Step 3 F'_n is freely generated by the non-trivial elements of the set:

$$\{[x_1^{d_1}\cdots x_j^{d_j}, x_k]^{x_k^{d_k}x_{k+1}^{d_{k+1}}\cdots x_n^{d_n}}\ ;\ d_l\in\mathbb{Z},\ j<k\}$$

where all subscripts are in ascending order.

Step 4 F'_n is freely generated by the non-trivial elements of the set:

$$\{[x_i^{d_i}, x_k]^{x_{i+1}^{d_{i+1}}\cdots x_k^{d_k}\cdots x_n^{d_n}}\ ;\ d_l\in\mathbb{Z}, i<k\}$$

where all subscripts in the exponent are in ascending order.

Again, since $x_i^t = x_i x_i^{t-1}$ we obtain the required result stated in the Theorem.

Step 5 F'_n is freely generated by the non-trivial elements of the set:

$$\{[x_i, x_k]^{x_i^{d_i}\cdots x_n^{d_n}}\ ;\ d_l\in\mathbb{Z}, i<k\}$$

where all subscripts in the exponent are in ascending order, which proves the Theorem.

3 Proofs

In proofs, we will use the common symbol ν_i for any element in $\langle x_i\rangle$, that is ν_i denotes some power of x_i, possibly trivial. For example, the set $\mathfrak{P}$ from Step 1 can be written in ν-symbols, as follows:

$$\mathfrak{P} = \{[\nu_1\cdots\nu_{i-1}, \nu_i\cdots\nu_n]\ ;\ \nu_j = x_j^d, d\in\mathbb{Z}, 2\le i\le n\}.$$

Proof of Step 1 We use a modified Nielsen-Schreier method (the so-called Kurosh method [3, 4.3], [4, 6.3]) based on the formula $Kx(\overline{Kx})^{-1}$, with different systems of representatives for every generator x_i.

Since $d\in\mathbb{Z}$, the set $\mathfrak{P}$ is a set of commutators of the form

$$[\nu_1^{-1}\cdots\nu_{i-1}^{-1}, \nu_i^{-1}\cdots\nu_n^{-1}] = [(\nu_{i-1}\cdots\nu_1)^{-1}, (\nu_n\cdots\nu_i)^{-1}] = [B_i^{-1}, A_i^{-1}],$$

where $B_1 = 1, A_i = \nu_n\nu_{n-1}\ldots\nu_i, B_i = \nu_{i-1}\ldots\nu_2\nu_1, 2\le i\le n$.

So we have to prove that F_n' is freely generated by non-trivial elements of the form $[B_i^{-1}, A_i^{-1}]$.

Every word $w \in F_n$ can be uniquely written, modulo F_n', in the form:

$$w_0 = \nu_n \nu_{n-1} \cdots \nu_2 \nu_1, \quad (\nu_j = x_j^d, d \in \mathbb{Z}, j = 1, \ldots, n),$$

then $\forall i \in \{1, \ldots, n\}$ we have:

$$A_i B_i = (\nu_n \nu_{n-1} \cdots \nu_i)(\nu_{i-1} \cdots \nu_2 \nu_1) = w_0.$$

We choose the i-th system of representatives ${}^i w$, corresponding to the generator x_i, for a word $w \neq 1$ modulo F_n', as:

$${}^i w := B_i A_i = (\nu_{i-1} \cdots \nu_1)(\nu_n \cdots \nu_i), \quad i = 2, \ldots, n,$$

and the neutral system of representatives as:

$${}^* w := \nu_n \nu_{n-1} \cdots \nu_1 = w_0.$$

The system of representatives is the extended Schreier's system (see [3, 4.3]). Hence, by [3, Lemma 4.2], F_n' is freely generated by elements, which have the form ${}^i w x_i \big({}^i({}^i w x_i)\big)^{-1}$ and ${}^i w({}^* w)^{-1}$, for $i = 1, \ldots, n$. To calculate ${}^i w x_i \big({}^i({}^i w x_i)\big)^{-1}$, we denote $\nu_i x_i$ by ν_i', then

$${}^i w x_i \big({}^i({}^i w x_i)\big)^{-1} = (\nu_{i-1} \cdots \nu_1)(\nu_n \cdots \nu_i) x_i \big((\nu_{i-1} \cdots \nu_1)(\nu_n \cdots \nu_i')\big)^{-1} = 1,$$

and since $w_0 = A_i B_i$, $\forall i$ we get ${}^i w({}^* w)^{-1} = B_i A_i (A_i B_i)^{-1} = [B_i^{-1}, A_i^{-1}]$, which finishes the proof. ■

We need now the so-called elementary simultaneous Nielsen transformations [2]. Let $\mathcal{B} = \{b_i \,;\, i \in I\}$ be a basis of a free group F, and let $\mathcal{B}$ be a disjoint union of its proper subsets U and V, so $\mathcal{B} = U \cup V$. Then we have two types of elementary simultaneous Nielsen transformations τ_1, τ_2. The transformation τ_1 permutes elements of the basis and changes some of them into their inverses. The transformation τ_2 multiplies elements from V by elements from the set U, and does not change elements from the set U. Thus for $b \in \mathcal{B}$, $u \in U$, $w \in V$ we have:

(i) $\tau_1(b_i) = b_{\pi(i)}^{\varepsilon_i}$, where π is a permutation of the set I, $\varepsilon_i = \pm 1$, $\forall b_i \in \mathcal{B}$;
(ii) $\tau_2(u) = u$, $\tau_2(w) = wu$ or uw.

These transformations are invertible and hence change any basis of F into a new basis. We call these transformations, in short, Nielsen transformations.

Proof of Step 2 We say that the word $\nu_{i_1} \nu_{i_2} \cdots \nu_{i_k}$ has syllable length k if all ν_{i_j} are nontrivial. We split the basis $\mathfrak{P}$ obtained in Step 1, into subsets with respect to the syllable length of the second entry of the commutator:

$$\mathfrak{P} = \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \cdots \cup \mathfrak{P}_{n-1}.$$

So $\mathfrak{P}_1 = \{[\nu_1 \cdots \nu_j, \nu_k], k > j\}$ and generally $\mathfrak{P}_t = \{[\nu_1 \cdots \nu_j, \nu_{k_1} \cdots \nu_{k_t}], j < k_1 < \cdots < k_t\}$, is the set of commutators having exactly t non-trivial factors ν in the second entry.

Each element in $\mathfrak{P}_t$ can be written as $[a, b\nu_{k_t}]$. We can replace it by $[a, b]^{\nu_{k_t}}$, which can be done by Nielsen transformation with use $[a, \nu_{k_t}] \in \mathfrak{P}_1$ (we use the identity $[a, b]^{\nu_{k_t}} \equiv [a, \nu_{k_t}]^{-1}[a, b\nu_{k_t}]$). So we can replace in our basis $\mathfrak{P}$ each element $[a, b\nu_{k_t}] \in \mathfrak{P}_t$ by $[a, b]^{\nu_{k_t}}$, where $[a, b] \in \mathfrak{P}_{t-1}$. It looks like we moved ν_{k_t} from the second entry to an exponent. We shall continue this moving, and hence we get a set $\mathfrak{P}_t^{(k)}$:

$$\mathfrak{P}_t^{(k)} := \{[a, b]^{\nu_{i_1}\nu_{i_2}\cdots\nu_{i_k}} ; [a, b] \in \mathfrak{P}_t, [a, b\nu_{i_1}\nu_{i_2} \cdots \nu_{i_k}] \in \mathfrak{P}_{t+k}\}.$$

After our first step we replaced each $\mathfrak{P}_t$ for $\mathfrak{P}_{t-1}^{(1)}$ and got:

$$\mathfrak{P}^{(1)} := \mathfrak{P}_1 \cup \mathfrak{P}_1^{(1)} \cup \cdots \cup \mathfrak{P}_{n-2}^{(1)}.$$

We repeat the step using the identity $[a, b]^{\nu_{k_{t-1}}\nu_{k_t}} = [a, \nu_{k_{t-1}}]^{-\nu_{k_t}}[a, b\nu_{k_{t-1}}]^{\nu_{k_t}}$, where $[a, \nu_{k_{t-1}}]^{-\nu_{k_t}} \in \mathfrak{P}_1^{(1)}$ acts (as τ_2) on $[a, b\nu_{k_{t-1}}]^{\nu_{k_t}} \in \mathfrak{P}_{t-1}^{(1)}$. In this way we can replace $\mathfrak{P}_t^{(1)}$ $(t > 1)$ for $\mathfrak{P}_{t-1}^{(2)}$ and get:

$$\mathfrak{P}^{(2)} := \mathfrak{P}_1 \cup \mathfrak{P}_1^{(1)} \cup \mathfrak{P}_1^{(2)} \cup \mathfrak{P}_2^{(2)} \cup \cdots \cup \mathfrak{P}_{n-3}^{(2)}.$$

After $n - 2$ steps we have a basis:

$$\mathfrak{R} := \mathfrak{P}^{(n-2)} = \mathfrak{P}_1 \cup \mathfrak{P}_1^{(1)} \cup \mathfrak{P}_1^{(2)} \cup \cdots \cup \mathfrak{P}_1^{(n-2)}.$$

The basis $\mathfrak{R}$ consists of commutators having one symbol ν in the second entry and having a product of symbols ν in the exponent appearing in the ascending order, so:

$$\mathfrak{R} = \{[\nu_1 \cdots \nu_j, \nu_k]^{\nu_{k_1}\cdots\nu_{k_t}} ; j < k < k_1 < \cdots < k_t\}. \qquad \blacksquare$$

In the basis $\mathfrak{R}$ every commutator has one factor $\nu_k = x_k^d$ in the second entry, and $x_k^d = x_k x_k^{d-1} = x_k \nu_k'$. In the next step we move $\nu_k' = x_k^{d-1}$ from the second entry to the exponent. To do it we need some special transformations of the infinite basis.

Lemma 1 *Let F be the free group of infinite countable rank and let A be the basis of F. Let N be the transformation of A in F, such that for every finite subset $X \subseteq A$ there exists a finite subset $Y \subseteq A$, such that $X \subseteq Y$ and $N|_Y$ is the Nielsen transformation in the subgroup $F(Y) = gp(Y)$ freely generated by Y. Then N maps A into the new basis A' of F.*

Proof We shall prove that every element $w \in F$ is an unique product of elements of A'. For any $w \in F$ there exists k, such that w is a product of elements from $X = \{a_1, \ldots, a_k\} \subseteq A$. Then, by the assumption, there exists a finite subset $Y \supseteq X$, for which N is a Nielsen transformation of $F(Y)$ and hence w belongs to $F(Y)$, so w can be expressed as the product of elements from the set A' and the expression is unique. $\blacksquare$

We will call the transformations from Lemma 1 *finite Nielsen transformations*.

Proof of Step 3 We will use the basis $\mathfrak{R}$ obtained in Step 2:

$$\mathfrak{R} = \{[\nu_1 \cdots \nu_j, \nu_k]^{\nu_{k_1} \cdots \nu_{k_t}} \; ; \; j < k < k_1 < \cdots < k_t\}.$$

The set $\mathfrak{R}$ consists of the elements which have the form $[a, x_k^d]^b$. Now we can consider the transformation N, which maps every element $[a, x_k^{\varepsilon d}]^b$ onto $[a, x_k^{\varepsilon}]^{x_k^{\varepsilon(d-1)} b}$, where $d > 0, \varepsilon \in \{-1, 1\}$, and we denote by $\mathfrak{R}'$ the image of $\mathfrak{R}$. Since we have

$$[a, x_k^{\varepsilon d}]^b \equiv [a, x_k^{\varepsilon}]^b [a, x_k^{\varepsilon}]^{x_k^{\varepsilon} b} \cdots [a, x_k^{\varepsilon}]^{x_k^{\varepsilon(d-1)} b},$$

N is Nielsen transformation in the finitely generated free group $F(Y)$, where $Y = \{[a, x_k^{\varepsilon}]^b, [a, x_k^{\varepsilon 2}]^b, \ldots, [a, x_k^{\varepsilon d}]^b\}$. So N is a finite Nielsen transformation, and by Lemma 1 the set $\mathfrak{R}'$ is a basis of F_n'. Finally, since $[a, x_k^{-1}]^{x_k^{-(d-1)} b} \equiv [a, x_k]^{-x_k^{-d} b}$, we change $[a, x_k]^{-x_k^{-d} b}$ onto $[a, x_k]^{x_k^{-d} b}$ and we get the required basis of F_n', and the proof is complete. ■

Proofs of Steps 4 and 5 are exactly the same as the proofs of Steps 2 and 3. We start with the basis $\mathfrak{R}'$ obtained in Step 3 and then we change it by the same technique as in Step 2, but we move elements from the first entry in commutators to exponents. The last step is the same as the proof of Step 3.

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