9

Gauge-invariance and order parameters

For the pure gauge theory without fermions, the formulation of Wilson emphasizes the analogy of lattice gauge theory with models of magnetism in statistical mechanics. The U_{ij} are much like 'spins' located on the bonds of the crystal. These variables then interact through the four-spin coupling in the Wilson action. Further pursuing this analogy, one might ask whether a lattice gauge theory can ever develop a spontaneous magnetization. In a ferromagnet, the spins develop a non-vanishing expectation value in the direction of the magnetization. Thus we might look for phases of lattice gauge theory where

$$\langle U_{ij} \rangle \neq 0. \tag{9.1}$$

We will now show that this is impossible in the Wilson theory.

In an ordinary magnet, such an expectation value represents a spontaneous breaking of a global symmetry. The magnetization has to choose some direction in which to point. This may be determined either with appropriate boundary conditions or with a limit on a vanishingly small applied magnetic field. Once a direction is selected, it remains stable because of the infinite number of degrees of freedom in the thermodynamic limit. Thermal fluctuations cannot coherently shift the magnetization of a large crystal.

In lattice gauge theory, however, an expectation value as indicated in eq. (9.1) breaks the local symmetry of gauge invariance. Because the Wilson action is unchanged under the substitution

$$U_{ij} \to g_i\, U_{ij} (g_j)^{-1}, \tag{9.2}$$

one can arbitrarily rotate the direction of U_{ij}. As this can be done without changing an infinite number of degrees of freedom, unlike the ferromagnet, thermal fluctuations will induce such rotations and ultimately average over all gauges (Elitzur, 1975). More formally, if we change variables on all other links emanating from site i

$$U_{ik} \to U_{ij}\, U_{ik}, \quad k \neq j, \tag{9.3}$$

then all dependence on U_{ij} cancels from the action and we have

$$\langle U_{ij} \rangle = \int \mathrm{d}U_{ij}\, U_{ij}, \tag{9.4}$$

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which vanishes if U_{ij} contains only non-trivial irreducible representations of the group. The magnetization vanishes in pure lattice gauge theory.

This is unfortunate because in a spin model the magnetization provides a useful order parameter for distinguishing phases. At high temperatures the system is disordered and the magnetization vanishes identically. If at lower temperatures the spins have an expectation value, then we are by definition in a ferromagnetic state. If we can show that at sufficiently low temperatures such a state exists, then we have proven that the system has a phase transition. In lattice gauge theory the expectation of U_{ij} always vanishes and therefore cannot be used to monitor phase changes.

As the problem is intimately entwined with gauge invariance, we should look for a gauge-invariant order parameter. Indeed, as the path integral runs over all gauges, the gauge non-invariant parts of any operator are removed from its expectation value. Thus we will concentrate our attention on quantities which are invariant under eq. (9.2). In the pure gauge theory, the simplest example of such an object is the trace of the product of four links around a plaquette, or essentially the action for the given plaquette. It expectation value represents the internal energy of the corresponding thermodynamic system and is given by a derivative of the partition function

$$P = \langle 1 - (1/n) \operatorname{Tr} U_{\square} \rangle = \tfrac{1}{6}(\partial/\partial\beta) \log Z. \tag{9.5}$$

The factor 1/6 is the ratio of the number of sites to number of plaquettes on a four-dimensional lattice.

The 'average plaquette' P is an order parameter in the sense that it must exhibit singularities of the bulk thermodynamics. However it lacks the useful property of a magnetization in that it never vanishes identically except exactly at zero temperature. We cannot distinguish phases with the average plaquette vanishing in one and not another. Indeed, gauge-invariance precludes any local order parameter from having this property of a magnetization in a spin system. By local we mean involving the expectation of a function of gauge variables in a fixed finite domain of the crystal. Several years before Wilson's work, Wegner (1971) used lattice gauge theory based on the group $Z_2 = \{\pm 1\}$ as an example of a class of models lacking local order parameters and yet having a non-trivial phase structure.

Despite its shortcomings as an order parameter, the average plaquette plays a major role in numerical work where it is the simplest variable to evaluate. Indeed, many transitions are easily seen as jumps or singularities in P as a function of the coupling. For example, in figure 9.1 we show P versus the inverse temperature β for the gauge group Z_2 on a four-

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dimensional lattice. The points are from Monte Carlo analysis and the curves are based on strong coupling series and duality, all subjects of later discussion. The large jump in P is indicative of the strong first-order phase transition in this model.

A hypothetical unconfined phase of a gauge theory based on a continuous group should contain massless gauge bosons. Using a transfer matrix formalism to determine energies, we define the mass gap as the energy difference between the ground state and the first excited state. This quantity will vanish exactly in an unconfined phase with its free gluons.

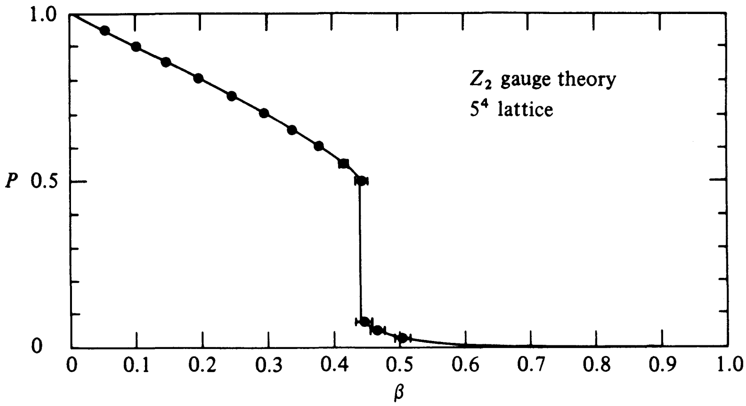

Fig. 9.1. The average plaquette for Z_2 lattice gauge theory. The points are from Monte Carlo simulation and the curves from strong and weak coupling analysis. Note the discontinuity in P at the phase transition at $\beta = \frac{1}{2}\log(1+\sqrt{2})$ (Creutz, 1980*a*).

In contrast, in a phase displaying confinement of massive quarks, we should have a spectrum of massive glueballs and bound states of quarks. Thus the mass gap is an order parameter which is expected to vanish in one phase but not another. In statistical mechanics language, the mass gap is the inverse of the correlation length. The expectation of two separated operators in a statistical system will generally display a correlation between the operators which falls with the distance between them. If for asymptotic separations this falloff is exponential, then the coefficient of the decrease is the mass gap m

$$C(r) \sim \exp(-mr). \tag{9.6}$$

This may be justified using a transfer matrix along the separation r. More physically, this equation represents a Yukawa exchange of the lightest particles on the theory. When the mass gap vanishes, we obtain power law forces as familiar in electrodynamics. Note that as an order parameter the

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mass gap is not local in that its definition involves correlations between asymptotically separated operators.

The use of the mass gap as an order parameter becomes somewhat more complicated if in the confinement phase the hadronic spectrum happens to display a massless particle. This is not simply an academic point because such a behavior is expected when bare quark masses vanish. In this chiral limit, alluded to in chapter 3, γ_5 symmetry is probably manifested in a Nambu–Jona-Lasinio (1961) Goldstone (1961) mode with a vanishing pion mass. In this case a discussion of confinement in terms of the mass gap requires a spin analysis of the massless quanta.

For the pure gluon theory without quarks. Wilson has proposed another non-local order parameter. The trace of a product of links around a closed loop is a gauge-invariant construction. Its expectation value is called the Wilson loop

$$W(C) = \langle \mathrm{Tr} \prod_{ij \in C} U_{ij} \rangle. \tag{9.7}$$

Here C denotes the loop in question and the group elements are ordered as encountered in circumnavigation of the contour. The simplest non-trivial Wilson loop is the average plaquette, defined in eq. (9.5) with an extra additive constant.

If a quark were to pass around the contour C, its wave function would pick up an internal symmetry rotation given by the product of the link variables encountered. The Wilson loop essentially measures the response of the gauge fields to an external quarklike source passing around its perimeter. For a timelike loop, this represents the production of a quark pair at the earliest time, moving them along the world lines dictated by the sides of the loop, and then annihilating at the latest time. If the loop is a rectangle of dimensions T by R, a transfer matrix argument suggests that for large T

$$W(R, T) \underset{T \to \infty}{\sim} \exp(-E(R)\,T), \tag{9.8}$$

where $E(R)$ is the gauge field energy associated with static quark–antiquark sources separated by distance R. If the interquark energy for large separations grows linearly

$$E(R) \underset{R \to \infty}{\to} KR, \tag{9.9}$$

then we expect for large loops of long rectangular shape

$$W(R, T) \sim \exp(-KRT). \tag{9.10}$$

The loop expectation falls with the exponential of the area of the loop and the coefficient of this area law is the coefficient of the linear potential. Physically, this area law represents the action of the world sheet of a flux tube connecting the sources. This picture suggests that this area law

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behavior should hold for arbitrarily shaped loops as long as they are larger than the cross sectional dimensions of a flux tube. In general we expect that with linear confinement

$$W(C) \sim \exp(-KA(C)), \tag{9.11}$$

where $A(C)$ is the minimal surface area enclosed with the loop C.

In a theory without confinement, the energy of a quark pair should not grow indefinitely with separation, but rather approach twice the self energy of an isolated quark. In such a situation the expectation value of the Wilson loop will decrease more slowly with loop size, in particular exponentially with the perimeter of the contour

$$W(C) \sim \exp(-kp(C)). \tag{9.12}$$

Here $p(C)$ is the perimeter and k is the self energy contained in the gauge fields around an isolated quarklike source. Some perimeter law behavior should always be present, even in a confining phase where an area law behavior dominates for large enough loops.

The coefficient of the area law provides another order parameter for lattice gauge theory. It vanishes identically in unconfined phases while remaining non-zero whenever quark sources experience a linear long-range potential. It has been extensively studied partly because of its simple flux tube interpretation and partly because of the ease of its evaluation in the strong coupling limit, to be discussed later. As it is directly related to the inter-quark potential, this coefficient is a physically meaningful parameter. In particular, it should be finite in the continuum limit of the pure gluonic theory. This is in contrast with the perimeter law behavior which should contain self energy divergences as the cutoff is removed. The area law is similar to the mass gap in that it represents a non-local order parameter. This is because of its definition in terms of the asymptotic behavior of a correlation function. It has the advantage over the mass gap in that it may be of value even for non-continuous groups such as Z_2 which may lose confinement without the appearance of a massless particle.

The area law criterion for confinement loses its value when quarks are introduced as dynamical variables. In this situation widely separated sources will reduce their energy by creating a pair of quarks from the vacuum fluctuations and screening their long range gauge fields. Effectively, a large Wilson loop measures the potential between two mesons rather than simple bare quarks. If we knew how to calculate with the full theory, however, we would not need a criterion for confinement. All we need to do is calculate the mass spectrum and see if it agrees with laboratory experiments. Hopefully we will soon reach this stage.

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Similar interesting questions regarding order parameters arise in gauge theories of the weak interaction, where a Higgs (1964) mechanism generates masses for the gauge bosons. In these theories lattice techniques have played almost no role, primarily because perturbative methods are more than adequate for relevant phenomenology. In the standard presentation, an expectation value for the Higgs field first results in a massless Goldstone (1961) boson which is subsequently 'eaten' by the gauge field and becomes the longitudinal component of a massive vector boson.

On more detailed inspection, this concept of the Higgs field acquiring a vacuum expectation value is overly simplistic. In particular, this field, and thereby its expectation, is not gauge-invariant. In some gauges such as the temporal one the Higgs expectation value is necessarily zero (Creutz and Tudron, 1978; Frohlich, Morchio and Strocchi, 1981) and the vector meson mass is related to the behavior of the vacuum under time-independent gauge transformations which are non-trivial at spatial infinity.

In lattice gauge theory one usually integrates over all gauges. When a Higgs field is present, its direction is thus averaged over. We conclude that the Higgs phase of the theory does not possess a local order parameter in the sense discussed at the beginning of this chapter. As with the confinement question, we could use the mass gap as a non-local order parameter distinguishing the Higgs phase from the massless vector meson phase. But this raises a rather peculiar question. What is the difference between the Higgs and confinement phases? Indeed, both are expected to have mass gaps. Fradkin and Shenker (1979) have shown that in certain cases these phases are not distinct and one can analytically continue from one to the other. This occurs when the Higgs field is in the fundamental representation of the gauge group. In this case the concept of confinement becomes obscured by the fact that an external source can always be screened by Higgs particles. This phenomenon gives rise to an alternative set of words to describe the states in a weak interaction theory when the Higgs fields are in the fundamental representation. For example, the electron would be a confined bound state of a bare electron and a Higgs particle (Abbott and Farhi, 1981*a*, *b*).

We now leave the discussion of order parameters and turn to the question of gauge fixing in the lattice theory. In Wilson's formulation, quantization does not require a choice of gauge. The integrals over the link variables are each over a compact domain and thus there cannot be any divergences arising from an integral over all gauges. This contrasts with

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usual continuum formulations where the volume of the gauge orbits is infinite and some sort of gauge fixing becomes a necessity. In addition to regulating the conventional ultraviolet divergences of field theory, the Wilson prescription also cuts off the total gauge volume. On the other hand, the gauge invariance of the action still permits working within a fixed gauge without affecting the expectations of gauge-invariant operators, such as the Wilson loop. We will now discuss a special class of gauges which are particularly simple in the lattice theory (Creutz, 1977).

Let $P(U)$ be some polynomial in the link variables which is invariant under the general gauge transformation of eq. (9.2). The following discussion goes through unchanged with other fields, such as those of quarks, present; however, for simplicity we consider only the pure gauge theory. Associated with this polynomial is a Green's function

$$G(P) = Z^{-1}\int (\mathrm{d}U)\,\mathrm{e}^{-S(U)}P(U). \tag{9.13}$$

We begin the discussion with the consideration of a single link from site i to site j. Suppose that in evaluating the expectation in eq. (9.13) we forgot to integrate over that one link variable. Remarkably, we will now see that the result for $G(P)$ would not be affected by our sloppiness. To see this formally we introduce a delta function on the gauge group. This has the properties

$$\int \mathrm{d}g\,\delta(g',g)f(g) = \int \mathrm{d}g\,\delta(g,g')f(g) = f(g')$$

$$\delta(g,g') = \delta(g_0 g g_1, g_0 g' g_1) \tag{9.14}$$

for arbitrary g_0 and g_1. Leaving link U_{ij} fixed at the element g rather than integrating over it as instructed in eq. (9.13) gives for the expectation of P

$$I(P,g) = Z^{-1}\int (\mathrm{d}U)\,\delta(U_{ij},g)\,\mathrm{e}^{-S(U)}P(U). \tag{9.15}$$

Clearly if we integrate over g we get back to eq. (9.13)

$$G(P) = \int \mathrm{d}g I(P,g), \tag{9.16}$$

If we now consider the gauge transformation of eq. (9.2) and note the invariance of $S(U)$, $P(U)$, and the measure, we obtain

$$I(P,g) = I(P,g_i^{-1}gg_j). \tag{9.17}$$

Since g_i and g_j are arbitrary, we conclude that $I(P,g)$ is actually independent of g. Eq. (9.16) then tells us

$$I(P,g) = G(P), \tag{9.18}$$

which is what we set out to prove. To calculate a gauge-invariant Green's

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function we can set any particular link variable to an arbitrary group element and only integrate over the remaining variables.

The above process can be repeated to fix more link variables. The final result is that we can arbitrarily neglect to integrate over any set of U_{ij} as long as this set contains no closed loops. The fixed links should form a tree, which may be disconnected. The gauge is completely determined if we have a maximal tree, a tree to which the addition of any more links would create a closed loop. An example of such a maximal tree is shown

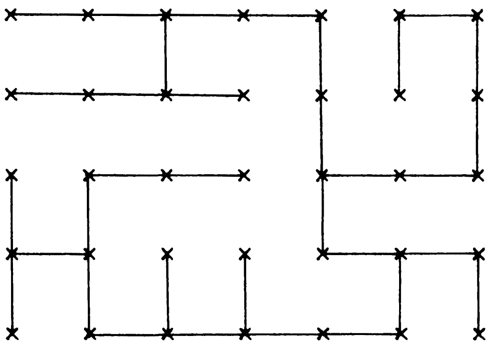

Fig. 9.2. An example of a maximal tree. All links on the tree can be set arbitrarily by the gauge fixing process.

in figure 9.2. The U_{ij} can be set to arbitrary group elements g_{ij}. The general formula for the Green's function of our gauge-invariant operator is

$$G(P) = Z^{-1} \int (\mathrm{d}U) \prod_{\{ij\} \in T} \delta(U_{ij}, g_{ij}) \, \mathrm{e}^{-S(U)} P(U). \qquad (9.19)$$

Here T denotes the tree in question and $\{ij\}$ refers to the link connecting sites i and i with arbitrary orientation.

A particularly simple gauge corresponds to setting all links in a particular direction to unity. This corresponds to an axial gauge where one component of the vector potential vanishes. Choosing the time direction, we obtain the $A_0 = 0$ or temporal gauge. This gauge will be useful for the construction of a transfer matrix and a Hamiltonian formulation of the lattice gauge theory. This gauge is illustrated in figure 9.3 and still leaves the freedom of time-independent gauge transformations.

Note that in an axial gauge plaquettes parallel to that axis represent a simple two-spin coupling of the unfixed variables. The theory reduces to a set of one-dimensional spin chains interacting with each other via the four-spin coupling of the remaining plaquettes. In two space-time dimensions there is no interchain coupling and the pure gauge theory is equivalent to an exactly solvable one-dimensional spin system.

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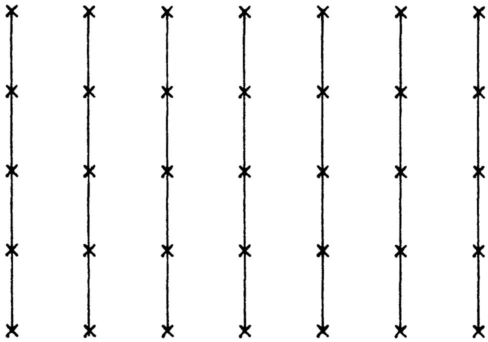

Fig. 9.3. A tree corresponding to the temporal gauge. Here the vertical direction represents time.

Problems

1. Solve two-dimensional lattice gauge theory for pure gauge fields. Find an expression for the average plaquette in terms of simple integrals over the gauge group. Show that the model has no phase transitions. Show that the Wilson loops always exhibit an area law.

2. Consider lattice gauge theory defined by replacing $U_{\Box}$ by the product of links around one-by-two rectangles and with the action being a sum over all such rectangles. Show that the two-dimensional model is no longer trivial. Show that the two-dimensional Z_2 model has a phase transition.

3. Find a gauge fixing tree such that most of the unfixed links have a non-vanishing expectation value, even on an infinite lattice.

4. Given an arbitrary gauge fixing function $f(U)$, show that our gauge-invariant Green's function is given by

$$G(P) = Z^{-1}\int(\mathrm{d}U)(f(U)/\phi(U))\,\mathrm{e}^{-S}P(U),$$

where the Fadeev–Popov (1967) correction factor $\phi(U)$ is an integral of f over all gauges (Kerler, 1981*b*)

$$\phi(U) = \int(\prod_i \mathrm{d}g_i)f(g_i\,U_{ij}\,g_j^{-1}).$$

Show that $\phi = 1$ for the gauge fixing function in eq. (9.19).

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