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REGULAR POLYGONS AND A PROPELLOR PROBLEM FOR CONVEX SETS IN THE PLANE

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We find the convex planar set of maximal area which is partitioned into subsets of equal area by a given *propellor*. The argument depends on the characterisation of an n-sided polygon which is circumscribed about a regular n-gon P_n, and inscribed in a regular n-gon Q_n, where P_n and Q_n are homothetic about their common centre.

1. Introduction

In 1963 Davis [1] described the following problem attributed to Ungar. A *cross* is defined as the union of two perpendicular segments in the plane. The boundary of a convex set K passes through the endpoints of the cross. If K is partitioned by the cross into four subsets of equal area, find the maximal area of K. Davis shows that the maximal area is assumed by a rectangle with sides parallel to the segments of the cross, and occurs when the segments of the cross bisect each other.

In this paper, we look at a natural analogue of the Ungar-Davis problem, the proof of which leads to some pretty geometry.

A set of n unit segments $OA_1, OA_2, \dots, OA_n$ in the plane E^2 meeting at equal angles of $2\pi/n$ at common endpoint O is said to form an *n-propeller*. Point O is the *centre* and points A_i $(i = 1, 2, \dots, n)$, are the *endpoints* of the propeller. To fix the scale, we shall assume that $OA_i = 1$ for all i.

Let K be a compact convex set in E^2 having area $A(K)$, and such that the boundary of K passes through the endpoints of the propeller.

We prove:

Theorem. *If K is partitioned by an n-propeller into n subsets of equal area, then $A(K) \leqslant n \tan \pi/n$.*

This result is best possible, and is assumed when and only when K is the regular n-gon with edges perpendicular to the segments of the propeller.

We observe that our result agrees with the conclusion of Davis in the case where the four arms of the cross have equal length. To establish this theorem we shall need the following lemma about regular polygons.

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2. A Lemma about polygons

We say that the n-sided polygon S_n is *inscribed in* the n-sided polygon Q_n if just one vertex of S_n lies on each edge of Q_n, and *circumscribed about* the n-sided polygon P_n if just one vertex of P_n lies on each edge of S_n.

Lemma. *Let $P_n = A_1A_2 \dots A_nA_1$ be a regular n-sided polygon with centre O, and $Q_n = B_1B_2 \dots B_nB_1$ a homothet of P_n about O having scale factor λ where $1 < \lambda \leqslant \lambda^* = \sec^2(\pi/n)$. Then for $1 < \lambda < \lambda^*$ there are exactly two n-sided polygons S_n and S'_n which can be circumscribed about polygon P_n and inscribed in polygon Q_n: these polygons are congruent and regular, centred at O, and each is the image of the other under reflection in any of the common lines of symmetry of polygons P_n, Q_n. For $\lambda = \lambda^*$, S_n and S'_n coincide.*

We shall see that for $\lambda > \lambda^*$ it is not possible for an n-sided polygon to be circumscribed about polygon P_n and inscribed in polygon Q_n.

Proof: Let the vertices of P_n, Q_n be labelled in an anti-clockwise direction (see Figure 1). Let the edges of P_n which are adjacent to A_iA_{i+1} be extended to meet the edge B_iB_{i+1} of Q_n in points L_i, R_i, where B_i, L_i, R_i, B_{i+1} now lie in order along this edge. We shall take the subscripts modulo n, reading $n+1$ as 1. Suppose now that $S_n = X_1X_2 \dots X_nX_1$ is an n-sided polygon as described in the statement of the lemma, with X_i lying on edge B_iB_{i+1} of Q_n. It is clear from the definition of inscribed and circumscribed that X_i must lie between L_i and R_i.

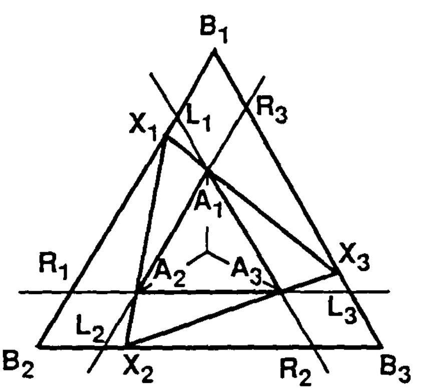

Figure 1. The case when $n = 3$.

For the purpose of this proof, we scale the figure so that $B_iL_i = R_iB_{i+1} = 1$ and set $L_iR_i = l$ for all i. Setting $L_iX_i = x_i$, then $X_iR_i = l - x_i$ for all i. Now for each i, $\triangle X_iR_iA_{i+1}$ is similar to $\triangle X_iB_{i+1}X_{i+1}$, since R_iA_{i+1} is parallel to $B_{i+1}X_{i+1}$. Hence $X_iR_i/R_iB_{i+1} = X_iA_{i+1}/A_{i+1}X_{i+1}$. In the same way $\triangle X_iB_{i+1}X_{i+1}$ is similar to $\triangle A_{i+1}L_{i+1}X_{i+1}$, since $A_{i+1}L_{i+1}$ is parallel to X_iB_{i+1}. Hence $X_iA_{i+1}/A_{i+1}X_{i+1} = B_{i+1}L_{i+1}/L_{i+1}X_{i+1}$.

Combining these results gives

$$\frac{X_iR_i}{R_iB_{i+1}} = \frac{B_{i+1}L_{i+1}}{L_{i+1}X_{i+1}},$$

or

$$\frac{l - x_i}{1} = \frac{1}{x_{i+1}}.$$

We thus obtain for all i,

$$x_{i+1} = \frac{1}{(l - x_i)}.$$

Beginning at the point X_1, we now get:

$$L_1X_1 = x_1 = x \text{ say}, \quad X_1R_1 = l - x,$$
$$L_2X_2 = x_2 = \frac{1}{l - x}, \quad X_2R_2 = l - \frac{1}{l - x} = [[l, l - x]]_2 \text{ say}.$$

The double square bracket denotes a continued fraction, but with minus signs in place of the usual plus signs. We append the subscript 2 to denote the number of terms inside the square brackets. Similarly,

$$L_3X_3 = x_3 = \frac{1}{[[l, l - x]]_2}, \quad X_3R_3 = l - \frac{1}{[[l, l - x]]_2} = [[l, l, l - x]]_3,$$

and in general, we have

$$L_iX_i = x_i = \frac{1}{[[l, l, \dots, l, l - x]]_{i-1}},$$
$$X_iR_i = l - \frac{1}{[[l, l, \dots, l, l - x]]_{i-1}} = [[l, l, \dots, l, l - x]]_i.$$

Since the polygon R_n closes, after n steps we must have $L_{n+1}X_{n+1} = L_1X_1$, or

$$x = \frac{1}{[[l, l, \dots, l, l - x]]_n}.$$

Suppose now that x^* is a solution of

$$x = \frac{1}{l - x}.$$

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We observe that if x^* is also a solution of

$$x = \frac{1}{[[l, l, \dots, l, l - x]]_k},$$

then x^* is a solution of

$$x = \frac{1}{l - \dfrac{1}{[[l, l, \dots, l, l - x]]_k}} = \frac{1}{[[l, l, \dots, l, l - x]]_{k+1}}.$$

It follows by induction that for all values of n, the required condition for the polygon R_n to close is that x be a positive solution x^* of the equation $x = (l - x)^{-1}$, that is, a solution of

$$x^2 - lx + 1.$$

Hence

$$x = \frac{1}{2}\left(l \pm \sqrt{l^2 - 4}\right).$$

It is now clear that there are two positive solutions $x_1 = x^*$, symmetrically placed about the centre of the segment L_1R_1. These solutions are imaginary for $l < 2$ and coincide when $l = 2$. In this last case, the vertices of S_n are the midpoints of the segments L_iR_i, and it is easily checked that in this case the enlargement factor in obtaining Q_n from P_n is λ^*. From the proof, $x^* = x = x_1 = x_2 = \dots = x_n$, so the polygons S_n, S'_n are regular, and by symmetry, centred at O as required.

It is interesting to observe that the equation $x^2 - lx + 1$ can be written as

$$x + \frac{1}{x} = l.$$

Thus if x^* is a solution, then so is $1/x^*$.

3. Proof of the Theorem

Since K is convex, it is bounded by support lines $a_1, \dots, a_n$ passing through $A_1, \dots, A_n$ respectively, forming a polygon R_n having $A(R_n) \geqslant A(K)$. We first show that it is sufficient to establish the theorem when K is such a polygon R_n.

Let the 'sectors' $A_1OA_2, A_2OA_3, \dots, A_nOA_1$ of K each have area α. Since $A(R_n) \geqslant A(K)$, it follows that $\beta = A(R_n)/n \geqslant \alpha$. We first rotate line a_2 about A_2 to position a'_2 where the portion of R_n bounded by segments OA_1, OA_2, and lines a_1, a'_2 has area β. Next we rotate line a_3 about A_3 to position a'_3 until the portion of R_n bounded by segments OA_2, OA_3 and lines a'_2, a'_3 has area β. Continue around the polygon, finally rotating a_n about A_n to position a'_n until the portion of R_n bounded by segments OA_{n-1}, OA_n and lines a'_{n-1}, a'_n has area β. We now observe that the portion of R_n bounded by segments OA_n, OA_1 and lines a'_n, a_1 has the required area β. Hence it is sufficient to prove the theorem for polygon R_n.

Let lines a_i, a_{i+1} meet in the vertex X_i of R_n. The condition of the theorem is that R_n is partitioned into n subsets of equal area by the segments of the propeller. Since the triangles OA_iA_{i+1} all have the same area, this means that the triangles $A_iX_iA_{i+1}$ all have the same area. Noting that the base segments A_iA_{i+1} of the triangles $A_iX_iA_{i+1}$ all have the same length, we deduce that the altitudes of these triangles (perpendicular to A_iA_{i+1}) must all be equal. Thus for each i the vertex X_i lies on a line, parallel to A_iA_{i+1}, and at a distance d from A_iA_{i+1} which is independent of i. Thus polygon R_n circumscribes the regular polygon $P_n = A_1A_2 \dots A_nA_1$, and is inscribed in the homothetic regular polygon Q_n determined by the lines $a_1, \dots, a_n$.

According to the lemma, polygon R_n is a regular n-gon, centred at point O. We can gauge the size of R_n by measuring the distance of an edge from the centre O. Clearly the greatest distance from O of an edge through A_i occurs when such an edge is perpendicular to OA_i. Hence the set of maximal area satisfying the conditions of the theorem is the regular n-sided polygon having its edges perpendicular to the segments of the propeller. The area of this polygon is $A(K) = n \tan \pi/n$.

This completes the proof of the theorem. $\square$

References

[1] C. Davis, 'An extremal problem for plane convex curves', *Proc. Sympos. Pure Math.* **VII** (1963), 181–186.

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