ANNIHILATORS OF RELATION MODULES

Dedicated to the memory of Hanna Neumann

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1. Introduction

Let $1 \to R \to F \xrightarrow{\alpha} G \to 1$ be a non-cyclic free presentation of a group G, $R = R_1 > R_2 > R_3 > \cdots > R_n > \cdots$ the lower central series of R. Then R_n/R_{n+1}, $n \geqq 1$, can be regarded as a right G-module by defining the action of G via conjugation in F. We wish to investigate the annihilators of these modules which we call *higher relation modules.*

Our main result is that if $\text{Ann}\, R/R_2 = (0)$, then, for all $n \geqq 1$,

$$\text{Ann}\, R_n/R_{n+1} = (0) \text{ (Ann = annihilator).}$$

We prove that there always exists an integer $c \geqq 1$, independent of n, such that

$$A_G^c \cdot \text{Ann}\, R_n/R_{n+1} \leqq \text{Ann}\, R/R_2,$$

where A_G is the augmentation ideal of ZG, the integral group ring of G. If G is periodic, then we prove that

$$\text{Ann}\, R_n/R_{n+1} \leqq \text{Ann}\, R/R_2 \text{ for all } n \geqq 1.$$

2. Free differential calculus

If F is a free group and $D_1, D_2, \cdots, D_n : ZF \to ZF$ are *right* derivations, we write $D_n \cdots D_2 D_1(u)$ for $D_n(D_{n-1}(\cdots D_2(D_1(u))\cdots))$ and $D^n(u)$ for $\underbrace{D(D(\cdots(D}_{n}(u))\cdots))$, $u \in ZF$. For $n = 0$, the expression $D_n \cdots D_1(u)$ is interpreted simply as u.

Let $1 \to R \to F \xrightarrow{\alpha} G \to 1$ be a free presentation of a group G. We extend α by linearity to a ring homomorphism $\alpha : ZF \to ZG$. Let $\varepsilon : ZF \to Z$ be the unit augmentation.

LEMMA 2.1. *Let n, i be integers with $n > i \geqq 0$, $u \in A_F^i$, $v \in A_F^{n-i-1} A_R$, $D_1, \cdots, D_n : ZF \to ZF$ arbitrary derivations. Then*

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PROOF. Extend α by linearity to $\alpha: ZF \to ZG$. Let $r \in R_n$, $z \in \text{Ann } R_n/R_{n+1}$, $\alpha(u) = z$, $u \in ZF$. Applying θ_n to $rR_{n+1} \cdot z = 0$ gives $(r-1)u \in A_F A_R^n$. Since $R \leqq F_c$, $A_R \leqq A_F^c$. Hence $(r-1)u \in A_F^{c(n-1)+1} A_R$. Therefore, for arbitrary derivations $D_1, D_2, \cdots, D_{c(n-1)+1}$,

$$\alpha D_{c(n-1)+1} \cdots D_1((r-1)u) = 0 \text{ [1]}.$$

This gives

$$\alpha(D_{c(n-1)+1} \cdots D_1(r-1))\,\alpha(u) + \varepsilon D_{c(n-1)} \cdots D_1(r-1)\,\alpha D_{c(n-1)+1}(u) + \cdots$$
$$+ \varepsilon(r-1)\alpha D_{c(n-1)+1} \cdots D_1(u) = 0.$$

Since $r - 1 \in A_R^n \leqq A_F^{cn}$,

$\varepsilon D_k D_{k-1} \cdots D_1(r-1) = 0$ for $k < cn$. Hence we get

$$\alpha D_{c(n-1)+1} \cdots D_1(r) \cdot z = 0.$$

THEOREM 3.2. *Let $1 \to R \to F \xrightarrow{\alpha} G \to 1$ be a non-cyclic free presentation of a group G. If for every given partial derivation $d: ZF \to ZF$ we can find an element r of R and a partial derivation $d_1: ZF \to ZF$ such that $\alpha d(r) = 0$ and $\varepsilon d_1(r) \neq 0$, then*

$$\text{Ann } R_n/R_{n+1} \leqq \text{Ann } R/R_2.$$

PROOF. It is enough [4] to prove that for every $z \in \text{Ann } R_n/R_{n+1}$, $s \in R$ and partial derivation $d: ZF \to ZF$, $\alpha d(s) \cdot z = 0$. Let $s \in R$ and d be a partial derivation. By hypothesis we can find an element $r \in R$ and a partial derivation d_1 such that $\varepsilon d_1(r) \neq 0$, $\alpha d(r) = 0$. If $r = s$, then $\alpha d(s) \cdot z = 0$ trivially. Suppose that $r \neq s$. Let $i \geqq 0$ be the least integer such that

$$\alpha d d_1^i([\cdots[[s,\underbrace{r],r],\cdots,r}_{i}]) \cdot z = 0. \tag{3.3}$$

By Lemma 3.1 (case $c = 1$), this is certainly true for $i = n-1$. This settles the question of the existence of i. If $i \neq 0$, then applying Lemma 2.2, (3.3) gives

$$\{\varepsilon d_1^i([\cdots[[s,\underbrace{r],r],\cdots,r}_{i-1}])\,\alpha d(r) - \varepsilon d_1(r)\alpha d d_1^{i-1}([\cdots[s,\underbrace{r],\cdots,r}_{i-1}])\} \cdot z = 0.$$

But $\alpha d(r) = 0, \varepsilon d_1(r) \neq 0$. Hence this implies that

$$\alpha d d_1^{i-1}([\cdots[[s,\underbrace{r],r],\cdots,r}_{i-1}]) \cdot z = 0,$$

contradicting the minimality of i. Hence the least integer i for which (3.3) holds is 0 i.e. $\alpha d(s) \cdot z = 0$.

COROLLARY 3.4. *If G is periodic, then* $\text{Ann } R_n/R_{n+1} \leqq \text{Ann } R/R_2$.

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PROOF. Given d to be the partial derivation with respect to the free generator x_j, say, of F, we take $r = x_i^m$, where x_i is a free generator $\neq x_j$ and $\alpha(x_i)$ is of order m in G. For d_1 we take the partial derivation with respect to x_i. This choice of r and d_1 is possible since F is non-cyclic and G is periodic. Then $\alpha d(r) = 0$ and $\varepsilon d_1(r) = m \neq 0$. Hence by Theorem 3.2, $\text{Ann } R_n/R_{n+1} \leqq \text{Ann } R/R_2$.

COROLLARY 3.5. *If G is finite, then* $\text{Ann } R_n/R_{n+1} = (0)$ *for all* $n \geqq 1$.

PROOF. This follows form Corollary 3.4 and the result that $\text{Ann } R/R_2 = (0)$ when G is finite [5].

LEMMA 3.6. *Let $R \leqq F_c$, $R \nleqq F_{c+1}$. Then for a given $n \geqq 2$*

either (a) $\text{Ann } R_n/R_{n+1} \leqq \text{Ann } R/R_2$
or (b) *there exists an integer $i \geqq 2$ such that*

(i) $R_{i-1} \nleqq F_{c(i-1)+1}$ *and*
(ii) $\alpha d_{c(i-1)+1} \cdots d_1(s) \cdot z = 0$ *for every $s \in R_i$, $z \in \text{Ann } R_n/R_{n+1}$ and all partial derivations $d_1, d_2, \cdots, d_{c(i-1)+1}$.*

PROOF. Let K be the set of natural numbers i which satisfy (b(ii)), By Lemma 3.1, K is not empty. Let i be the least member of K. If $i = 1$, then $\alpha d(s) \cdot z = 0$ for every $s \in R$, $z \in \text{Ann } R_n/R_{n+1}$ and every partial derivation d. Therefore $z \in \text{Ann } R/R_2$ [4].

Suppose $i \geqq 2$. We assert that i satisfies (b(i)). For, let $R_{i-1} \leqq F_{c(i-1)+1}$. Since $R \nleqq F_{c+1}$ and $R \leqq F_c$, we can find an element $r \in R$, $r \notin F_{c+1}$, $r \in F_c$. By ([1], 4.6) it is possible to choose partial derivations $d_1, d_2, \cdots, d_c$ such that

$$\varepsilon d_c d_{c-1} \cdots d_1(r) \neq 0 .$$

Let t be an arbitrary element of R_{i-1} and $d_{c+1}, d_{c+2}, \cdots, d_{c(i-1)+1}$ be arbitrary partial derivations. Then $[r,t] \in R_i$ and, since i satisfies the requirement (b(ii)) of the Lemma, we have

$$\alpha d_{c(i-1)+1} \cdots d_1([r,t]) \cdot z = 0$$

for all $z \in \text{Ann } R_n/R_{n+1}$. By Lemma 2.2 this gives

$$\{\varepsilon d_c \cdots d_1(r)\alpha d_{c(i-1)+1} \cdots d_{c+1}(t) - \varepsilon d_{c(i-1)} \cdots d_1(t)\alpha d_{c(i-1)+1}(r)\} \cdot z = 0 .$$

But $t \in R_{i-1} \leqq F_{c(i-1)+1}$. Hence by ([1],4.6) $\varepsilon d_{c(i-1)} \cdots d_1(t) = 0$. Also $\varepsilon d_c \cdots d_1(r) \neq 0$. Hence $\alpha d_{c(i-1)+1} \cdots d_{c+1}(t) \cdot z = 0$ for every $t \in R_{i-1}$, $z \in \text{Ann } R_n/R_{n+1}$ and arbitrary partial derivations $d_{c+1}, \cdots, d_{c(i-1)+1}$ i.e. $i-1$ satisfies the condition (b(ii)). This contradicts the choice of i. Hence it must satisfy (b(i)).

LEMMA 3.7. *Let $R \leqq F_c, R \nleqq F_{c+1}$. Then $\alpha d(r) \cdot z = 0$ for all $z \in \text{Ann } R_n/R_{n+1}$, $r \in R \cap F_{c+1}$ and all partial derivations d.*

PROOF. By Lemma 3.6 either 3.6(a) or 3.6(b) holds. If 3.6(a) holds, then there is nothing to prove [4]. Suppose 3.6(b) holds. Choose s in R_{i-1} which is not in $F_{c(i-1)+1}$. By ([1], 4.6) we can find partial derivations $d_1, \cdots, d_{c(i-1)}$ such that $\varepsilon d_{c(i-1)} \cdots d_1(s) \neq 0$. Let $r \in R \cap F_{c+1}$. Then $[r,s] \in R_i$ and therefore

$$\alpha d d_{c(i-1)} \cdots d_1([r,s]) \cdot z = 0$$

for all $z \varepsilon \operatorname{Ann} R_n / R_{n+1}$ and arbitrary partial derivations d. Hence by Lemma 2.2

$$\{\varepsilon d_c \cdots d_1(r)\, \alpha d d_{c(i-1)} \cdots d_{c+1}(s) - \varepsilon d_{c(i-1)} \cdots d_1(s)\, \alpha d(r)\} \cdot z = 0 .$$

But $r \in F_{c+1}$ and therefore $\varepsilon d_c \cdots d_1(r) = 0$. Also $\varepsilon d_{c(i-1)} \cdots d_1(s) \neq 0$. Hence $\alpha d(r) \cdot z = 0$ for $r \in R \cap F_{c+1}$ and arbitrary partial derivations d.

We can now prove

THEOREM 3.8. *There exists an integer c, independent of n, such that*

$$A_G^c \cdot \operatorname{Ann} R_n / R_{n+1} \leqq \operatorname{Ann} R / R_2 .$$

PROOF. Let $z \in \operatorname{Ann} R_n / R_{n+1}$. Since F is residually nilpotent, we can find an integer $c \geqq 1$ such that $R \leqq F_c$, $R \nleqq F_{c+1}$. By Lemma 3.7 $\alpha d(s) \cdot z = 0$ for all $s \in R \cap F_{c+1}$ and all partial derivations d. Let $r \in R$ and $f_1, f_2, \cdots, f_c \in F$. Then $[\cdots[[r,f_1],f_2],\cdots,f_c] \in R \cap F_{c+1}$. Hence

$$\alpha d([\cdots[[r,f_1],f_2],\cdots,f_c]) \cdot z = 0 . \tag{3.9}$$

It is easy to check that

$$\alpha d([\cdots[[r,f_1],f_2],\cdots,f_c]) = \alpha d(r)(\alpha(f_1) - 1) \cdots (\alpha(f_c) - 1) .$$

Hence (3.9) gives

$$\alpha d(r)(g_1 - 1)(g_2 - 1) \cdots (g_c - 1) \cdot z = 0$$

where $\alpha(f_i) = g_i$, $i = 1, 2, \cdots, c$. Since this is true for all $r \in R$ and all partial derivations d, it follows that $(g_1 - 1)(g_2 - 1) \cdots (g_c - 1) \cdot z \in \operatorname{Ann} R / R_2$ [4]. Hence $A_G^c \cdot \operatorname{Ann} R_n / R_{n+1} \leqq \operatorname{Ann} R / R_2$.

COROLLARY 3.10. *If* $\operatorname{Ann} R / R_2 = (0)$, *then* $\operatorname{Ann} R_n / R_{n+1} = (0)$ *for all* $n \geqq 1$.

PROOF. By Corollary 3.5 we can assume that G is infinite. Since the integral group ring of an infinite group does not have any non-zero element which annihilates its augmentation ideal, the corollary follows from Theorem 3.8.

REMARK 3.11. For a non-cyclic free presentation $1 \to R \to F \xrightarrow{\alpha} G \to 1$ of a group G, the annihilator of the relation module R / R_2 is known ([4],[5]) to be trivial in the following cases:

(i) ZG is without zero-divisors;
(ii) the free presentation under consideration is the standard free presentation;
(iii) G is finite;
(iv) G is residually finite and centre of G is infinite;
(v) G is nilpotent;
(vi) the rank of F is infinite.

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