COMPOSITIO MATHEMATICA

K_1 of products of Drinfeld modular curves and special values of L-functions

Ramesh Sreekantan

Compositio Math. **146** (2010), 886–918.

doi:10.1112/S0010437X10004720

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

Compositio Math. 146 (2010) 886–918
doi:10.1112/S0010437X10004720

K_1 of products of Drinfeld modular curves and special values of L-functions

Ramesh Sreekantan

Abstract

Beilinson [*Higher regulators and values of L-functions*, Itogi Nauki i Tekhniki Seriya Sovremennye Problemy Matematiki Noveishie Dostizheniya (Current problems in mathematics), vol. 24 (Vserossiisky Institut Nauchnoi i Tekhnicheskoi Informatsii, Moscow, 1984), 181–238] obtained a formula relating the special value of the L-function of H^2 of a product of modular curves to the regulator of an element of a motivic cohomology group, thus providing evidence for his general conjectures on special values of L-functions. In this paper we prove a similar formula for the L-function of the product of two Drinfeld modular curves, providing evidence for an analogous conjecture in the case of function fields.

1. Introduction

1.1 Beilinson's conjectures and a function field analogue

The algebraic K-theory of a smooth projective variety over a field has a finite, increasing filtration called the Adams filtration. For a variety over a number field, Beilinson [Beĭ84] formulated conjectures which relate the graded pieces of this filtration, the motivic cohomology groups $H^*_{\mathcal{M}}$, to special values of the Hasse–Weil L-function of a cohomology group of the variety.

The conjectures are of the following nature: corresponding to the motivic cohomology group $H^*_{\mathcal{M}}$ there is a real vector space $H^*_{\mathcal{D}}$, called the real Deligne cohomology, whose dimension is the order of the pole, at a specific point, of the Archimedean factor of the L-function.

Beilinson defined a regulator map from the $H^*_{\mathcal{M}}$ to $H^*_{\mathcal{D}}$ and conjectured that its image determines a $\mathbb{Q}$-structure on the $H^*_{\mathcal{D}}$. The real vector space $H^*_{\mathcal{D}}$ has another $\mathbb{Q}$-structure induced by de Rham and Betti cohomology groups. Beilinson conjectured further that the determinant of the change of basis between these two $\mathbb{Q}$ structures is, up to a non-zero rational number, the first non-zero term in the Taylor expansion of the L-function at a specific point. More details can be found in the book [RSS88] or in the paper [Ram89].

Beilinson's conjectures have been proved only in a few special cases. In [Beĭ84], Beilinson proved them for the product of two modular curves and as a result for the product of two non-isogenous elliptic curves over $\mathbb{Q}$. It is these results that we generalize to the function field case.

Since the conjectures deal with the transcendental part of the value of the L-function and involve the Archimedean L-factor they can be viewed as conjectures for the Archimedean place. It is natural to ask whether one can formulate a similar question for the other finite places.

Received 26 November 2008, accepted in final form 3 September 2009, published online 8 June 2010.
2000 Mathematics Subject Classification 11F52 (primary), 11G40 (secondary).
Keywords: Drinfeld modular curves, motivic cohomology, special values of L-functions.

In [Sre08], we formulated a function field analogue of the Beilinson conjectures. In particular we defined a group which, at a finite place, plays the role of the real Deligne cohomology.

This group, called the ν-adic Deligne cohomology, is a rational vector space whose dimension was shown by Consani [Con98], assuming some standard conjectures, to coincide with the order of the pole, at a certain integer, of the local L-factor at the place ν.

In [Sre08] we defined a regulator map $r_{\mathcal{D},\nu}$ from the motivic cohomology to the ν-adic Deligne cohomology and, in analogy with the Beilinson conjectures, conjectured that the image is a full lattice. Finally, in some cases, we made a conjecture on the special value of the L-function.

One such case is that of the L-function of a surface at the integer $s = 1$. It is a generalization of the Tate conjecture for a variety over a function field. The precise statement of this conjecture is as follows.

CONJECTURE 1.1. Let X be a smooth proper surface over a function field K and $\mathcal{X}$ a semi-stable model of X over A, its ring of integers. Let $\Lambda(H^2(\bar{X}, \mathbb{Q}_\ell), s)$ be the completed L-function of H^2, namely the product of the local L-factors at all places of K, where $\bar{X} = X \times \mathrm{Spec}(\mathbb{C}_\infty)$. Then, there is a 'thickened' regulator map $R_{\mathcal{D}} = \bigoplus_\nu r_{\mathcal{D},\nu} \oplus cl$,

$$R_{\mathcal{D}} : H^3_{\mathcal{M}}(X, \mathbb{Q}(2)) \oplus B^1(X) \longrightarrow \bigoplus_\nu PCH^1(X_\nu),$$

where $B^1(X) = CH^1(X)/CH^1_{\mathrm{hom}}$ and $PCH^1(X_\nu)$ is a subgroup of the Chow group of the special fibre at v, which provides an integral structure on the ν-adic Deligne cohomology $H^3_{\mathcal{D}}(X_{/\nu}, \mathbb{Q}(2))$ defined below. We conjecture $R_{\mathcal{D}}$ satisfies the following properties, where Λ^* denotes the first non-zero value in the Laurent expansion and $|\,|$ of a finite set denotes its cardinality.

(A) The regulator map $R_{\mathcal{D}}$ is a pseudo-isomorphism; namely it has a finite kernel and co-kernel.

(B) The order of the pole is the dimension of the space of Tate cycles – that is, the Tate conjecture holds

$$-\mathrm{ord}_{s=2}\Lambda(H^2(\bar{X}, \mathbb{Q}_\ell), s) = \dim_{\mathbb{Q}} B^1(X) \otimes \mathbb{Q}.$$

(C) The completed L-function of H^2 is such that

$$\Lambda^*(H^2(\bar{X}, \mathbb{Q}_\ell), 1) = \pm \frac{|\mathrm{coker}(R_{\mathcal{D}})|}{|\ker(R_{\mathcal{D}})|} \cdot \log(q)^{\mathrm{ord}_{s=1}\Lambda(H^2(\bar{X}),s)}.$$

In other words, the conjecture asserts that the regulator map provides an isomorphism of the rational motivic cohomology with the sum of all the ν-adic Deligne cohomology groups. The special value then measures the obstruction to this map being an isomorphism of integral structures.

This conjecture comes from the localization sequence for motivic cohomology which relates the motivic cohomologies of $X, \mathcal{X}$ and $\mathcal{X}_\nu$. The regulator map is the boundary map in the localization sequence. We stated the conjecture for surfaces. For points, when $X = \mathrm{Spec}(K)$, this is simply a combination of the function field class number formula and units theorem; the special value conjecture in this case implies the well-known formula

$$\Lambda^*(H^0(\mathrm{Spec}(X), 0)) = -\frac{h_K}{(q-1)\log(q)}$$

where h_K is the class number and $(q-1)$ is the number of roots of unity which can be interpreted as the orders of the kernel and cokernel of the regulator map respectively and the power of $\log(q)$ that appears corresponds to the well-known fact that the zeta function has a simple pole at $s = 1$.

Beilinson's theorem [Beĭ84] follows from a formula relating the cohomological L-function of $h^1(M_f) \otimes h^1(M_g)$, where $h^1(M_f)$ and $h^1(M_g)$ are the motives of eigenforms of weight two and some level N, to the regulator of an element of a certain motivic cohomology group evaluated on the $(1,1)$-form

$$\omega_{f,g} = f(z_1)\overline{g(z_2)}(dz_1 \otimes d\bar{z}_2 - d\bar{z}_1 \otimes dz_2).$$

We show an analogous formula in the Drinfeld modular case with the Archimedean place being replaced by the prime ∞. More precisely, since our L-functions essentially take rational values, we have an exact formula for the value analogous to the main theorem of [BS04].

Our main result is the following theorem.

Theorem 1.2. *Let I be a square-free element of $\mathbb{F}_q[T]$ and $\Gamma_0(I)$ the congruence subgroup of level I. Let f and g be Hecke eigenforms for $\Gamma_0(I)$ and $\Lambda(h^1(M_f) \otimes h^1(M_g), s)$ denote the completed (that is, with the L-factor at ∞ included) L-function of the motive $h^1(M_f) \otimes h^1(M_g)$. Then one has*

$$\Lambda(h^1(M_f) \otimes h^1(M_g), 1) = \frac{q}{2(q-1)\kappa}(r_{\mathcal{D},\infty}(\Xi_0(I)), \mathcal{Z}_{f,g}) \tag{1}$$

where $\Xi_0(I)$ is an element of motivic cohomology group $H^3_{\mathcal{M}}(X_0(I) \times X_0(I), \mathbb{Q}(2))$, $r_{\mathcal{D},\infty}$ is the ∞-adic regulator map, κ is an explicit integer constant and $\mathcal{Z}_{f,g}$ is a special cycle in the special fibre at ∞ and $(,)$ denotes the intersection pairing the Chow group of the special fibre.

1.2 Outline of the paper

In the first few sections we introduce some of the background on Drinfeld modular curves. This is perhaps well known to people working with function fields, but perhaps not so well known to people working in the area of algebraic cycles, hence it has been included.

We then study the analytic side of the problem, namely the special value of the L-function. We use the Drinfeld uniformization and an analogue of the Rankin–Selberg method to get an integral formula for the L-function. We also formulate and prove an analogue of Kronecker's first limit formula and use it to get an integral formula for the special value at 1 of the L-function.

Following that we study the algebraic side of the problem. We introduce the motivic cohomology group of interest to us and define a regulator map on it. This regulator map is the boundary map in a localization sequence relating the motivic cohomology groups of the generic fibre and special fibre. The result is that the regulator of an element of our motivic cohomology group is a certain 1-cycle on the special fibre.

We then construct an explicit element in this motivic cohomology group using analogues of the classical modular units and compute its regulator. The regulator of this element is then related to our integral formula using the relation between components of the associated reduction of the Drinfeld modular curve and vertices on the Bruhat–Tits tree.

In the classical case the regulator is a current on $(1,1)$-forms and one obtains the special value by evaluating this current on a specific form. Here, the regulator is a 1-cycle and one obtains the special value by computing the intersection pairing with a specific cycle supported on the special fibre. Finally we relate our formula with the conjecture made above.

Curiously, the formulae are almost identical to the number field case, though the objects involved are quite different. It suggests, however, that there should be some underlying structure on which all these results case be proved and the case of number field and function fields arise by specializing to the case of $\mathbb{Z}$ or $\mathbb{F}_q[T]$.

2. Notation

Throughout this paper we use the following notation:

- $\mathbb{F}_q$, the finite field with $q = p^n$ elements, where p is a prime number;
- $A = \mathbb{F}_q[T]$, the polynomial ring in one variable;
- $K = \mathbb{F}_q(T)$, the quotient field of A;
- $\pi_\infty = T^{-1}$, a uniformizer at the infinite place ∞;
- $K_\infty = \mathbb{F}_q((\pi_\infty))$, the completion of K at ∞;
- K_∞^{sep}, the separable closure of K_∞;
- K_∞^{ur}, the maximal unramified extension of K_∞;
- $\mathbb{C}_\infty$, the completed algebraic closure of K_∞;
- $\mathrm{ord}_\infty = -\deg$, the negative value of the usual degree function;
- $\mathcal{O}_\infty = \mathbb{F}_q[[\pi_\infty]]$, the ∞-adic integers;
- $|\cdot|$, the ∞-adic absolute value on K_∞, extended to $\mathbb{C}_\infty$;
- $|\cdot|_i$, the 'imaginary part' of $|\cdot|$, $|z|_i = \inf_{x\in K_\infty}\{|z-x|\}$;
- G, the group scheme GL_2;
- B, the Borel subgroup of G;
- Z, the center of G;
- $\mathcal{K} = G(\mathcal{O}_\infty)$;
- $\mathcal{I} = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathcal{K} \text{ such that } c \equiv 0 \bmod \infty\right\}$;
- $\mathcal{T}$, the Bruhat–Tits tree of $\mathrm{PGL}_2(K_\infty)$;
- $V(\mathfrak{G})$, the set of vertices of a graph $\mathfrak{G}$;
- $Y(\mathfrak{G})$, the set of oriented edges of an oriented graph $\mathfrak{G}$ (if e is an edge, $o(e)$ and $t(e)$ denote the origin and terminus of the edge);
- $\mathfrak{m}$, a divisor of K with degree $\deg(\mathfrak{m})$ (this is different from $\deg(m) = -\mathrm{ord}_\infty(m)$ for $m \in K$).

3. Preliminaries on Drinfeld modular curves

In the function field setting, there are two analogues of the complex upper half-plane: the Bruhat–Tits tree and the Drinfeld upper half-plane. These sets capture different aspects of the classical upper half-plane. The Bruhat–Tits tree has a transitive group action, but does not have a manifold structure, whereas the Drinfeld upper half-plane has the structure of a rigid analytic manifold, but no transitive group action. These two sets are related by means of the building map. We first describe the Bruhat–Tits tree. We refer to the paper [GR92] for further details.

3.1 The Bruhat–Tits tree

The Bruhat–Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(K_\infty)$ is an oriented graph. It has the following description.

3.1.1 *Vertices and ends of $\mathcal{T}$.* The vertices of $\mathcal{T}$ consist of similarity classes $[L]$, where L is an $\mathcal{O}_\infty$-lattice in $(K_\infty)^2$. Recall that a lattice L is said to be similar to L' ($L \equiv L'$) if and only if there exists an element $c \in K_\infty^*$ such that $L = cL'$. Two vertices $[L]$ and $[L']$ are joined by an edge if they are represented by lattices L and L' with $L \subset L'$ and $\dim_{\mathbb{F}_q}(L'/L) = 1$. Each vertex v has exactly $(q+1)$-adjacent vertices and this set is in bijection with $\mathbb{P}^1(\mathbb{F}_q)$. More generally, the set of vertices of $\mathcal{T}$ which are adjacent to a fixed vertex $[L]$ by at most k edges is in bijection with $\mathbb{P}^1(L/\pi_\infty^k L)$. This makes $\mathcal{T}$ into a $(q+1)$-regular tree.

A half-line is an infinite sequence of adjacent non-repeating vertices $\{v_i\}$ starting with an initial vertex v_0. Two half-lines are said to be equivalent if the symmetric difference of the two sets of vertices is a finite set. An end is an equivalence class of half-lines.

Let $\partial\mathcal{T}$ be the set of the ends of $\mathcal{T}$. There is a bijection (independent of L)

$$\partial\mathcal{T} \xrightarrow{\simeq} \varprojlim_k \mathbb{P}^1(L/\pi_\infty^k L) \simeq \mathbb{P}^1(\mathcal{O}_\infty) = \mathbb{P}^1(K_\infty).$$

The left-action of $G(K_\infty)$ on $\mathcal{T}$ extends to an action on $\partial\mathcal{T}$ which agrees with the action of $G(K_\infty)$ on $\mathbb{P}(K_\infty)$ by fractional linear transformations.

3.1.2 *Orbit spaces.* For $i \in \mathbb{Z}$, let $v_i \in V(\mathcal{T})$ be the vertex $[\pi_\infty^{-i}\mathcal{O}_\infty \oplus \mathcal{O}_\infty]$. As the vertex v_0 has stabilizer $\mathcal{K} \cdot Z(K_\infty)$ in $G(K_\infty)$, one obtains the following identification:

$$G(K_\infty)/\mathcal{K} \cdot Z(K_\infty) \xrightarrow{\sim} V(\mathcal{T}) \quad g \mapsto g(v_0).$$

Similarly, let e_i be the edge $\overrightarrow{v_i v_{i+1}}$ (i.e. $o(e_i) = v_i$, $t(e_i) = v_{i+1}$); then

$$G(K_\infty)/\mathcal{I} \cdot Z(K_\infty) \xrightarrow{\sim} Y(\mathcal{T}) \quad g \mapsto g(e_0).$$

These identifications allow one to consider functions on vertices and on edges of $\mathcal{T}$ as equivariant functions on matrices.

Let $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We set

$$S_V = \left\{ \begin{pmatrix} \pi_\infty^k & u \\ 0 & 1 \end{pmatrix} \,\middle|\, k \in \mathbb{Z},\ u \in K_\infty,\ u \bmod \pi_\infty^k \mathcal{O}_\infty \right\}$$

and

$$S_U = \left\{ w \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \,\middle|\, c \in \mathbb{F}_q \right\} \cup \{1\}, \quad S_Y = \{gh \mid g \in S_V, h \in S_U\}.$$

Then S_V is a system of representatives for $V(\mathcal{T})$ and S_Y is a system of representatives for $Y(\mathcal{T})$ [Pap02]. We will use these systems to define functions on the vertices and the edges of the tree.

3.1.3 *Orientation.* The choice of an end ∞ representing the equivalence class of the half-line $\{v_0, v_1, \ldots\}$, where v_i are as above, defines an orientation on $\mathcal{T}$ in the following manner. If $e = w_0 w_1$ is an edge, e is said to be positively oriented if there is a half-line in the equivalence class of ∞ starting with initial vertex w_0 and subsequent vertex w_1 and negatively oriented if the half-line has initial vertex w_1 and subsequent vertex w_0. For a positively oriented edge, $e = w_0 w_1$, let $o(e) = w_0$ denote the origin of e and $t(e) = w_1$ denote the terminus. This determines a decomposition $Y(\mathcal{T}) = Y(\mathcal{T})^+ \cup Y(\mathcal{T})^-$. We say that $\operatorname{sgn}(e) = +1$ if $e \in Y(\mathcal{T})^+$ and $\operatorname{sgn}(e) = -1$ if $e \in Y^-(\mathcal{T})$.

At a vertex v there is precisely one positively oriented edge with origin v and there are q positively oriented edges with terminus v. That determines a bijection of S_V with the set of positively oriented edges $Y(\mathcal{T})^+$. We will use the notation $v(k, u)$ and $e(k, u)$ to denote the vertex and the positively oriented edge represented by the matrix $\left(\begin{smallmatrix} \pi_\infty^k & u \\ 0 & 1 \end{smallmatrix}\right)$ respectively. The edge $e(k, u)$ has origin $o(e) = v(k, u)$ and terminus $t(e) = v(k-1, u)$.

3.1.4 *Realizations and norms.* The realization $\mathcal{T}(\mathbb{R})$ of the unoriented tree $\mathcal{T}$ is a topological space consisting of a real unit interval for every unoriented edge of $\mathcal{T}$, glued together at the end points according to the incidence relations on $\mathcal{T}$. If e is an edge, we denote by $e(\mathbb{R})$ the corresponding interval on the realization. Let $\mathcal{T}(\mathbb{Z})$ denote the points on $\mathcal{T}(\mathbb{R})$ corresponding to the vertices of $\mathcal{T}$. The set of points $\{t[L]+(1-t)[L'] \mid t \in \mathbb{Q}\}$ lying on edges $([L],[L'])$ will be denoted by $\mathcal{T}(\mathbb{Q})$.

A norm on a K_∞-vector space W is a function $\nu : W \to \mathbb{R}$ satisfying the following properties.

- $\nu(v) \geqslant 0;\ \nu(v)=0 \Leftrightarrow v=0.$
- $\nu(xv)=|x|\nu(v),$ for all $x \in K_\infty$.
- $\nu(v+w) \leqslant \max\{\nu(v),\nu(w)\},$ for all $v,w \in W$.

Two norms ν_1 and ν_2 are said to be similar if there exist non-zero real constants c_1 and c_2 such that

$$c_1 v_1 \leqslant v_2 \leqslant c_2 v_1.$$

The right action of $\mathrm{GL}(W)$ on W induces an action on the set of norms as

$$\gamma(\nu)(v)=\nu(v\gamma).$$

This action descends to similarity classes. The following theorem relates norms to the realization of the tree.

THEOREM 3.1 (Goldman–Iwahori). *There is a canonical $G(K_\infty)$-equivariant bijection b between the set $\mathcal{T}(\mathbb{R})$ and the set of similarity classes of norms on $W=K_\infty^2$.*

This bijection is defined as follows. With a vertex $[L]$ in $\mathcal{T}(\mathbb{Z})=V(\mathcal{T})$ we associate $b([L])$, the class of the norm ν_L defined by

$$\nu_L(v)=\inf\{|x| : x \in K_\infty, v \in xL\}.$$

This norm makes L a unit ball. If P is a point of $\mathcal{T}(\mathbb{R})$ which lies on the edge $([L],[L'])$ with $\pi_\infty L' \subset L \subset L'$ and $P=(1-t)[L]+t[L']$, then $b(P)$ is the class of the norm defined by

$$\nu_P(v)=\sup\{\nu_L(v), q^t \nu_{L'}(v)\}.$$

3.2 Drinfeld's upper half-plane and the building map

The set $\Omega=\mathbb{P}^1(\mathbb{C}_\infty)-\mathbb{P}^1(K_\infty)=\mathbb{C}_\infty-K_\infty$ is called the Drinfeld upper half-plane. This space has the structure of a rigid analytic space over K_∞. There is a canonical $G(K_\infty)$-equivariant map

$$\lambda : \Omega \longrightarrow \mathcal{T}(\mathbb{R}) \tag{2}$$

called the building map. It is defined as follows. With $z \in \Omega$, we associate the similarity class of the norm ν_z on $(K_\infty)^2$ defined by

$$\nu_z((u,v))=|uz+v|.$$

Since $|\cdot|$ takes values in $q^{\mathbb{Q}}$, the image of λ is contained in $\mathcal{T}(\mathbb{Q})$, and in fact one shows that $\lambda(\Omega)=\mathcal{T}(\mathbb{Q})$.

3.3 The pure covering and its associated reduction

The Drinfeld upper half-space is a rigid analytic space. We need to study its reduction at the prime ∞. However, there is no canonical reduction, but there is a natural one obtained by the associated analytic reduction of a certain pure cover of Ω. This is described in detail in [GR92, pp. 33–34] and in fact, we essentially copy from there.

The pure cover is described as follows. For $n \in \mathbb{Z}$, let D_n denote the subset of $\mathbb{C}_\infty$ defined by:

(1) $D_n = \{z \in \mathbb{C}_\infty : |\pi_\infty|^{n+1} \leqslant |z| \leqslant |\pi_\infty|^n\}$;

(2) $|z - c\pi_\infty^n| \geqslant |\pi_\infty|^n, |z - c\pi_\infty^{n+1}| \geqslant |\pi_\infty^{n+1}|$ for all $c \in \mathbb{F}_q^* \subset K_\infty$.

Equivalently,

(2′) $|z| = |z|_i$.

Condition (2′) shows $D_n \subset \Omega$ and is independent of the choice of π_∞. This is an affinoid space over K_∞ with ring of holomorphic functions

$$A_n = K_\infty\langle \pi_\infty^{-n} z, \pi_\infty^{n+1} z^{-1}, (\pi_\infty^{-n} z - c)^{-1}, (\pi_\infty^{-(n+1)} z - c)^{-1} \mid c \in \mathbb{F}_q^*\rangle$$

which is the algebra of 'strictly convergent power series' in $\pi_\infty^{-n} z$. This allows one to define the canonical reduction $(D_n)_\infty$, and this is isomorphic to the union of two projective lines meeting at an $\mathbb{F}_q$-rational point, with all other rational points deleted.

For $\mathbf{i} = (n, x)$, $n \in \mathbb{Z}$, $x \in K_\infty$, let $D_{\mathbf{i}} = D_{(n,x)} = x + D_n$. Then one can see that, if $\mathbf{i}' = (n', x')$,

$$D_{\mathbf{i}} = D_{\mathbf{i}}' \Leftrightarrow n = n' \quad \text{and} \quad |x - x'| \leqslant |\pi_\infty|^{n+1}.$$

So if $I = \{(n, x) \mid n \in \mathbb{Z}, x \in K_\infty/\pi_\infty^{n+1}\mathcal{O}_\infty\}$, where, for each n, x runs through a set of representatives, then

$$\Omega = \bigcup_{\mathbf{i} \in I} D_{\mathbf{i}}$$

is a pure covering of Ω. For any $\mathbf{i}$, there are only finitely many $\mathbf{i}'$ such that $D_{\mathbf{i}} \cap D_{\mathbf{i}'} \neq \phi$.

With respect to this covering one has an associated analytic reduction,

$$R : \Omega \longrightarrow \Omega_\infty$$

where Ω_∞ consists of a union of $\mathbb{P}^1_{\mathbb{F}_q}$ each of which meets $q + 1$ other ones at $\mathbb{F}_q$ rational points. Conversely, any $\mathbb{F}_q$ rational point s of a component M determines a component M' such that $M' \cap M = \{s\}$. For adjacent M and M', let $M^* = M - M(\mathbb{F}_q)$ and

$$(M \cup M')^* = M \cup M' - (M(\mathbb{F}_q) \cup M'(\mathbb{F}_q)) \cup (M \cap M').$$

Then there exits $\mathbf{i}, \mathbf{i}'$ such that

$$R^{-1}(M^*) = D_{\mathbf{i}} \cap D_{\mathbf{i}'} \quad \text{and} \quad R^{-1}((M \cup M')^*) = D_{\mathbf{i}}.$$

The intersection graph of Ω_∞ is the graph whose vertices are the components M of Ω_∞. Two vertices M and M' are joined by an oriented edge if and only if M and M' are adjacent components of Ω_∞, that is, if $M \cap M' \neq \phi$. The map λ in (2) determines a canonical identification of this graph with the Bruhat–Tits tree: given a component M there exists a unique $[L] \in \mathcal{T}(\mathbb{Z})$ such that

$$\lambda^{-1}([L]) = R^{-1}(M^*)$$

and this association is compatible with the group actions and identifies the two graphs. We will use this identification rather crucially in the final step of the proof.

3.4 Drinfeld modular curves of level I

For a monic polynomial I in A let

$$\Gamma_0(I)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma \;\middle|\; c\equiv 0 \bmod I\right\}.$$

$\Gamma_0(I)$ acts discretely on Ω via Möbius transformations: for $z\in\Omega$ and $\gamma=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_0(I)$, define

$$\gamma z=\frac{az+b}{cz+d}.$$

The Drinfeld modular curve $X_0(I)$ of level I is a smooth, proper, irreducible algebraic curve, defined over K, such that its $\mathbb{C}_\infty$ points have the structure of a rigid analytic space and there is a canonical isomorphism of analytic spaces over $\mathbb{C}_\infty$

$$X_0(I)(\mathbb{C}_\infty)\simeq\Gamma_0(I)\backslash\Omega\cup\{\text{cusps}\}$$

where the cusps are finitely many points in bijection with $\Gamma_0(I)\backslash\mathbb{P}^1(K)$. Entirely analogous to the classical construction over a number field, the Drinfeld modular curve $X_0(I)$ parameterizes Drinfeld modules of rank two with level I structure.

Let

$$\mathcal{T}_0(I)=\Gamma_0(I)\backslash\mathcal{T}$$

denote the corresponding quotient of the Bruhat–Tits tree by the left action of $\Gamma_0(I)$. Let $X(\mathcal{T}_0(I))$ and $Y(\mathcal{T}_0(I))$ denote the vertices and edges of the graph $\mathcal{T}_0(I)$ respectively. The graph $\mathcal{T}_0(I)$ is an infinite graph consisting of a finite graph $\mathcal{T}_0(I)^0$ and a finite number of ends corresponding to the finitely many cusps [GR92, § 2.6].

The curve $X_0(I)$ is a totally split curve over K_∞. The pure covering of Ω induces a pure covering of $X_0(I)$ and the associated analytic reduction R is a scheme $X_0(I)_\infty$ over $\mathbb{F}_q$ which is a finite union of $\mathbb{P}^1_{\mathbb{F}_q}$ intersecting at $\mathbb{F}_q$ rational points. The intersection graph of this scheme is the finite part $\mathcal{T}_0(I)^0$ of the graph $\mathcal{T}_0(I)$.

3.5 Harmonic cochains on the Bruhat–Tits tree

In the function field setting there are two notions of modular forms, corresponding to the two analogues of the complex upper half-plane. One notion deals with certain equivariant functions on the Drinfeld upper half-plane while the other refers to certain invariant harmonic cochains on the Bruhat–Tits tree. The latter are sometimes called automorphic forms of Jacquet–Langlands–Drinfeld (JLD) type. It is these that we will be concerned with and will review their definition and properties in this section.

If R is a commutative ring, an R-valued harmonic cochain on $Y(\mathcal{T})$ is a map $\phi:Y(\mathcal{T})\longrightarrow R$ satisfying the harmonic conditions:

- $\phi(e)+\phi(\overline{e})=0$;
- $\sum_{t(e)=v}\phi(e)=0$

where, for e in $Y(\mathcal{T})$, $\overline{e}$ denotes the same edge with the opposite orientation. The second condition can also be stated as follows. First, notice that there is precisely one edge e_0 with $t(e_0)=v$ and $\operatorname{sgn}(e_0)=-1$. The second condition is then equivalent to

$$\phi(e_0)=\sum_{\substack{t(e)=v\\ \operatorname{sgn}(e)=1}}\phi(e).$$

If Γ is a subgroup of $G(A)$ we will consider cochains satisfying the further condition of Γ-invariance, namely:

- $\phi(\gamma e)=\phi(e)$, for all $\gamma\in\Gamma$.

The group of Γ-invariant, R-valued harmonic cochains on the edges of $\mathcal{T}$ is denoted by $H(Y(\mathcal{T}),R)^{\Gamma}$. The harmonic functions on the edges of $\mathcal{T}$ are the analogues of classical cusp forms of weight two. In fact, if $\ell\neq p$ is a prime number, the Γ-invariant harmonic cochains detect 'half' of the étale cohomology group $H^1_{\text{ét}}(X(\Gamma),\mathbb{Q}_\ell)$ [Tei92, p. 272].

An R-valued harmonic cochain f is said to be of level I if $f\in H(Y(\mathcal{T}),R)^{\Gamma_0(I)}$. If f has finite support as a function on $\Gamma_0(I)\backslash Y(\mathcal{T})$, it is called a cusp form or said to be cuspidal. Usually we will deal with $\mathbb{Z}$, $\mathbb{C}$ or $\mathbb{Q}_\ell$ valued functions. In analogy with the classical case, we sometimes will use the word 'form' to denote these functions.

3.5.1 *Fourier expansions.* A harmonic function on the set of positively oriented edges $Y(\mathcal{T})^+$ which is invariant under the action of the group

$$\Gamma_\infty=\left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\in G(A)\right\}$$

has a Fourier expansion. This statement is a consequence of the general theory of Fourier analysis on adèle groups. Details can be founds in [Gek95]. This expansion has the following description. Let $\eta:K_\infty\to\mathbb{C}^*$ be the character defined as

$$\eta\left(\sum_j a_j\pi_\infty^j\right)=\exp\left(\frac{2\pi i\ \mathrm{Tr}(a_1)}{p}\right)$$

where Tr is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$. Then, the Fourier expansion of a Γ_∞-invariant function f on $Y(\mathcal{T})^+$ is given by

$$f\left(\begin{pmatrix}\pi_\infty^k & u\\ 0 & 1\end{pmatrix}\right)=c_0(f,\pi_\infty^k)+\sum_{\substack{0\neq m\in A\\ \deg(m)\leqslant k-2}}c(f,\mathrm{div}(m)\cdot\infty^{k-2})\eta(mu).$$

The constant Fourier coefficient $c_0(f,\pi_\infty^k)$ is the function of $k\in\mathbb{Z}$ given by

$$c_0(f,\pi_\infty^k)=\begin{cases} f\left(\begin{pmatrix}\pi_\infty^k & 0\\ 0 & 1\end{pmatrix}\right) & \text{if } k\leqslant 1,\\[2ex] q^{1-k}\displaystyle\sum_{u\in(\pi_\infty)/(\pi_\infty^k)} f\left(\begin{pmatrix}\pi_\infty^k & u\\ 0 & 1\end{pmatrix}\right) & \text{if } k\geqslant 1.\end{cases}$$

For a non-negative divisor $\mathfrak{m}$ on K, with $\mathfrak{m}=\mathrm{div}(m)\cdot\infty^{\deg(\mathfrak{m})}$, the non-constant Fourier coefficient is

$$c(f,\mathfrak{m})=q^{-1-\deg(\mathfrak{m})}\sum_{u\in(\pi_\infty)/(\pi_\infty^{2+\deg(\mathfrak{m})})} f\left(\begin{pmatrix}\pi_\infty^{2+\deg(\mathfrak{m})} & u\\ 0 & 1\end{pmatrix}\right)\eta(-mu).$$

3.5.2 *Petersson inner product.* There is an analogue of the Petersson inner product for invariant functions on the tree $\mathcal{T}$.

If f and g are complex valued harmonic cochains for $\Gamma_0(I)$, one of which is cuspidal, define

$$\delta(f,g)(e) = f(e)\overline{g(e)}\,d\mu(e) \qquad \text{for } e \in Y(\mathcal{T}_0(I))$$

where $\mu(\cdot)$ is the Haar measure on the discrete set $Y(\mathcal{T}_0(I))$ defined by $\mu(e) = (q-1)/2|\mathrm{Stab}_{\Gamma_0(I)}(e)|^{-1}$, where $|\mathrm{Stab}_{\Gamma_0(I)}(e)|$ is the cardinality of the stabilizer of $e \in Y(\mathcal{T}_0(I))$. The Petersson inner product of f and g is defined as

$$\langle f, g\rangle = \int_{Y(\mathcal{T}_0(I))} \delta(f,g) = \int_{Y(\mathcal{T}_0(I))} f(e)\overline{g(e)}\,d\mu(e).$$

3.5.3 *Hecke operators and Hecke eigenforms.* Let $\mathfrak{p}$ be a prime and I a fixed level. The Hecke operator $T_{\mathfrak{p}}$ is the operator on $H(Y(\mathcal{T}),\mathbb{C})^{\Gamma_0(I)}$ defined by

$$T_{\mathfrak{p}}(f)(e) = \begin{cases} f\left(e\begin{pmatrix}\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right) + \displaystyle\sum_{r \bmod \mathfrak{p}} f\left(e\begin{pmatrix}1 & r\\ 0 & \mathfrak{p}\end{pmatrix}\right) & \text{if } \mathfrak{p} \nmid I,\\ \displaystyle\sum_{r \bmod \mathfrak{p}} f\left(e\begin{pmatrix}1 & r\\ 0 & \mathfrak{p}\end{pmatrix}\right) & \text{if } \mathfrak{p} | I. \end{cases}$$

A Hecke eigenform f is a harmonic cochain of level I which is an eigenfunction of all the Hecke operators $T_{\mathfrak{p}}$. f is called a newform if in addition it lies in the orthogonal complement, with respect to the Petersson inner product, of the space generated by all cusp forms of level I' for all levels I' properly dividing I.

If f is a non-zero newform, then the coefficient $c(f,1)$ in the Fourier expansion is not zero. The form f is said to be normalized if one further assumes that $c(f,1)=1$. Let $\lambda_{\mathfrak{p}}$ denote the eigenvalue of the Hecke operator $T_{\mathfrak{p}}$. The Fourier coefficients of a cuspidal, normalized newform f have the following special properties.

- $c_0(f, \pi_\infty^k) = 0$, for all $k \in \mathbb{Z}$.
- $c(f,1) = 1$.
- $c(f,\mathfrak{m})c(f,\mathfrak{n}) = c(f,\mathfrak{m}\mathfrak{n})$, whenever $\mathfrak{m}$ and $\mathfrak{n}$ are relatively prime.
- $c(f,\mathfrak{p}^{n-1}) - \lambda_{\mathfrak{p}} c(f,\mathfrak{p}^n) + |\mathfrak{p}|c(f,\mathfrak{p}^{n+1}) = 0$, if $\mathfrak{p} \nmid I\cdot\infty$.
- $c(f,\mathfrak{p}^{n+1}) - \lambda_{\mathfrak{p}} c(f,\mathfrak{p}^n) = 0$, if $\mathfrak{p}|I$.
- $c(f,\infty^{n-1}) = q^{-n+1}$, if $n \geqslant 1$.

If f and g are normalized Hecke eigenforms and $f \neq g$, then $\langle f, g\rangle = 0$. Further, since the Hecke operators are self adjoint, $f = \bar{f}$.

3.5.4 *Logarithms and the logarithmic derivative.* Let f be a $\mathbb{C}_\infty$-valued invertible function on Ω. There is a notion of the logarithm of $|f|$ defined as follows. Let v be a vertex of $\mathcal{T}$ and $\tau_v \in \Omega$ an element of $\lambda^{-1}(v)$ where λ is the building map defined in (2). From § 3.3 one can see that the function $|\cdot|$ factors through the building map, so the quantity $|f|$ depends only on v and not on the choice of τ_v.

Define

$$\log|f|(v) = \log_q |f(\tau_v)|. \tag{3}$$

This function takes values in $\mathbb{Z}$.

If g is a function on the vertices of the tree $\mathcal{T}$, then the derivative of g is a function on the edges of $\mathcal{T}$ defined to be

$$\partial g(e) = g(t(e)) - g(o(e)). \tag{4}$$

The logarithmic derivative of an invertible function f on Ω is the composite of these two maps, namely,

$$\partial \log|f|(e) = \log|f|(t(e)) - \log|f|(o(e)). \tag{5}$$

3.5.5 *The cohomology of a Drinfeld modular curve.* The cohomology of a Drinfeld modular curve has a decomposition, due to Drinfeld, which is analogous to the classical decomposition of the cohomology of a modular curve into eigenspaces of modular forms of weight two.

Let ℓ be a prime, $\ell \neq p$. There is a two-dimensional ℓ-adic representation $\mathfrak{sp}_\ell$ of the Galois group $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$, called the special representation, which acts through a quotient isomorphic to $\hat{\mathbb{Z}} \ltimes \mathbb{Z}_\ell(1)$. The group $\hat{\mathbb{Z}}$ is isomorphic to $\mathrm{Gal}(K_\infty^{ur}/K_\infty)$. The canonical generator of $\hat{\mathbb{Z}}$ corresponds to F_∞, the Frobenius automorphism of K_∞^{ur}/K_∞. The group $\mathbb{Z}_\ell(1)$ is isomorphic to $\mathrm{Gal}(E_\ell/K_\infty^{ur})$, where E_ℓ/K_∞^{ur} is the field extension obtained by adjoining all the ℓ^rth roots of the uniformizer π_∞ to K_∞^{ur}. The action of $F_\infty = 1 \in \hat{\mathbb{Z}}$ on $\mathbb{Z}_\ell(1)$ is given by $F_\infty u F_\infty^{-1} = u^q$, for $u \in \mathbb{Z}_\ell(1)$. Choose an isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$, then

$$\mathfrak{sp}_\ell : \mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty) \twoheadrightarrow \hat{\mathbb{Z}} \ltimes \mathbb{Z}_\ell \to \mathrm{Gl}(2, \mathbb{Q}_\ell)$$

where the right-hand side arrow is defined as

$$(1,0) = F_\infty \mapsto \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad (0,1) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

We recall the following theorem of Drinfeld [GR92].

Theorem 3.2 (Drinfeld). *Let $X_0(I)$ be a Drinfeld modular curve with level I structure and let $\bar{X}_0(I) = X_0(I) \times \mathrm{Spec}(\mathbb{C}_\infty)$. Then*

$$H^1_{\acute{e}t}(\bar{X}_0(I), \mathbb{Q}_\ell) \cong H(Y(\mathcal{T}), \mathbb{Q}_\ell)^{\Gamma_0(I)} \otimes \mathfrak{sp}_\ell. \tag{7}$$

This isomorphism is compatible with the action of the local Galois group $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ and the action of the Hecke operators.

A consequence of this theorem and Eichler–Shimura type relations [GR92, (4.13.2)] is the decomposition of the L-function of a Drinfeld modular curve into a product of L-functions of Hecke eigenforms:

$$L(H^1_{\text{ét}}(\bar{X}_0(I)), s) = \prod L(h^1(M_f), s), \quad s \in \mathbb{C}, \tag{8}$$

where $L(h^1(M_f), s) = L(f, s)$ is the L-function of the motive $h^1(M_f)$ corresponding to the Hecke eigenform f [Pap02, p. 332].

In this paper, we focus on the study of the L-function of $H^2(\bar{X}_0(I) \times \bar{X}_0(I), \mathbb{Q}_\ell)$. Applying the Künneth formula we get the following decomposition:

$$L(H^2_{\text{ét}}(\bar{X}_0(I) \times \bar{X}_0(I)), s) = L(H^2_{\text{ét}}(\bar{X}_0(I)), s)^2 L(H^1_{\text{ét}}(\bar{X}_0(I)) \otimes H^1_{\text{ét}}(\bar{X}_0(I)), s). \tag{9}$$

The incomplete (here we omit the local factor at ∞) L-function of $H^2_{\text{ét}}(\bar{X}_0(I))$ is $\zeta_A(s-1) = 1/(1-q^{2-s})$. Under the assumption that the level is square-free and using the decomposition (8), the L-function of the last factor in (9) can be expressed as a product

$$L(H^1(\bar{X}_0(I)) \otimes H^1(\bar{X}_0(I)), s) = \zeta_A(2s)^{-1} \prod_{f,g} L_{f,g}(s)$$

where f and g are normalized newforms of JLD type and level I. Also, $L_{f,g}(s)$ is the Rankin–Selberg convolution L-function defined in the next section. It is essentially the L-function of the tensor product of the motives $h^1(M_f) \otimes h^1(M_g)$.

4. The Rankin–Selberg convolution

The main goal of this section is the computation of a special value of the convolution L-function of two automorphic forms of JLD-type verifying certain prescribed conditions.

We begin by studying certain Eisenstein series on the Bruhat–Tits tree $\mathcal{T}$. The classical Eisenstein–Kronecker–Lerch series are real analytic functions on the upper half-plane, invariant under the action of a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. They are related to logarithms of modular units on the associated modular curve via the Kronecker limit formulas.

There are function field analogues of these series, as well as an analogue of Kronecker's first limit formula. These results follow from the work of Gekeler [Gek95] and they are the crucial steps in the process of relating the regulators of elements in K-theory to special values of L-functions.

4.1 Eisenstein series

The real analytic Eisenstein series for $\Gamma_0(I)$ is defined as

$$E_I(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(I)} |\gamma(\tau)|_i^s, \quad \tau \in \Omega, \ s \in \mathbb{C}.$$

This series converges absolutely for $Re(s) \gg 0$ and from the definition one can see that it is $\Gamma_0(I)$ invariant.

The 'imaginary part' function $|z|_i = \inf\{|z - x| \,;\, x \in K_\infty\}$ factors through the building map, so the Eisenstein series can be thought of as a function defined on the vertices of the Bruhat–Tits tree. In terms of the matrix representatives S_V, $E_I(\tau, s)$ can be expressed as follows.

Let $m, n \in A$, $(m, n) \neq (0, 0)$ and let $v \in V(\mathcal{T})$ be a vertex represented by $v = \left(\begin{smallmatrix} \pi^k & u \\ 0 & 1 \end{smallmatrix}\right)$. For $\omega = \mathrm{ord}_\infty(mu + n)$ and $s \in \mathbb{C}$, define

$$\phi^s_{m,n}(v) = \phi^s_{m,n}\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} = \begin{cases} q^{(k-2\deg(m))s} & \text{if } \omega \geqslant k - \deg(m), \\ q^{(2\omega - k)s} & \text{if } \omega < k - \deg(m). \end{cases} \tag{10}$$

Then, using an explicit set of representatives for $\Gamma_\infty \backslash \Gamma_0(I)$, we have (see [Pap02, § 4] for details)

$$E_I(v, s) = q^{-ks} + \sum_{\substack{m \in A \\ m \text{ monic} \\ m \equiv 0 \bmod I}} \sum_{\substack{n \in A, \\ (m,n)=1}} \phi^s_{m,n}(v).$$

Let $E(v, s) = E_1(v, s)$. In [Pap02], it is shown that $E_I(v, s)$ has an analytic continuation to a meromorphic function on the entire complex plane, with a simple pole at $s = 1$. Lemma 3.4 of [Pap02] relates the two series E and E_I through the formula

$$\zeta_I(2s) E_I(v, s) = \frac{\zeta(2s)}{|I|^s} \sum_{\substack{d|I \\ d \text{ monic}}} \frac{\mu(d)}{|d|^s} E((I/d)v, s) \tag{11}$$

where $\zeta(s)=1/(1-q^{1-s})$ is the zeta function of A, $\zeta_I(s)=\prod_{\mathfrak{p}\nmid I}(1-|\mathfrak{p}|^{-s})^{-1}$, $\mu(\cdot)$ is the Möbius function of A, defined entirely analogously to the usual Möbius function using the monic prime factorization of an element of A, and $(I/d)v$ denotes the action of the matrix $\left(\begin{smallmatrix} I/d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on v.

Notice that a function F on the vertices of $\mathcal{T}$ can be considered as a function on the edges of the tree by defining $F(e)=F(o(e))$. In particular, if we define

$$E_I(e,s)=E_I(o(e),s), \quad e\in Y(\mathcal{T}), \tag{12}$$

we recover the definition given in [Pap02, § 3].

4.1.1 *Functional equation.* The Eisenstein series $E(e,s)$ satisfies a functional equation analogous to that satisfied by the classical Eisenstein–Kronecker–Lerch series. The analogue of the 'Archimedean factor' of the zeta function of A is

$$L_\infty(s)=(1-|\infty|^s)^{-1}=\frac{1}{1-q^{-s}}. \tag{13}$$

We recall the following result.

Theorem 4.1. *Define $\Lambda(e,s)=-L_\infty(s)E(e,s)$. Then, $\Lambda(e,s)$ has a simple pole at $s=1$ with residue $-(\log q)^{-1}$ and satisfies the functional equation*

$$\Lambda(e,s)=-\Lambda(e,1-s).$$

Proof. [Pap02, Theorem 3.3]. □

4.2 The Rankin–Selberg convolution

In [Pap02], Papikian describes a function field analogue of the Rankin–Selberg formula. In this section we will apply this result by using the interpretation of the Eisenstein series $E_I(v,s)$ as an automorphic form on the edges of the Bruhat–Tits tree $\mathcal{T}$.

Let f and g be two automorphic forms of level I on $\mathcal{T}$. Consider the series

$$L_{f,g}(s)=\zeta_I(2s)\sum_{\substack{\mathfrak{m}\text{ effective divisors}\\(\mathfrak{m},\infty)=1}}\frac{c(f,\mathfrak{m})\bar{c}(g,\mathfrak{m})}{|\mathfrak{m}|^{s-1}}.$$

If f and g are normalized newforms, using the decomposition $\mathfrak{m}=\mathfrak{m}_{\text{fin}}\infty^d$ $(d\geqslant 0)$ and the last property of the Fourier coefficients listed in § 3.5.3, namely,

$$c(f,\mathfrak{m})=c(f,\mathfrak{m}_{\text{fin}}\cdot\infty^d)=c(f,\mathfrak{m}_{\text{fin}})q^{-d},$$

we can pull out the Euler factor at ∞ and we have

$$\zeta_I(2s)\sum_{\mathfrak{m}\text{ effective divisors}}\frac{c(f,\mathfrak{m})\bar{c}(g,\mathfrak{m})}{|\mathfrak{m}|^{s-1}}=L_\infty(s+1)L_{f,g}(s).$$

Proposition 4.2 ('Rankin's trick'). *Let f and g be two cusp forms of level I. Then*

$$\zeta_I(2s)\langle fE_I(e,s),\ g\rangle \ = \ \zeta_I(2s)\int_{Y(\mathcal{T}_0(I))}E_I(e,s)f(e)\overline{g(e)}\,d\mu(e)=q^{1-2s}L_\infty(s+1)L_{f,g}(s).$$

Proof. See [Pap02, § 4]. □

We set $\Phi(s) = \Phi_{f,g}(s)$ where

$$\Phi_{f,g}(s) := -\frac{\zeta_I(2s)L_\infty(s)|I|^s}{\zeta(2s)} \int_{Y(\mathcal{T}_0(I))} E_I(e,s) f(e)\overline{g(e)}\, d\mu(e). \tag{14}$$

It follows from the proposition above, (11), (12) and Theorem 4.1 that $\Phi(s)$ has the following description:

$$\begin{aligned}\Phi(s) &= \sum_{\substack{d|I \\ d \text{ monic}}} \frac{\mu(d)}{|d|^s} \int_{Y(\mathcal{T}_0(I))} \Lambda((I/d)e, s) f(e)\overline{g(e)}\, d\mu(e) \\ &= -q^{1-2s}|I|^s L_{f,g}(s) L_\infty(s) L_\infty(s+1)\zeta(2s)^{-1}. \end{aligned} \tag{15}$$

The function $\Phi(s-1)$ is the *completed* (namely with the factors at ∞ included) L-function of the motive $h^1(M_f) \otimes h^1(M_g)$,

$$\Phi_{f,g}(s-1) = \Lambda(h^1(M_f) \otimes h^1(M_g), s).$$

For future use, we recall the following result.

THEOREM 4.3 (Rankin). *The L-function $L_{f,g}(s)$ has a simple pole at $s = 1$ with residue a non-zero multiple of $\langle f, g\rangle$, the Petersson inner product of f and g. In particular, if f and g are normalized newforms and $f \neq g$, then $\langle f, g\rangle = 0$, so $L_{f,g}(s)$ does not have a pole at $s = 1$.*

It follows that if f and g are normalized newforms and $f \neq g$, then $\Phi(1) = q|I|L_{f,g}(1)/(1-q^2)$. To compute the value $\Phi(0)$ we will make use of the following theorem.

THEOREM 4.4. *Let f and g be two cuspidal eigenforms of square-free levels I_1, I_2 respectively, with I_1 and I_2 co-prime monic polynomials. Let $I = I_1 I_2$. Then, the function $\Phi(s)$ as defined in (14) satisfies the functional equation*

$$\Phi(s) = -\Phi(1-s).$$

Proof. The method of the proof is similar to that of Ogg [Ogg69, § 4]. Using the Atkin–Lehner operators, we can simplify the integral in (14). Let $\mathfrak{p}$ be a monic prime element of A such that $\mathfrak{p}|I$, say $\mathfrak{p}|I_1$. The Atkin–Lehner operator $W_\mathfrak{p}$ corresponding to $\mathfrak{p}$ is represented by

$$\beta = \begin{pmatrix} a\mathfrak{p} & -b \\ I & \mathfrak{p} \end{pmatrix}, \quad \det(\beta) = \mathfrak{p}, \quad \beta \in \Gamma_0(I/\mathfrak{p}) \begin{pmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for some } a, b \in A.$$

Let d be a monic divisor of $I/\mathfrak{p}$, then

$$\begin{pmatrix} I/\mathfrak{p}d & 0 \\ 0 & 1 \end{pmatrix} \beta \begin{pmatrix} I/d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in \Gamma.$$

As $\Lambda(e, s)$ of Theorem 4.1 is Γ-invariant we have

$$\Lambda((I/\mathfrak{p}d)\beta e, s) = \Lambda((I/d)e, s)$$

where $(I/\mathfrak{p}d) = \left(\begin{smallmatrix} I/\mathfrak{p}d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since β normalizes $\Gamma_0(I_1)$ and f is a newform, we obtain

$$f|_\beta = f_{W_\mathfrak{p}} = c(f, \mathfrak{p}) f$$

where $c(f,\mathfrak{p}) = \pm 1$. Further, if $h = g|_\beta$, then $h|_\beta = g|_{\beta^2} = g$. We then have

$$\begin{aligned}\int_{Y(\mathcal{T}_0(I))} \Lambda((I/\mathfrak{p}d)e, s)\delta(f,g) &= \int_{\beta^{-1}(Y(\mathcal{T}_0(I)))} \Lambda((I/d)e, s)c(f,\mathfrak{p})\delta(f, g|_\beta)\\ &= \int_{Y(\mathcal{T}_0(I))} \Lambda((I/d)e, s)c(f,\mathfrak{p})\delta(f,h) \qquad (16)\end{aligned}$$

as $\beta^{-1}(Y(\mathcal{T}_0(I)))$ is a fundamental domain for $\beta^{-1}\Gamma_0(I)\beta = \Gamma_0(I)$. We deduce that

$$\Phi(s) = \sum_{\substack{d|(I/\mathfrak{p})\\ d \text{ monic}}} \frac{\mu(d)}{|d|^s} \int_{Y(\mathcal{T}_0(I))} \Lambda((I/d)e, s)(\delta(f,g) + c(f,\mathfrak{p})\delta(f,h)). \qquad (17)$$

Now, we repeat this process with h in the place of g. It follows from the Fourier expansion that

$$L_{f,h}(s) = c(f,\mathfrak{p})|\mathfrak{p}|^{-s}L_{f,g}(s).$$

Substituting this expression in (17), we have

$$\begin{aligned}c(f,\mathfrak{p})|\mathfrak{p}|^{-s}\Phi(s) &= \sum_{\substack{d|(I/\mathfrak{p})\\ d \text{ monic}}} \int_{Y(\mathcal{T}_0(I))} (\Lambda((I/d)e, s) - |\mathfrak{p}|^{-s}\Lambda((I/\mathfrak{p}d)e, s))\delta(f,h)\\ &= \sum_{\substack{d|(I/\mathfrak{p})\\ d \text{ monic}}} \frac{\mu(d)}{|d|^s} \int_{Y(\mathcal{T}_0(I))} \Lambda((I/d)e, s)\bigg(\delta(f,h) + \frac{c(f,\mathfrak{p})}{|\mathfrak{p}|^s}\delta(f,g)\bigg). \qquad (18)\end{aligned}$$

We denote by $\mathfrak{S}$ the sum of the terms involving $\delta(f,h)$. Comparing (17) and (18), we obtain

$$\mathfrak{S}\bigg(\frac{c(f,\mathfrak{p})}{|\mathfrak{p}|^s} - 1\bigg) = 0.$$

The only way this equation can hold for all s in $\mathbb{C}$ is if $\mathfrak{S} = 0$. Hence we obtain

$$\Phi(s) = \sum_{\substack{d|(I/\mathfrak{p})\\ d \text{ monic}}} \frac{\mu(d)}{|d|^s} \int_{Y(\mathcal{T}_0(I))} \Lambda((I/d)e, s)\delta(f,g).$$

Repeating this process for all primes $\mathfrak{p}$ dividing I, keeping in mind the assumption that a prime divides I_1 or I_2 but not both, we get

$$\Phi(s) = \int_{Y(\mathcal{T}_0(I))} \Lambda((I)e, s)\delta(f,g).$$

As $\Lambda((I)e, s)$ satisfies the functional equation $\Lambda((I)e, s) = -\Lambda((I)e, 1-s)$, we finally obtain

$$\Phi(s) = -\Phi(1-s). \qquad \square$$

4.3 Kronecker's limit formula and the delta function

For the computation of the value $\Phi(0)$ we introduce the Drinfeld discriminant function Δ.

4.3.1 *The discriminant function and the Drinfeld modular unit.* Let τ be a coordinate function on Ω and let $\Lambda_\tau = \langle 1, \tau\rangle$ be the rank-two free A-submodule of $\mathbb{C}_\infty$ generated by 1 and τ. Consider the following product:

$$e_{\Lambda_\tau}(z) = z \prod_{\lambda \in \Lambda_\tau \setminus \{0\}} \bigg(1 - \frac{z}{\lambda}\bigg) = z \prod_{\substack{a,b \in A\\ (a,b) \neq (0,0)}} \bigg(1 - \frac{z}{a\tau + b}\bigg).$$

This product converges to give an entire, $\mathbb{F}_q$-linear, surjective, Λ_τ-periodic function $e_{\Lambda_\tau} : \mathbb{C}_\infty \to \mathbb{C}_\infty$, called the Carlitz exponential function attached to Λ_τ. This is the function field analogue of the classical $\wp$-function and it provides the structure of a Drinfeld A-module to the additive group scheme $\mathbb{C}_\infty/\Lambda_\tau$.

The discriminant function $\Delta : \Omega \to \mathbb{C}_\infty$ is the analytic function defined by

$$\Delta(\tau) = \prod_{\substack{\alpha,\beta \in T^{-1}A/A \\ (\alpha,\beta)\neq(0,0)}} e_{\Lambda_\tau}(\alpha z + \beta).$$

This is a modular form of weight $q^2 - 1$. For $I \neq 1$, a monic polynomial in A, let Δ_I be the function

$$\Delta_I(\tau) := \prod_{\substack{d|I \\ d\,\text{monic}}} \Delta((I/d)\tau)^{\mu(d)}. \tag{19}$$

Here $\mu(\cdot)$ denotes the Möbius function on A. Since

$$\sum_{\substack{d|I \\ d\,\text{monic}}} \mu(d) = 0$$

one has that the weight of Δ_I is zero so it is in fact a $\Gamma_0(I)$-invariant function on Ω. As we will see later in equation (30), it is a modular unit, that is, its divisor is supported on the cusps and defined over K . We call this the Drinfeld modular unit for $\Gamma_0(I)$.

4.3.2 *The Kronecker limit formula.* The classical Kronecker limit formula links the Eisenstein series to the logarithm of the discriminant function Δ. In this section, we prove an analogue of this result in the function field case.

We first compute the constant term $a_0(v)$ in the Taylor expansion of $E(v, s) = E_1(v, s)$ around $s = 1$. We have

$$E(v, s) = \frac{a_{-1}}{s-1} + a_0(v) + a_1(v)(s-1) + \cdots, \tag{20}$$

where a_{-1} is a constant independent of v. To compute explicitly the coefficient function $a_0(v)$ we differentiate 'with respect to v', namely we apply the ∂ operator defined in § 3.5.4 and then evaluate the result at $s = 1$. This computation gives

$$\partial E(\cdot, s)|_{s=1} = \partial a_0(\cdot).$$

It follows from (4) that

$$a_0(v) = \int_{v_0}^{v} \partial E(\cdot, s)(e)|_{s=1}\, d\mu(e) + C$$

where v_0 is any vertex on the tree and C is a constant. For definiteness we can choose v_0 to be the vertex corresponding to the lattice $[\mathcal{O}_\infty \oplus \mathcal{O}_\infty]$. The integration is to be understood as the weighted sum of the value of the function on the edges lying on the unique path joining v_0 and v.

The function $\partial E(\cdot, s)$ on the edges in $Y(\mathcal{T}_0)$ is related to the logarithmic derivative of the discriminant function Δ through an improper Eisenstein series studied by Gekeler in [Gek97].

We first define Gekeler's series [Gek95]. For $e = e(k, u) \in Y(\mathcal{T})$, $s \in \mathbb{C}$, let $\psi^s(e) = \mathrm{sgn}(e)q^{-ks}$. Consider the following Eisenstein series:

$$F(e, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi^s(\gamma(e)).$$

This series converges for $\mathrm{Re}(s) \gg 0$. Let $m, n \in A$, $(m, n) \neq (0, 0)$. For $\omega = \mathrm{ord}_\infty(mu + n)$ let

$$\psi^s_{m,n}(e) = \psi^s_{m,n}(e(k, u)) = \psi^s_{m,n}\left(\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} -q^{(k-2\deg(m)-1)s} & \text{if } \omega \geqslant k - \deg(m), \\ q^{(2\omega-k)s} & \text{if } \omega < k - \deg(m). \end{cases}$$

We have

$$F(e, s) = \psi^s(e) + \sum_{\substack{m \in A \\ m \text{ monic}}} \sum_{\substack{n \in A \\ (m,n)=1}} \psi^s_{m,n}(e).$$

One can consider the limit as $s \to 1$ but the resulting function $F(e, 1)$, while $G(A)$ invariant, is not harmonic. The series for $F(e, 1)$ does not converge.

Gekeler [Gek95, (4.4)] defines a conditionally convergent improper Eisenstein series $\tilde{F}(e)$ as follows:

$$\tilde{F}(e) = \sum_{\substack{m \in A \\ \text{monic}}} \sum_{\substack{n \in A \\ (m,n)=1}} \psi_{m,n}(e) + \psi(e). \tag{21}$$

This function is harmonic, but not $G(A)$ invariant. The relation between these two functions is given as follows [Gek95, Corollary 7.11]:

$$F(e, 1) = \tilde{F}(e) - \mathrm{sgn}(e)\frac{q+1}{2q}. \tag{22}$$

This equation helps us relate the functions $E(\cdot, 1)$ and $\log|\Delta|$ as they are connected to F and $\tilde{F}$ through their derivatives. Precisely, we have the following theorem.

Theorem 4.5 (Gekeler). *Let $\tilde{F}$ be as above. We have*

$$\partial \log|\Delta|(e) = (1 - q)\tilde{F}(e). \tag{23}$$

Proof. See [Gek97, Corollary 2.8]. □

Moreover the following lemma describes a relation between the series $E(v, s)$ and $F(e, s)$.

Lemma 4.6. *Let $E(v, s)$ be the series in (20) and let $F(e, s)$ be the series defined in (22). Then*

$$\partial E(\cdot, s)(e) = (q^s - 1)F(e, s). \tag{24}$$

Proof. It follows from the definition of the derivative of a function given in (4) that

$$\partial E(\cdot, s)(e) = E(t(e), s) - E(o(e), s).$$

Set $e = e(k, u) = v(k, u)v(\vec{k} - 1, u)$. Let $\phi^s_{m,n}(v)$ be the function defined in (10). There are four cases to consider.

Case 0. For $e = e(k, u)$,

$$\begin{aligned} \phi^s(t(e)) - \phi^s(o(e)) &= q^{(k-1)s} - q^{-ks} \\ &= (q^s - 1)q^{-ks} \\ &= (q^s - 1)\psi^s(e). \end{aligned}$$

Case 1. If $\omega > k - 1 - \deg(m)$, then

$$\begin{aligned} \phi^s_{m,n}(t(e)) - \phi^s_{m,n}(o(e)) &= q^{(k-1-2\deg(m))s} - q^{(k-2\deg(m))s} \\ &= (1 - q^s)q^{(k-1-2\deg(m))s} \\ &= (q^s - 1)\psi^s_{m,n}(e). \end{aligned}$$

Case 2. If $\omega < k-1-\deg(m)$, then

$$\begin{aligned}\phi^s_{m,n}(t(e)) - \phi^s_{m,n}(o(e)) &= q^{(2\omega-(k-1))s} - q^{(2\omega-k)s}\\ &= (q^s-1)q^{(2\omega-k)s}\\ &= (q^s-1)\psi^s_{m,n}(e).\end{aligned}$$

Case 3. If $\omega = k-1-\deg(m)$, so $2\omega - k = 2k-2-2\deg(m)$, then

$$\begin{aligned}\phi^s_{m,n}(t(e)) - \phi^s_{m,n}(o(e)) &= q^{(k-2\deg(m)-1)s} - q^{(2\omega-k)s}\\ &= q^{(k-1-2\deg(m))s} - q^{(2k-2-2\deg(m))s}\\ &= (q^s-1)q^{(2\omega-k)s}\\ &= (q^s-1)\psi^s_{m,n}(e).\end{aligned}$$

These computations show that

$$\partial E(\cdot, s)(e) = (q^s-1)F(e,s).$$ □

Taking the limit as $s \to 1$ we have

$$\partial E(\cdot, 1)(e) = (q-1)F(e,1).$$

Let

$$\Lambda(v,s) = -L_\infty(s)E(v,s) = \frac{q}{q-1}E(v,s)$$

be the function defined in Theorem 4.1. We have the following theorem.

THEOREM 4.7 ('Kronecker's first limit formula'). *The function $\Lambda(v,s)$ has an expansion around $s=1$ of the form*

$$\Lambda(v,s) = \frac{b_{-1}}{s-1} + \frac{q}{1-q}\log|\Delta|(v) - \frac{q-1}{2}\log|\cdot|_i(v) + C + b_1(v)(s-1) + \cdots, \tag{25}$$

where b_{-1} and C are constants independent of v.

Proof. It follows from Lemma 4.6 that

$$\partial\Lambda(\cdot,1)(e) = qF(e,1).$$

Using (22) we obtain

$$\partial\Lambda(\cdot,1)(e) = q\biggl(\tilde{F}(e) - \frac{q+1}{2q}\,\mathrm{sgn}(e)\biggr).$$

To obtain the constant term in the Laurent expansion we integrate the right-hand side of this expression from v_0 to v. From (23) we have that the first term is

$$q\tilde{F}(e) = \frac{q}{1-q}\partial\log|\Delta|(e)$$

so its integral is $\log|\Delta|(v) - \log|\Delta|(v_0)$. The integral of the second term is

$$\int_{v_0}^{v} \frac{q+1}{2q}\,\mathrm{sgn}(e)\,d\mu(e) = \frac{q+1}{2q}k(v) = \frac{q+1}{2q}(-\log|\cdot|_i(v))$$

where $|\cdot|_i$ is the 'imaginary part', namely the distance from K_∞ of any element τ_v in $\lambda^{-1}(v)$, which descends to a function on the tree. This follows from the discussion on [Gek95, pp. 371–372].

Combining these two expressions we get the theorem. □

Observe that each term that appears here is analogous to a term which appears in the classical Kronecker limit formula.

4.4 A special value of the L-function

Using the functional equation for $\Lambda(v, s)$ as stated in Theorem 4.1 and the expansion (25), we obtain the following result.

Theorem 4.8. *Let f and g be two newforms of levels I_1 and I_2 respectively with I_1 and I_2 relatively prime polynomials in A. Let $I = I_1 I_2$. Then*

$$\Phi_{f,g}(0) = \frac{q}{1-q} \int_{Y(\mathcal{T}_0(I)} \log|\Delta_I| \delta(f, g).$$

Here, $\log|\Delta_I|$ is interpreted as a function on $Y(\mathcal{T})$ by $\log|\Delta_I|(e) = \log|\Delta_I|(o(e))$ and $\delta(f, g)$ is as in § 3.5.2.

Proof. From the description of $\Phi(s)$ given in (15) we have

$$\Phi(0) = \lim_{s \to 0} \sum_{\substack{d|I \\ d \text{ monic}}} \frac{\mu(d)}{|d|^s} \int_{Y(\mathcal{T}_0(I))} \Lambda((I/d)e, s) \delta(f, g).$$

From the functional equation of $\Lambda(e, s)$ and the limit formula (25), we have

$$\Lambda(e, s) = \frac{b_{-1}}{s} + \frac{q}{1-q} \log|\Delta|(e) - \frac{q^2-1}{2} \log|\cdot|_i(e) + C + \text{h.o.t.(s)}$$

where C is a constant and h.o.t.(s) denotes higher order terms in s. Since

$$\sum_{\substack{d|I \\ d \text{ monic}}} \mu(d) = 0 \quad \text{and} \quad \langle f, g \rangle = 0$$

we have

$$\sum_{\substack{d|I \\ d \text{ monic}}} \Lambda((I/d)e, 0) = \frac{q}{1-q} \log|\Delta_I(e)|$$

as the sum of the residues of the poles is 0 , the constant term C gets multiplied by 0 and finally

$$\sum_{\substack{d|I \\ d \text{ monic}}} \mu(d) \log|\cdot|_i((I/d)e) = 0$$

as $|(I/d)e|_i = |(I/d)||e|_i$ and one can easily check that $\prod_{\substack{d|I \\ d \text{ monic}}} (I/d)^{\mu(d)} = (1)$, so its logarithm is 0. It follows that

$$\Phi_{f,g}(0) = \Phi(0) = -\frac{q}{q-1} \int_{Y(\mathcal{T}_0(I))} \log|\Delta_I|(o(e)) \delta(f, g). \qquad \square$$

Since $\delta(f, g)$ is orientation invariant, we can replace the usual integration on edges by integration over positively oriented edges to get

$$\Phi_{f,g}(0) = \Phi(0) = -\frac{q}{q-1} \int_{Y^+(\mathcal{T}_0(I))} (\log|\Delta_I|(o(e)) + \log|\Delta_I|(t(e))) \delta^+(f, g). \tag{26}$$

Here $\delta^+(f, g) = f(e)g(e)\, d\mu^+(e)$ and μ^+ denotes the Haar measure on the positively oriented edges: $\mu^+(e) = (q-1)/|\text{Stab}_e(\Gamma_\infty)|$.

5. Elements in K-theory

5.1 The group $H^3_{\mathcal{M}}(X, \mathbb{Q}(2))$

Let X be an algebraic surface over a field F. The second graded piece of the Adams filtration on $K_1(X) \otimes \mathbb{Q}$ is usually denoted by $H^3_{\mathcal{M}}(X, \mathbb{Q}(2))$. It has the following description in terms of generators and relations.

The elements of this group are represented by finite formal sums

$$\sum_i (C_i, f_i)$$

where C_i are curves on X and f_i are F-valued rational functions on C_i satisfying the cocycle condition

$$\sum_i \operatorname{div}(f_i) = 0. \tag{27}$$

Relations in this group are given by the tame symbol of functions. Precisely, if C is a curve on X and f and g are two functions on X, the tame symbol of f and g at C is defined by

$$T_C(f, g) = (-1)^{\operatorname{ord}(g)\operatorname{ord}(f)} \frac{f^{\operatorname{ord}(g)}}{g^{\operatorname{ord}(f)}}, \quad \operatorname{ord}(\cdot) = \operatorname{ord}_C(\cdot).$$

Elements of the form $\sum_C (C, T_C(f, g))$ are said to be zero in $H^3_{\mathcal{M}}(X, \mathbb{Q}(2))$.

5.2 The regulator map on surfaces

We define the regulator as the boundary map in a localization sequence. We use the formalism of Consani [Con98].

Let Λ be a Henselian discrete valuation ring with fraction field F and let X be a smooth, proper surface defined over F. We set $\bar{X} = X \times \operatorname{Spec}(\bar{F})$ for $\bar{F}$ an algebraic closure of F. By a *semi-stable model* of X (or semi-stable fibration) we mean a flat, proper morphism $\mathcal{X} \to \operatorname{Spec}(\Lambda)$ of finite type over Λ, with generic fibre $\mathcal{X}_\eta \cong X$ and special fibre $\mathcal{X}_\nu = Y$, a reduced divisor with normal crossings in $\mathcal{X}$. The letters η and ν denote respectively the generic and closed points of $\operatorname{Spec}(\Lambda)$. The scheme $\mathcal{X}$ is assumed to be non-singular and the residue field at v is assumed to be finite.

The scheme Y is a finite union of irreducible components: $Y = \bigcup_{i=1}^r Y_i$, with Y_i smooth, proper, irreducible surfaces. Let J be a subset of $\{1, 2, \ldots, r\}$ whose cardinality is denoted by $|J|$. We set $Y_J = \bigcap_{j \in J} Y_j$ and define

$$Y^{(j)} = \begin{cases} \mathcal{X} & \text{if } j = 0, \\ \coprod_{|J|=j} Y_J & \text{if } 1 \leqslant j \leqslant 3, \\ \emptyset & \text{if } j > 3. \end{cases}$$

Let $\iota : Y \to \mathcal{X}$ denote the subscheme inclusion map. This map ι induces a push-forward homomorphism $\iota_* : CH_1(Y^{(1)}) \to CH_1(\mathcal{X})$ and a pullback map $\iota^* : CH^2(\mathcal{X}) \to CH^2(Y^{(1)})$. Let $J = \{j_1, j_2\}$, with $j_1 < j_2$ and $I = J - \{j_t\}$, for $t \in \{1, 2\}$. Then the inclusions $\delta_t : Y_J \to Y_I$ induce push-forward maps δ_{t*} on the Chow homology groups. The Gysin morphism $\gamma : CH_1(Y^{(2)}) \to CH_1(Y^{(1)})$ is defined by $\gamma = \sum_{t=1}^2 (-1)^{t-1} \delta_{t*}$.

Let

$$PCH^1(Y) = \frac{\ker[\iota^*\iota_* : CH_1(Y^{(1)}) \to CH^2(Y^{(1)})]}{\operatorname{im}[\gamma : CH_1(Y^{(2)}) \to CH_1(Y^{(1)})]} \Big/ \{\text{torsion}\}$$

and define the ν-adic Deligne cohomology to be

$$H^3_{\mathcal{D}}(X_{/\nu}, \mathbb{Q}(2)) = PCH^1(Y) \otimes \mathbb{Q}.$$

If certain 'standard conjectures' are satisfied, it follows from [Con98, Theorem 3.5] that

$$\dim_{\mathbb{Q}} H^3_{\mathcal{D}}(X_{/\nu}, \mathbb{Q}(2)) = -\mathrm{ord}_{s=1} L_\nu(H^2(\bar{X}, \mathbb{Q}_\ell), s), \quad s \in \mathbb{C},$$

where $L_\nu(H^2(\bar{X}), s)$ is the local Euler factor at ν of the Hasse–Weil L-function of X.

There is a localization sequence which relates the motivic cohomology of $\mathcal{X}, X$ and $Y = \mathcal{X}_\nu$ [Blo86]. When specialized to our case it is as follows:

$$\cdots \longrightarrow H^3_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(2)) \longrightarrow H^3_{\mathcal{M}}(X, \mathbb{Q}(2)) \xrightarrow{\partial} H^3_{\mathcal{D}}(X_{/\nu}, \mathbb{Q}(2)) \longrightarrow H^4_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(2)) \longrightarrow \cdots.$$

The ν-adic regulator map $r_{\mathcal{D},\nu}$ is defined to be the boundary map ∂ in this localization sequence. If $\sum_i (C_i, f_i)$ is an element of $H^3_{\mathcal{M}}(X, \mathbb{Q}(2))$ then

$$r_{\mathcal{D},\nu}\bigg(\sum_i (C_i, f_i)\bigg) = \sum_i \mathrm{div}(\bar{f}_i),$$

where $\bar{f}_i$ is the function f_i extended to the Zariski closure of C in $\mathcal{X}$. The condition $\sum \mathrm{div}(f_i) = 0$ shows that the 'horizontal' divisors cancel each other out and so the image of the regulator map is supported in the special fibre $\mathcal{X}_v$.

Explicitly, one has the following formula for the regulator:

$$r_{\mathcal{D},\nu}\bigg(\sum_i (C_i, f_i)\bigg) = \sum_i \sum_Y \mathrm{ord}_Y(f_i) Y, \tag{28}$$

where Y runs through the components of the reduction of the Zariski closure of the curves C_i.

This regulator map clearly depends on the choice of model. However, Consani's work shows that the dimension of the target space does not depend on the model since the local L-factor does not. Since the regulator map is simply the boundary map of a localization sequence it satisfies the expected functoriality properties.

While all our calculations below are with respect to a particular model, perhaps the correct framework to work with is the non-Archimedean Arakelov theory of Bloch–Gillet–Soule [BGS95].

5.3 The case of products of Drinfeld modular curves

We now apply the results of the previous section to the case of the self-product of a Drinfeld modular curve $X_0(I)$ and the prime ∞. In [Tei92, p. 280], Teitelbaum describes how to construct a model $\mathcal{X}_0(I)$ of the curve $X_0(I)$ over $\mathcal{O}_\infty$. This model has semi-stable reduction at ∞, and Teitelbaum describes a covering by affinoids which have a canonical reduction. The special fibres of the affinoids covering $\mathcal{X}_0(I)$ are made up of two types of components; either of the type $(T_i \cup T_j)$ where the T_i and T_j are isomorphic to $\mathbb{P}^1_{\mathbb{F}_q}$ with all but one rational point deleted and meet at that point $T_{ij} = T_i \cap T_j$, or of the form T_i where T_i is isomorphic to $\mathbb{P}^1_{\mathbb{F}_q}$ with all but one rational point deleted.

The self-product $\mathcal{X}_0(I) \times \mathcal{X}_0(I)$ has, therefore, a covering by products of the affinoids covering $\mathcal{X}_0(I)$ so there are four possibilities for the special fibre:

(i) $(T_1 \cup T_2) \times T_3$;
(ii) $T_1 \times (T_3 \cup T_4)$;
(iii) $T_1 \times T_3$;
(iv) $(T_1 \cup T_2) \times (T_3 \cup T_4)$,

depending on whether the reduction is of the first or second type above. Therefore it is made up of components of the form

$$\begin{array}{ll} T_1 \times T_4, & T_1 \times T_3, \\ T_2 \times T_4, & T_2 \times T_3. \end{array}$$

One has the following schematic representation.

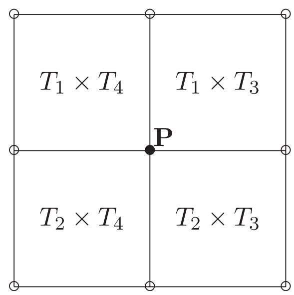

FIGURE 1.

This reduction, however, is not semi-stable. In the first three cases there is no problem but in case (iv) above there are four components and all of them meet at the point $\mathbf{P} = (T_{12}, T_{34})$, hence it is not semi-stable.

However, if we blow up $\mathcal{X}_0(I) \times \mathcal{X}_0(I)$ at this point the special fibre of the blow-up is locally a normal crossing divisor. Locally, the special fibre consists of five components, $Y_1, \dots, Y_5$, where

$$\begin{array}{ll} Y_3 = \widetilde{T_1 \times T_4}, & Y_1 = \widetilde{T_1 \times T_3}, \\ Y_4 = \widetilde{T_2 \times T_4}, & Y_2 = \widetilde{T_2 \times T_3} \end{array}$$

are the strict transforms of the components $T_i \times T_j$ above and $Y_5 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is the exceptional fibre [Con99, Lemma 4.1].

We label the components in this curious way as it is important in what follows. If one thinks of the point of intersection as the origin, then Y_1 is the strict transform of the first quadrant, Y_2 of the one below it, Y_3 of the quadrant to the left of Y_1 and Y_4 the strict transform of the quadrant to the left of Y_2. The diagram in Figure 2 is a schematic representation of the situation.

Recall that this is the local picture; to obtain the semi-stable model we have to repeat this procedure for every point of intersection of the components $T_i \times T_j$, namely, at the points denoted by $\circ$ in Figure 1. So the components of special fibre consist of the the strict transforms of the $T_i \times T_j$ with all the four corners being blown up.

Observe that the labeling Y_i above is with respect to which corner of the $T_i \times T_j$ is being considered; so for example $T_i \times T_j$ will be labelled Y_1 if the south-west corner is blown up but will be labelled Y_4 if the north-east corner is blown up, Y_2 if the north-west corner and Y_3 if the south-east corner is blown up.

Let Y_{ij} denote the cycle $Y_i \cap Y_j$, if it exists. For example, one has cycles Y_{15}, Y_{12}, Y_{13} but no cycle Y_{14} as Y_1 and Y_4 do not intersect. Similarly, let Y_{ijk} denote the cycle $Y_i \cap Y_j \cap Y_k$, if it exists. From Figure 2 one can see that for any such cycle at least one of the i, j or k has to be 5, say $k = 5$, and the cycle is $Y_{ij5} = Y_{i5} \cap Y_{j5}$. Further, one does not have cycles Y_{145} and Y_{235}. Since the cycles Y_{i5} and Y_{j5} are rulings on $Y_5 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ their intersection number is either 1 or 0 and so the cycles Y_{ij5} have support on one point with multiplicity one.

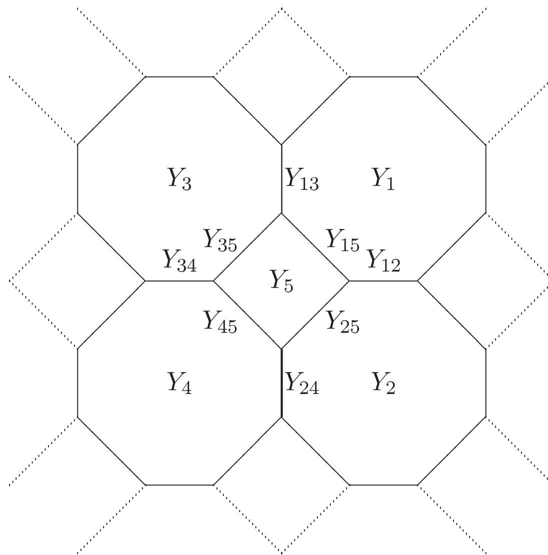

Figure 2.

In the group $H^3_{\mathcal{D}}((X_0(I) \times X_0(I))_{/\infty}, \mathbb{Q}(2))$ one has cycles coming from the restriction of the generic cycles as well as certain cycles supported in the exceptional fibres. Locally, the restriction of horizontal and vertical components give the cycles $Y_{12} + Y_{34} + (Y_{15} - Y_{45} + Y_{25} - Y_{35})$ and $Y_{13} + Y_{24} + (Y_{15} - Y_{45} - Y_{25} + Y_{35})$ [Con99, Lemma 4.1]. In the exceptional fibre Y_5 over $\mathbf{P}$ one also has the cycle $\mathcal{Z}_{\mathbf{P}} = Y_{15} + Y_{45} - Y_{25} - Y_{35}$. Computing the intersection with the other cycles show that this is not the restriction of a generic cycle; in fact, it is orthogonal to them and the cycles Y_{12}, Y_{13}, Y_{24} and Y_{34} as well.

There are relations in this group coming from the image of the Gysin map γ. For example, the difference of the image of the cycles Y_{15} in $CH^1(Y_1)$ and $CH^1(Y_5)$ lies in the image of the Gysin map, so is 0 in $H^3_{\mathcal{D}}((X_0(I) \times X_0(I))_{/\infty}, \mathbb{Q}(2))$. Hence, there is a well-defined Y_{15} in $H^3_{\mathcal{D}}((X_0(I) \times X_0(I))_{/\infty}, \mathbb{Q}(2))$. Similarly, the cycles Y_{ij}, which lie in both Y_i and $Y_j, i, j \in \{1, \dots 5\}$, are well-defined. Further, the cycle which is Y_{12} with respect to $\mathbf{P}$ is Y_{34} with respect to the point $\mathbf{P'}$ to the right of $\mathbf{P}$ and so is counted only once in $H^3_{\mathcal{D}}((X_0(I) \times X_0(I))_{/\infty}, \mathbb{Q}(2))$, and similarly for the others. So the local cycles described above coming from the restriction of the horizontal and vertical cycles patch up to give global cycles in $H^3_{\mathcal{D}}((X_0(I) \times X_0(I))_{/\infty}, \mathbb{Q}(2))$.

5.3.1 *A description in terms of the graph.* Using the relation between the Bruhat–Tits tree and the special fibre described at the end of § 3.3 or in [Tei92] one can also express this local picture in terms of the graph. Recall that components of the special fibre of Ω correspond to vertices on the tree and two components intersect at an edge. From that, we have that the graph $\mathcal{T}_0(I)$ consists of a finite graph $\mathcal{T}_0(I)^0$ and finitely many ends. The graph $\mathcal{T}_0(I)^0$ is the dual graph of the intersection graph of the special fibre of $\mathcal{X}_0(I)$. The situation where the canonical reduction of an affinoid has two components corresponds to an edge e with vertices $o(e)$ and $t(e)$, both of which are $\mathcal{T}_0(I)^0$. The situation when the canonical reduction has only one component corresponds to an edge e with a distinguished vertex which is in $\mathcal{T}_0(I)^0$.

The special fibre of the product then has the following local description; it corresponds to either two, one or four pairs of vertices depending on whether we have case (i) or (ii), (iii) or (iv) above. In case (iv), the four pairs of vertices correspond to the four pairs of components and the point $\mathbf{P} = (T_{12}, T_{34})$ corresponds to a pair of edges (e_{12}, e_{34}). Hence we can re-label the cycle $\mathcal{Z}_{\mathbf{P}}$ as $\mathcal{Z}_{(e_{12},e_{34})}$, where (e_{12}, e_{34}) is the point being blown up.

The regulator of an element supported on curves uniformized by the Drinfeld upper half-plane lying on $X_0(I) \times X_0(I)$ can also be expressed in terms of the graph. Since components in the special fibre correspond to vertices of the graph, one can rewrite the regulator in terms of vertices. Let Y_v denote the component corresponding to a vertex v. From the definition of $\log|\cdot|$ one has $\mathrm{ord}_{Y_v}(f) = \log|f|(v)$. So one can rewrite the expression (28) as

$$r_{\mathcal{D},\infty}\left(\sum_i (C_i, f_i)\right) = \sum_i \sum_v \log|f_i|(v) Y_v, \tag{29}$$

where v runs through the vertices of the Bruhat–Tits graphs of C_i. In § 5.6, the element we construct will be supported on curves isomorphic to $X_0(I)$ so we can express its regulator using (29).

5.4 The special cycle $\mathcal{Z}_{f,g}$

As mentioned before, the motivic cohomology group of the surface $X_0(I) \times X_0(I)$ can be decomposed with respect to eigenspaces for pairs of cusp forms (f, g) and this results in a decomposition of the ∞-adic Deligne cohomology group as well. We denote these groups by $H^3_{\mathcal{D}}(h^1(M_f) \otimes h^1(M_g)_{/\infty}, \mathbb{Q}(2))$. The local L-factor at ∞ (13) and Consani's theorem [Con98, Theorem 3.5] shows that this space is one dimensional.

There is a special cycle in this group which plays the role played by the $(1, 1)$-form

$$\omega_{f,g} = f(z_1)\overline{g(z_2)}(dz_1 \otimes d\bar{z}_2 - d\bar{z}_1 \otimes dz_2)$$

in the classical case. While $\omega_{f,g}$ is not represented by an algebraic cycle, in our case there is a special cycle, supported in the special fibre, which represents it. It is defined as follows. For f, g two cuspidal automorphic forms of JLD type and $\mathcal{Z}_{(e,e')}$ as above, we define

$$\mathcal{Z}_{f,g} = \sum_{e,e' \in Y(\mathcal{T}_0(I)^0)} f(e)\overline{g(e')}\mathcal{Z}_{(e,e')}$$

in $H^3_{\mathcal{D}}(h^1(M_f) \otimes h^1(M_g)_{/\infty}, \mathbb{Q}(2))$. The action of the Hecke correspondence is through its action on f and g and so that shows that this cycle lies in the (f, g) component with respect to the Hecke action.

Note that this cycle is orientation invariant as $(\bar{e}, \bar{e}') = (e, e')$ and $f(\bar{e})\bar{g}(\bar{e}') = f(e)\bar{g}(e')$. Further, as it is composed of the cycles $\mathcal{Z}_{(e,e')}$, it is orthogonal to the cycles which come by restriction from the generic Neron–Severi group.

5.5 A special element in the motivic cohomology group

In this section we will use the Drinfeld modular unit Δ_I defined in (19) on the diagonal $D_0(I)$ of $X_0(I)$ to construct a canonical element $\Xi_0(I)$ in the motivic cohomology $H^3_{\mathcal{M}}(X_0(I) \times X_0(I), \mathbb{Q}(2))$ of the self-product X of the Drinfeld curve $X_0(I)$. The trick is to 'cancel out' the zeroes and the poles of (a power of) Δ_I using certain functions supported on the vertical and horizontal fibres of X. The existence of these functions is a consequence of the function field analogue of the Manin–Drinfeld theorem proved by Gekeler in [Gek97]. Theorem 5.2 provides a more explicit description of them. As a corollary, we get an effective version of the Manin–Drinfeld theorem in the function field case.

5.5.1 *Cusps.* Let $I \in A$ be a monic, square-free polynomial. We first compute the divisor of the function Δ_I explicitly. For this we need to work with an explicit description of the set of the cusps of $X_0(I)$. It is well-known that the set of these points is in bijection with the set

$$\Gamma_0(I)\backslash\Gamma/\Gamma_\infty \stackrel{\simeq}{\to} \{[a:d] : d|I,\ a \in (A/tA)^*,\ t=(d, I/d),\ a, d \text{ monic, coprime}\}/\mathbb{F}_q^*.$$

We will denote the cusp corresponding to $[a:d]$ by P_d^a. Since I is square-free, the cusps are of the form $P_d = P_d^1$, where d is a monic divisor of I. For a function $F = F(\tau)$ on Ω and $f \in A$ let $F(f)$ denote the function $F(f\tau)$. For $a, b \in A$, let $(a, b) = \gcd\{a, b\}$ and $[a, b] = \operatorname{lcm}\{a, b\}$. As A is a Principal Ideal Domain they are both elements of A. If J is an element of A, the symbol $|J|$ denotes the cardinality of the set $A/(J)$, where (J) is the ideal generated by J.

Lemma 5.1. *Let $I \in A$ be square-free and monic and assume that I' and d are monic divisors of I. Then*

$$\operatorname{ord}_{P_d} \Delta(I') = |I| \frac{|(d, I')|}{|[d, I']|}$$

where the order at a cusp is computed in terms of a local uniformizer as in [GR92, § 2.7].

Proof. It follows from [Gek97, § 3] that

$$\operatorname{ord}_{P_d}\Delta = |(I/d)|, \quad \operatorname{ord}_{P_d}\Delta(I) = |d|.$$

To obtain an explicit description of the divisor of $\Delta(I')$ on $X_0(I)$ we need to compute the ramification index of P_d over $P_{d'}$, where $d' = \gcd\{d, I'\}$. It follows from [Gek97, Lemma 3.8] that

$$\operatorname{ram}_{P_{d'}}^{P_d} = \frac{|I||(d, I')|}{|d||I'|}.$$

Therefore, one gets

$$\begin{aligned}\operatorname{ord}_{P_d}\Delta(I') &= \operatorname{ram}_{P_{d'}}^{P_d} \cdot \operatorname{ord}_{P_{d'}}\Delta(I') \\ &= \frac{|I||(d, I')|}{|d||I'|} \cdot |(d, I')| \\ &= |I| \frac{|(d, I')|}{|[d, I']|}.\end{aligned}$$

□

It follows from the definition of the function Δ_I in (19) and Lemma 5.1 that

$$\operatorname{div}(\Delta_I) = \sum_{\substack{d|I \\ d \text{ monic}}} \mu(d) \operatorname{div}(\Delta(I/d)) = \prod_{\substack{f|I \\ f \text{ prime}}} (1 - |f|) \bigg(\sum_{\substack{d|I \\ d \text{ monic}}} \mu(d) P_d \bigg). \tag{30}$$

A simple modular unit is a (Drinfeld) modular unit whose divisor is of the form $k(P - Q)$, where P and Q are cusps of $X_0(I)$ and $k \in \mathbb{Z}$. The following theorem shows that there exists $\kappa \in \mathbb{N}$ such that the function Δ_I^κ can be decomposed into a product of such units.

Theorem 5.2. *Let I be a square-free, monic element of A and let $I = \prod_{i=0}^r f_i$ be the prime factorization of I, with the f_i monic elements of A. Let $\kappa = \prod_{i=0}^r (1 + |f_i|)$. Then*

$$\Delta_I^\kappa = \prod_{\substack{a|(I/f_0) \\ a \text{ monic}}} F_a$$

where the functions F_a are simple modular units and

$$\operatorname{div}(F_a) = \prod_{i=0}^{r} (1 - |f_i|^2)\mu(a)(P_a - P_{f_0 a}).$$

Proof. The proof will follow from the following lemmas.

LEMMA 5.3. *Let P_a be a cusp of $X_0(I)$. Then, the divisor of the form*

$$D_a = \prod_{\substack{d|I \\ d \text{ monic}}} \Delta(d)^{\mu(I/d)|I|(|(a,d)|/|[a,d]|)}$$

is

$$\operatorname{div}(D_a) = \prod_{\substack{f|I \\ f \text{ monic, prime}}} (1 - |f|^2)\mu(a)P_a.$$

Proof. Let P_b be a cusp of $X_0(I)$. Then, it follows from Lemma 5.1 that

$$\operatorname{ord}_{P_b}(D_a) = \sum_{\substack{d|I \\ d \text{ monic}}} \mu(I/d)|I|^2 \frac{|(a, d)(b, d)|}{|[a, d][b, d]|}.$$

We consider the following cases.

Case 1 ($a \neq b$). In this case there is a prime element $f \in A$ dividing a but not b (or *vice versa*). Assume that $f|a$ and $\gcd\{f, b\} = 1$. Then

$$\operatorname{ord}_{P_b}(D_a) = \sum_{\substack{d|(I/f) \\ d \text{ monic}}} \mu(I/d)|I|^2 \left(\frac{|(a, d)(b, d)|}{|[a, d][b, d]|} - \frac{|(a, fd)(b, fd)|}{|[a, fd][b, fd]|} \right).$$

Since $f|a$, it follows that $(a, fd) = f(a, d)$ and $[a, fd] = [a, d]$. Further, $(f, b) = 1$, $(b, fd) = (b, d)$ and $[b, fd] = f[b, d]$. Therefore

$$\frac{|(a, d)(b, d)|}{|[a, d][b, d]|} - \frac{|(a, fd)(b, fd)|}{|[a, fd][b, fd]|} = 0,$$

so we have $\operatorname{ord}_{P_b}(D_a) = 0$.

Case 2 ($a = b$). In this case we have to show that

$$\sum_{\substack{d|I \\ d \text{ monic}}} \mu(I/d)|I|^2 \frac{|(a, d)|^2}{|[a, d]|^2} = \mu(a) \prod_{\substack{f|I \\ f \text{ monic, prime}}} (1 - |f|^2). \tag{31}$$

The proof is by induction on a. If $a = 1$, then the left-hand side of (31) is

$$\sum_{\substack{d|I \\ d \text{ monic}}} \mu(I/d)|I|^2 \frac{|(1, d)|^2}{|[1, d]|^2} = \sum_{\substack{d|I \\ d \text{ monic}}} \mu(I/d) \left(\frac{|I|}{|d|} \right)^2 = \prod_{\substack{f|I \\ f \text{ monic, prime}}} (1 - |f|^2)$$

and the lemma follows.

Now, we assume that (31) holds for some $a|I$. Let f be a monic prime of A such that $f|I$ and $(f,a)=1$. We will show that (31) holds for fa. The left-hand side of (31) is now

$$\sum_{\substack{d|I\\ d\text{ monic}}} \mu(I/d)|I|^2\frac{|(fa,d)|^2}{|[fa,d]|^2} = \sum_{\substack{d|(I/f)\\ d\text{ monic}}} \mu(I/d)|I|^2\left(\frac{|(fa,d)|^2}{|[fa,d]|^2} - \frac{|(fa,fd)|^2}{|[fa,fd]|^2}\right).$$

If $d|(I/f)$, we have

$$(fa,d)=(a,d), \quad [fa,d]=f[a,d]$$

and

$$(fa,fd)=f(a,d), \quad [fa,fd]=(f)[a,d].$$

Hence

$$\begin{aligned}\sum_{\substack{d|I\\ d\text{ monic}}} \mu(I/d)|I|^2\frac{|(fa,d)|^2}{|[fa,d]|^2} &= \sum_{\substack{d|(I/f)\\ d\text{ monic}}} \mu(I/d)|I|^2\left(\frac{1}{|f|^2}-1\right)\left(\frac{|(a,d)|^2}{|[a,d]|^2}\right)\\ &= -(1-|f|^2)\sum_{\substack{d|(I/f)\\ d\text{ monic}}} \mu(I/fd)|(I/f)|^2\frac{|(a,d)|^2}{|[a,d]|^2}.\end{aligned}$$

By induction, this is

$$-(1-|f|^2)\mu(a)\prod_{\substack{g|(I/f)\\ g\text{ monic, prime}}}(1-|g|^2) = \mu(fa)\prod_{\substack{g|I\\ g\text{ monic, prime}}}(1-|g|^2).$$

This conclude the proof of the lemma. $\square$

Let f_0 be a prime element of A dividing I. For $a|(I/f_0)$, we set

$$F_a = D_aD_{f_0a}$$

where the functions D_a and D_{f_0a} are defined as in Lemma 5.3. Then, by applying that lemma, we have

$$\operatorname{div}(F_a) = \prod_{\substack{f|I\\ f\text{ monic, prime}}}(1-|f|^2)\mu(a)(P_a-P_{f_0a}).$$

Hence F_a is a simple modular unit. The statements of the theorem will follow by applying the next lemma.

Lemma 5.4. *Under the same hypotheses of Theorem 5.2, we have*

$$\prod_{\substack{a|(I/f_0)\\ a\text{ monic}}} F_a = \prod_{\substack{d|(I/f_0)\\ d\text{ monic}}}\prod_{\substack{a|(I/f_0)\\ a\text{ monic}}}\left(\frac{\Delta(d)}{\Delta(f_0d)}\right)^{\mu(I/d)|I|(1+1/|f_0|)|(a,d)|/|[a,d]|}.$$

Proof. From the definition of F_a we have

$$\prod_{\substack{a|(I/f_0)\\ a\text{ monic}}} F_a = \prod_{\substack{a|(I/f_0)\\ a\text{ monic}}}\prod_{\substack{d|I\\ d\text{ monic}}}\Delta(d)^{\mu(I/d)|I|(|(a,d)|/|[a,d]|+|(f_0a,d)|/|[f_0a,d]|)}.$$

If $(d,f_0)=1$, then $(f_0a,d)=(a,d)$ and $[f_0a,d]=f_0[a,d]$. Therefore, we get

$$\frac{|(a,d)|}{|[a,d]|}+\frac{|(f_0a,d)|}{|[f_0a,d]|} = \frac{|(a,d)|}{|[a,d]|}\left(1+\frac{1}{|f_0|}\right) = \frac{|(a,f_0d)|}{|[a,f_0d]|}+\frac{|(f_0a,f_0d)|}{|[f_0a,f_0d]|}.$$

Collecting together the terms with the same d, we obtain

$$\prod_{\substack{a|(I/f_0)\\ a \text{ monic}}} F_a = \prod_{\substack{d|(I/f_0)\\ d \text{ monic}}} \left(\frac{\Delta(d)}{\Delta(f_0 d)}\right)^{\mu(I/d)|I|(1+1/|f_0|)(\sum_{\substack{a|(I/f_0)\\ a \text{ monic}}} |(a,d)|/|[a,d]|)}.$$

Using an induction argument similar to the one used in the proof of Lemma 5.3, we have

$$\sum_{\substack{a|(I/f_0)\\ a \text{ monic}}} \frac{|(a,d)|}{|[a,d]|} = \prod_{\substack{f|(I/f_0)\\ f \text{ prime}}} \left(1 + \frac{1}{|f|}\right).$$ □

To finish the proof of the theorem, we notice that

$$\Delta_I = \prod_{\substack{d|I\\ d \text{ monic}}} \Delta(d)^{\mu(I/d)} = \prod_{\substack{d|(I/f_0)\\ d \text{ monic}}} \left(\frac{\Delta(d)}{\Delta(f_0 d)}\right)^{\mu(I/d)}.$$

Let $\kappa = \prod_{i=0}^{r}(1+|f_i|)$. Then, it follows from Lemma 5.4 that

$$\prod_{\substack{a|(I/f_0)\\ a \text{ monic}}} F_a = \Delta_I^{\kappa}.$$ □

As a corollary of Lemma 5.3, we obtain the following result of independent interest.

COROLLARY 5.5 (Effective Manin–Drinfeld theorem). *Let $I = \prod_{i=0}^{r} f_i$ be the monic, prime factorization of a square-free, monic polynomial I in A. Then the cuspidal divisor class group is finite and its order divides $\prod_{i=0}^{r}(1-|f_i|^2)$.*

Proof. If a and a' are two cusps of $X_0(I)$, then it follows from Lemma 5.3 that the function

$$F_{a,a'} = \frac{D_a}{D_{a'}^{\mu(a)/\mu(a')}}$$

has divisor

$$\operatorname{div}(F_{a,a'}) = \prod_{i=0}^{r}(1-|f_i|^2)\mu(a)(P_a - P_{a'}).$$ □

5.6 An element in $H^3_{\mathcal{M}}(X_0(I) \times X_0(I), \mathbb{Q}(2))$

Using the factorization in Theorem 5.2, we can construct an element of the motivic cohomology group as follows.

Let $D_0(I)$ denote the diagonal on $X_0(I) \times X_0(I)$ and let $I = \prod_{i=0}^{r} f_i$ be the monic prime factorization of I. Let $\kappa = \prod_{i=0}^{r}(1+|f_i|)$. Let $F_d = D_d D_{f_0 d}$ as in Lemma 5.3. Consider the element

$$\Xi_0(I) = (D_0(I), \Delta_I^{\kappa}) - \left(\sum_{d|(I/f_0)} (P_d \times X_0(I), P_d \times F_d) + (X_0(I) \times P_{f_0 d}, F_d \times P_{f_0 d})\right). \quad (32)$$

It follows from Theorem 5.2 that this element satisfies the cocycle condition (27), as the sum of the divisors of the functions is a sum of multiples of terms of the form

$$(P_d, P_d) - (P_{f_0 d}, P_{f_0 d}) - (P_d, P_d) + (P_d, P_{f_0 d}) + (P_{f_0 d}, P_{f_0 d}) - (P_d, P_{f_0 d}).$$

Hence $\Xi_0(I)$ determines an element of $H^3_{\mathcal{M}}(X_0(I) \times X_0(I), \mathbb{Q}(2))$.

5.6.1 *The regulator of* $\Xi_0(I)$. From the formula give in (29), the regulator of our element of $H^3_{\mathcal{M}}(X_0(I)\times X_0(I),\mathbb{Q}(2))$ is given by the formula

$$\begin{aligned} r_{\mathcal{D},\infty}(\Xi_0(I)) = & \sum_{v\in X(D_0(I))} \log|\Delta_I^{\kappa}|(v)Y_v \\ & + \sum_{d|(I/f_0)}\Bigg(\sum_{v\in X((P_d\times X_0(I)))} \log|P_d\times F_d|(v)Y_v \\ & + \sum_{v\in X((X_0(I)\times P_{f_0 d}))} \log|F_d\times P_{f_0 d}|(v)Y_v\Bigg). \end{aligned} \tag{33}$$

5.7 The final result

We have the following theorem which relates the special value of the L-function with the intersection pairing of certain cycles. This intersection pairing is the intersection pairing on the group $PCH^1(Y)$ obtained as the sum of the intersection pairings on the Chow groups of the components. It is well defined as it vanishes on the image of the Gysin map.

Theorem 5.6. *Let f and g be Hecke eigenforms for $\Gamma_0(I)$ and $\Phi_{f,g}$ the completed Rankin–Selberg L-function. Then one has*

$$\Phi_{f,g}(0)=\frac{q}{2(q-1)\kappa}(r_{\mathcal{D},\infty}(\Xi_0(I)),\mathcal{Z}_{f,g}) \tag{34}$$

where $\Xi_0(I)$ is the element of the higher Chow group constructed above, $r_{\mathcal{D},\infty}$ is the regulator map and $\mathcal{Z}_{f,g}$ is the special cycle described above.

Proof. We first compute the pairing of the regulator of $\Xi_0(I)$ with $\mathcal{Z}_{f,g}$. For this we have to compute the pairing of special fibre of the total transform of the diagonal $D_0(I)$ with $\mathcal{Z}_{f,g}$ as well as the pairing of the vertical and horizontal components with $\mathcal{Z}_{f,g}$. Since the pairing is the sum of all the pairings of the components one can compute it locally around a point $\mathbf{P}=(e,e')$ which is being blown up as in § 5.3.

Recall that $\mathcal{Z}_{\mathbf{P}}=Y_{15}+Y_{45}-Y_{25}-Y_{35}$. We have the following intersection numbers of $\mathcal{Z}_{\mathbf{P}}$ with the various cycles Y_{ij}:

- $(\mathcal{Z}_{\mathbf{P}},Y_{12})=(\mathcal{Z}_{\mathbf{P}},Y_{13})=(\mathcal{Z}_{\mathbf{P}},Y_{24})=(\mathcal{Z}_{\mathbf{P}},Y_{34})=0$;
- $(\mathcal{Z}_{\mathbf{P}},Y_{15})=(\mathcal{Z}_{\mathbf{P}},Y_{45})=-2$;
- $(\mathcal{Z}_{\mathbf{P}},Y_{25})=(\mathcal{Z}_{\mathbf{P}},Y_{35})=2$.

These can easily be computed using the fact $\mathcal{Z}_{\mathbf{P}}$ is the difference of rulings on Y_5.

Locally, $D_0(I)$ is the blow-up of the diagonal in $(T_1\cup T_3)\times(T_2\cup T_4)$, where the T_i are as in § 5.3. The part of the diagonal which passes through $\mathbf{P}$ is the sum of the diagonals in $T_1\times T_3$ and $T_2\times T_4$. Let Δ_1 and Δ_4 denote the strict transforms of these diagonals in Y_1 and Y_4. The total transform is

$$\Delta_1+Y_{15}+\Delta_4+Y_{45},$$

as the blow up of the diagonal in $T_1\times T_3$ has exceptional fibre Y_{15}, and similarly for the other diagonal. One has $(\mathcal{Z}_{\mathbf{P}},\Delta_i)=0$ since $\mathcal{Z}_{\mathbf{P}}$ is supported in the exceptional fibre.

For vertical or horizontal components, the total transform is [Con99, Lemma 4.1]

$$Y_{13}+(Y_{15}-Y_{35})+Y_{24}+(Y_{25}-Y_{45})$$

and

$$Y_{12} + (Y_{15} - Y_{25}) + Y_{34} + (Y_{35} - Y_{45})$$

respectively. Hence, using the intersection numbers computed above, we have:

- $(\mathcal{Z}_{\mathbf{P}}, Y_{13} + (Y_{15} - Y_{35}) + Y_{24} + (Y_{25} - Y_{45})) = 0$;
- $(\mathcal{Z}_{\mathbf{P}}, Y_{12} + (Y_{15} - Y_{25}) + Y_{34} + (Y_{35} - Y_{45})) = 0$.

The regulator of $\Xi_0(I)$ is

$$\sum_{v \in X(D_0(I))} \log |\Delta_I^{\kappa}|(v) Y_v + \sum_{d|(I/f_0)} \Bigg(\sum_{v \in X((P_d \times X_0(I)))} \log |P_d \times F_d|(v) Y_v \\ + \sum_{v \in X((X_0(I) \times P_{f_0 d}))} \log |F_d \times P_{f_0 d}|(v) Y_v \Bigg).$$

From the above we can see that the vertical and horizontal components have intersection number 0 with $\mathcal{Z}_{f,g}$, so it suffices to compute the intersection number of the diagonal component of the regulator with $\mathcal{Z}_{f,g}$.

Locally, at the picture corresponding to the point (e, e), the diagonal components appear with multiplicities:

- $\kappa \log |\Delta_0(I)|(o(e))$ for Δ_4 and Y_{45};
- $\kappa \log |\Delta_0(I)|(t(e))$ for Δ_1 and Y_{15},

as the vertex $o(e)$ corresponds to the component Δ_4 and the vertex $t(e)$ corresponds to the component Δ_1 of the diagonal. Hence the diagonal component is a sum of terms of the type

$$\kappa \log |\Delta_0(I)|(o(e))(\Delta_4 + Y_{45}) + \kappa \log |\Delta_0(I)|(t(e))(\Delta_1 + Y_{15}).$$

Using the fact that $t(e) = o(\bar{e})$ and the calculations above, we get

$$(r_{\mathcal{D},\infty}(\Xi_0(I)), \mathcal{Z}_{f,g}) = (-2\kappa) \int_{e \in Y_0^+(I)} (\log |\Delta_I|(o(e)) + \log |\Delta_I|(t(e))) f(e) g(e)\, d\mu^+(e).$$

This is a finite sum as f and g have finite support.

Comparing this with (26) gives us our final result:

$$\Phi_{f,g}(0) = \frac{q}{2(q-1)\kappa} (r_{\mathcal{D},\infty}(\Xi_0(I)), \mathcal{Z}_{f,g}). \tag{35}$$

As $\Phi_{f,q}(s-1) = \Lambda(h^1(M_f) \otimes h^1(M_g), s)$, we get Theorem 5.6. □

5.7.1 *An application to elliptic curves.* Theorem 5.6 provides some evidence for Conjecture 1.1 in the case of a product of two non-isogenous elliptic curves over K.

If E is a non-isotrivial (that is, $j_E \notin \mathbb{F}_q$), semi-stable elliptic curve over K with conductor $I_E = I \cdot \infty$ and split multiplicative reduction at ∞, by the work of Deligne [Del73], Drinfeld, Zarhin and eventually Gekeler–Reversat [GR92] we have that E is modular. This means that the Hasse–Weil L-function $L(E, s)$ is equal to the L-function of an automorphic form f of JLD-type with rational fourier coefficients,

$$L(E, s) = L(f, s) = \sum_{\mathfrak{m} \text{ pos.div}} \frac{c(f, \mathfrak{m})}{|\mathfrak{m}|^{s-1}}.$$

Furthermore, there exists a Drinfeld modular curve $X_0(I)$ of level I and a dominant morphism (the modular parametrization)

$$\pi_f : X_0(I) \longrightarrow E. \tag{36}$$

Now, let E and E' be two such modular elliptic curves with corresponding automorphic forms f and g of levels I_1 and I_2. Assume that $(I_1, I_2) = 1$ and that $I = I_1 I_2$ is square-free. Then, the L-function of $H^2(\bar{E} \times \bar{E}', \mathbb{Q}_\ell)$ can be expressed in terms of the L-function of the Rankin–Selberg convolution of f and g. Künneth's theorem gives the decomposition

$$L(H^2(\bar{E} \times \bar{E}'), s) = L(H^2(\bar{E}), s)^2 L(H^1(\bar{E}) \otimes H^1(\bar{E}'), s) = \zeta_A(s-1)^2 L(H^1(\bar{E}) \otimes H^1(\bar{E}'), s).$$

The completed L-function of $H^1(\bar{E}) \otimes H^1(\bar{E})$ is the function $\Phi(s-1) = \Phi_{f,g}(s-1)$ of (15). We set

$$\Lambda_{E,E'}(s) = L_\infty(s-1)^2 \zeta_A(s-1)^2 \Phi(s-1). \tag{37}$$

Then $\Lambda_{E,E'}(s)$ is the completed L-function of $H^2(\bar{E} \times \bar{E}', \mathbb{Q}_\ell)$.

The following result is an application of Theorem 5.6.

Theorem 5.7. *Let E and E' be elliptic curves over K satisfying the above conditions. Then, there is an element $\Xi \in H^3_{\mathcal{M}}(E \times E', \mathbb{Q}(2))$ such that*

$$\Lambda^*_{E,E'}(1) = \frac{q \deg(\Pi)^2}{2\kappa(1-q)^3 \log_e(q)^2} (r_{\mathcal{D},\infty}(\Xi), \mathcal{Z}_{E,E'}) \tag{38}$$

where Π is the restriction of the product of the modular parameterizations of E and E' to the diagonal $D_0(I)$ of $X_0(I)$ and $\Lambda^*_{E,E'}(1)$ is the first non-zero value in the Laurent expansion at $s = 1$.

Proof. Let $\pi_f \times \pi_g : X_0(I) \times X_0(I) \to E \times E'$ be the product of the modular parameterizations π_f and π_g. Let

$$\Xi = (\pi_f \times \pi_g)_*(\Xi_0(I)) \in H^3_{\mathcal{M}}(E \times E', \mathbb{Q}(2))$$

be the push-forward cocycle in motivic cohomology, where $\Xi_0(I)$ is the class defined in (32). Let $\mathcal{Z}_{E,E'} = (\pi_f \times \pi_g)_*(\mathcal{Z}_{f,g})$ be the push-forward cycle in the Chow group where $\mathcal{Z}_{f,g}$ is the 1-cycle considered in Theorem 5.6. The two push-forward maps contribute a factor $\deg(\Pi)^2$ to the equation. Moreover, the residue at $s = 1$ of the Archimedean factor in (37) is $\log_e(q)^{-2}$. The result then follows from Theorem 5.6. □

For a self-product of elliptic curves of the type considered in Theorem 5.7, part (C) of Conjecture 1.1 asserts that

$$\Lambda^*_{E,E'}(1) = \frac{|\mathrm{coker}(R_{\mathcal{D}})|}{|\ker(R_{\mathcal{D}})|} \log_e(q)^{-2}.$$

Note that (38) contains the correct power of $\log_e(q)$. Further, one has that the intersection number $(r_{\mathcal{D},\infty}(\Xi), Z_{E,E'})$ divides $\mathrm{coker}(R_{\mathcal{D}})$. Finally, the power $(1-q)^3$ in the denominator of (38) can be partly explained in terms of the kernel of the regulator map $R_{\mathcal{D}}$. The group $H^3_{\mathcal{M}}(E \times E', \mathbb{Q}(2))$ contains certain elements coming from

$$H^2_{\mathcal{M}}(E \times E', \mathbb{Q}(1)) \otimes H^1_{\mathcal{M}}(E \times E', \mathbb{Q}(1))$$

called decomposable elements. Note that

$$H^2_{\mathcal{M}}(E \times E', \mathbb{Q}(1)) \cong \mathrm{Pic}(E \times E') \quad \text{and} \quad H^1_{\mathcal{M}}(E \times E', \mathbb{Q}(1)) \cong K^*.$$

Elements of the form $D \otimes u$, for $D \in NS(E \times E')$ and u a torsion element in K^*, belong to $\ker(R_{\mathcal{D}})$. There are $(q-1)$ elements u coming from $\mathbb{F}_q^*$ and there are two independent elements D of $NS(E \times E')$ providing $(q-1)^2$ such elements.

6. Final remarks

Many of the arguments can be carried out in much greater generality; for example, the ground field could be any local field. The assumption $(I_1, I_2) = 1$ in Theorem 5.7 is not that essential. Along the lines of the arguments in [BS04], we can prove a similar result under the weaker assumption that I_1 and I_2 have some common factors, but are not identical.

As suggested by the referee, another direction in which this work can be generalized is that of higher weight forms. Scholl generalized the work of Beilinson for forms of weight greater than two; however, while in our case the analogue of weight-two forms are the $\mathbb{Q}_\ell$-valued harmonic cochains on the tree, it is not clear to me what the analogue of higher-weight forms is. One might expect that, perhaps, harmonic cochains with values in local systems might play the role.

As remarked earlier, since all the factors appearing are analogous to factors appearing in the classical number field case it would be interesting to know if there was some common underlying field over which the conjecture can be formulated and proved, for which the above work and the classical theorems are special cases.

ACKNOWLEDGEMENTS

I would like to thank S. Bloch, C. Consani, J. Korman, S. Kondo, M. Papikian, A. Prasad and M. Sundara for their help and comments on earlier versions of this manuscript. I would also like to thank the referee for his comments. I would like to thank the University of Toronto, the Max Planck Institute for Mathematics, Bonn, and the TIFR Centre for Applicable Mathematics in Bangalore for providing me with an excellent atmosphere in which to work. Finally I would like to dedicate this paper to the memory of my mother, Ratna Sreekantan.

REFERENCES

BS04 S. Baba and R. Sreekantan, *An analogue of circular units for products of elliptic curves*, Proc. Edinb. Math. Soc. (2) **47** (2004), 35–51.

Beĭ84 A. A. Beĭlinson, *Higher regulators and values of L-functions*, Itogi Nauki i Tekhniki Seriya Sovremennye Problemy Matematiki Noveishie Dostizheniya (Current problems in mathematics), vol. 24 (Vserossiisky Institut Nauchnoi i Tekhnicheskoi Informatsii, Moscow, 1984), 181–238.

Blo86 S. Bloch, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

BGS95 S. Bloch, H. Gillet and C. Soulé, *Non-Archimedean Arakelov theory*, J. Algebraic Geom. **4** (1995), 427–485.

Con98 C. Consani, *Double complexes and Euler L-factors*, Compositio Math. **111** (1998), 323–358.

Con99 C. Consani, *The local monodromy as a generalized algebraic correspondence*, Doc. Math. **4** (1999), 65–108. With an appendix by Spencer Bloch (electronic).

Del73 P. Deligne, *Les constantes des équations fonctionnelles des L-functions*, in *Modular functions of one variable, II (Proc. Int. Summer School, University of Antwerp)* Antwerp, 17 July–3 August 1972, Lecture Notes in Mathematics, vol. 349 (Springer, Berlin, 1973), 501–597.

Gek95 E.-U. Gekeler, *Improper Eisenstein series on Bruhat–Tits trees*, Manuscripta Math. **86** (1995), 367–391.

Gek97 E.-U. Gekeler, *On the Drinfeld discriminant function*, Compositio Math. **106** (1997), 181–202.

GR92 E.-U. Gekeler and M. Reversat, *Jacobians of Drinfeld modular curves*, J. Reine Angew. Math. **476** (1996), 27–93.

Ogg69 A. P. Ogg, *On a convolution of L-series*, Invent. Math. **7** (1969), 297–312.

Pap02 M. Papikian, *On the degree of modular parametrizations over function fields*, J. Number Theory **97** (2002), 317–349.

Ram89 D. Ramakrishnan, *Regulators, algebraic cycles, and values of L-functions*, in *Algebraic K- theory and algebraic number theory,* Honolulu, HI, 12–16 January 1987, Contemporary Mathematics, vol. 83 (American Mathematical Society, Providence, RI, 1989), 183–310.

RSS88 M. Rapoport, N. Schappacher and P. Schneider (eds), *Beilinson's conjectures on special values of L-functions*, Perspectives in Mathematics, vol. 4 (Academic Press, Boston, MA, 1988).

Sre08 R. Sreekantan, *A non-Archimedean analogue of the Hodge-$\mathcal{D}$-conjecture for products of elliptic curves*, J. Algebraic Geom. **17** (2008), 781–798.

Tei92 J. T. Teitelbaum, *Modular symbols for* $\mathbf{F}_q(T)$, Duke Math. J. **68** (1992), 271–295.

Ramesh Sreekantan rameshsreekantan@gmail.com
Indian Statistical Institute, 8th Mile, Mysore Road, Jnana Bharathi, Bangalore 560 059, India