

ON SOME GEOMETRY ASSOCIATED WITH
A GENERALISED TODA LATTICE

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We define the notion of Darboux integrability for linear second order partial differential operators,

$$L = \partial_{x_1 x_2}^2 + \alpha \partial_{x_1} + \beta \partial_{x_2} + \gamma.$$

We then build on certain geometric results of E. Vessiot related to the theory of Monge characteristics to show that the Darboux integrable operators L can be used to obtain a solution of the A_2 Toda field theory. This solution is parametrised by four arbitrary functions. The approach presented in this paper thereby represents an alternative means of linearising the A_2 Toda equations and may be contrasted with the known linearisation via the Lax pair.

1. INTRODUCTION

In an interesting paper [8], Leznov and Saveliev were able to obtain the zero curvature representation for the nonlinear system

$$(1.1) \quad \frac{\partial^2 u_\alpha}{\partial x_1 \partial x_2} = \sum_{\beta=1}^r k_{\alpha\beta} \exp(u_\beta)$$

for arbitrary real matrices $k = \{k_{\alpha\beta}\}_{\alpha,\beta=1}^r$. System (1.1) is encountered in a number of contexts in mathematical physics. For example, the cylindrically symmetric self dual Yang Mills fields in \mathbb{R}^4 are described by (1.1) when k coincides with the Cartan matrix of the relevant gauge group. If k coincides with the Cartan matrix of the series A_r of simple Lie algebras then (1.1) is equivalent to the generalised Toda chain with free end points. Leznov and Saveliev discovered, using the theory of Lie algebra representations, that in these special cases, that is, when k is the Cartan matrix of a simple Lie algebra of rank r , they could obtain an explicit solution of (1.1) involving $2r$ arbitrary functions which they called the general solution. In their paper they posed the question: "why is

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system (1.1) explicitly integrable in just these cases ?” The purpose of the present paper is to show that when k is the Cartan matrix of the Lie algebra A_2 , the general solution is strongly linked to the integrability in the *Darboux sense* of the linear hyperbolic partial differential equation

$$(1.2) \quad \frac{\partial^2 w}{\partial x_1 \partial x_2} + \alpha \frac{\partial w}{\partial x_1} + \beta \frac{\partial w}{\partial x_2} + \gamma w = 0$$

where

$$\alpha, \beta, \gamma \in C^\infty(\mathbb{R}^2, \mathbb{R}).$$

The main idea is that (1.2) is Darboux integrable if and only if a variant of the A_2 Toda equations are satisfied. This fact together with a fundamental result of Vessiot [12] on the classification of Darboux integrable semilinear hyperbolic partial differential equations allows one to construct an explicit solution in finite terms of four arbitrary functions each of one variable.

This result in fact bears out general remarks made by Hermann [7] and illustrates a paradigm concerning integrable nonlinear partial differential equations, namely that every integrable nonlinear partial differential equation is associated with a “linear problem” whose “integrability condition” is the nonlinear partial differential equation under study. In the usual discussions of completely integrable systems the linear problem of interest is derived from the Lax pair. Here we point out another type of linear problem, namely the equation (1.2) while the notion of “integrability condition” is replaced in this context by the notion of Darboux integrability.

Of course the idea of Darboux integrability which is to be described is a classical one and when it is applied to equation (1.2) one discovers that it is strongly linked to yet another classical notion, that of a *Laplace Transformation* (not to be confused with a Laplace transform). This topic together with its relevance to various nonlinear partial differential equation now known to be completely integrable was first discussed by Darboux in the second volume of his treatise [2] and effectively reviewed in Hermann [7, Chapter IX] More recently, the connection between the Laplace transformation and various Toda lattices has been discussed in Weiss [13]. In a sense the results of this paper are complementary to those presented in Weiss and it would be interesting to make the connections precise.

By now many papers have been written on the Toda equations and no attempt will be made here to place the results of this paper within the spectrum of known results. We shall be content here to show what the classical notions of Darboux integrability and Laplace transformations have to do with the A_2 Toda field theory.

2. LAPLACE TRANSFORMATION

In this section we briefly review some notions associated with the idea of a Laplace

transformation of equation (1.2). For related material on this topic the reader is referred to Hermann [7].

Let \mathcal{D} denote the set of second order linear partial differential operators of the form

$$L = \partial_{x_1 x_2}^2 + \alpha \partial_{x_1} + \beta \partial_{x_2} + \gamma$$

where

$$\alpha, \beta, \gamma \in C^\infty(\mathbb{R}^2, \mathbb{R}).$$

There is an interesting Lie pseudogroup of transformations Ψ which acts on \mathcal{D} :

$$\Psi : G \times \mathcal{D} \rightarrow \mathcal{D}, \quad G = C^\infty(\mathbb{R}^2, \mathbb{R}),$$

defined by

$$\Psi(\lambda, L) = T_\lambda(L) = \lambda^{-1} L \lambda, \quad \lambda \in G, L \in \mathcal{D},$$

and where the group operation in G is point multiplication of functions. A straightforward computation then gives

$$(2.1) \quad T_\lambda(L) = \partial_{x_1 x_2}^2 + (\alpha + \lambda^{-1} \partial_{x_2} \lambda) \partial_{x_1} + (\beta + \lambda^{-1} \partial_{x_1} \lambda) \partial_{x_2} + (\lambda^{-1} \partial_{x_1 x_2}^2 \lambda + \alpha \lambda^{-1} \partial_{x_1} \lambda + \beta \lambda^{-1} \partial_{x_2} \lambda + \gamma).$$

Next, consider the set of all maps $\mathcal{D} \rightarrow C^\infty(\mathbb{R}^2, \mathbb{R})$. Denote this set by \mathcal{F} . From the above G -action on \mathcal{D} , there is an induced action on \mathcal{F} defined by

$$\tau_\lambda(f)(L) = f(T_\lambda L), \quad f \in \mathcal{F}, \lambda \in G.$$

It is now of interest to enquire about the invariants of this G -action on \mathcal{F} . That is, we seek functions $f : \mathcal{D} \rightarrow C^\infty(\mathbb{R}^2)$ such that

$$\tau_\lambda(f)(L) = f(L), \quad \lambda \in G, L \in \mathcal{D}.$$

Classically, two such invariants are known:

$$(2.2) \quad h(L) = \partial_{x_1} \alpha + \alpha \beta - \gamma,$$

$$(2.3) \quad k(L) = \partial_{x_2} \beta + \alpha \beta - \gamma.$$

It is straightforward to show from equation (2.1) that

$$\tau_\lambda(h)(L) = h(T_\lambda L) = h(\lambda^{-1} L \lambda) = h(L), \quad \text{for all } \lambda \in G, L \in \mathcal{D}$$

and similarly for k . This establishes h and k as differential invariants in \mathcal{F} known classically as the Laplace invariants of L . The Laplace invariants will play a key role in our subsequent considerations.

THE LAPLACE TRANSFORMATION AS A BACKLUND MAP.

We are now in a position to introduce the Laplace transformation and to establish a number of properties which will later prove useful. Suppose we consider a set of partial differential equations

$$(2.4) \quad u_1 - \partial_{x_2} u - \alpha u = 0,$$

$$(2.5) \quad \partial_{x_1} u_1 + \beta u_1 - \zeta u = 0$$

for two unknown functions $u, u_1 \in C^\infty(\mathbb{R}^2, \mathbb{R})$, where α, β, ζ are given real-valued functions of x_1 and x_2 . We ask the question: what should the function $\zeta(x_1, x_2)$ be in order that the function $u(x_1, x_2)$ satisfy the partial differential equation $Lu = 0$? The answer is $\zeta = h(L)$.

Matters being so, one finds that the partial differential equation satisfied by the function $u_1(x_1, x_2)$ is $L_1 u_1 = 0$, where,

$$(2.6) \quad L_1 = \partial_{x_1 x_2}^2 + (\alpha - \partial_{x_2} \ln |h(L)|) \partial_{x_1} + \beta \partial_{x_2} + (\gamma - \partial_{x_1} \alpha + \partial_{x_2} \beta - \beta \partial_{x_2} \ln |h(L)|).$$

Clearly, then, equations (2.4) and (2.5) define a nonauto Backlund transformation relating the solutions of $L\phi = 0$ to those of $L_1\phi = 0$.

There is one further nonauto Backlund transformation that may be written down analogous to (2.4) and (2.5) but involving the other Laplace invariant, namely

$$(2.7) \quad u_{-1} - \partial_{x_1} u - \beta u = 0,$$

$$(2.8) \quad u_{-1} - k(L)u - \alpha u_{-1}.$$

Once more, by eliminating the unknown function u_{-1} , one finds that u satisfies the partial differential equation $Lu = 0$ while u_{-1} satisfies $L_{-1}u_{-1} = 0$ where,

$$(2.9) \quad L_{-1} = \partial_{x_1 x_2}^2 + (\beta - \partial_{x_1} \ln |k(L)|) \partial_{x_2} + \alpha \partial_{x_1} + (\gamma - \partial_{x_2} \beta + \partial_{x_1} \alpha - \alpha \partial_{x_1} \ln |k(L)|)$$

as may easily be verified. The operators L_1 and $L_{-1} \in \mathcal{D}$ are known as the *Laplace transformations* of $L \in \mathcal{D}$. The notation used here in fact goes back to Darboux and is a little misleading because it gives the impression that the Laplace transformations $\mathcal{L}_1 : L \rightarrow L_1$ and $\mathcal{L}_{-1} : L \rightarrow L_{-1}$ are inverses, which they are not. However, if $\widehat{\mathcal{D}}$ denotes the space of orbits in \mathcal{D} under G then it may be shown that \mathcal{L}_1 and \mathcal{L}_{-1} map G -orbits to G -orbits, that is, if Δ_1, L_1 are the images of $\Delta, L \in \mathcal{D}$ respectively, then if Δ and L share an orbit, there exists $\lambda, \mu \in G$ such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{T_\lambda} & \Delta \\ \mathcal{L}_1 \downarrow & & \downarrow \mathcal{L}_1 \\ L_1 & \xrightarrow{T_\mu} & \Delta_1 \end{array}$$

commutes. Similarly for \mathcal{L}_{-1} . Hence \mathcal{L}_1 and \mathcal{L}_{-1} pass to the quotient to act on the orbit space $\widehat{\mathcal{D}}$. It is then easily shown that

$$\mathcal{L}_{-1} \circ \mathcal{L}_1(L) = k(L)Lk(L)^{-1},$$

and

$$\mathcal{L}_1 \circ \mathcal{L}_{-1}(L) = h(L)Lh(L)^{-1},$$

which proves that the Laplace transformations \mathcal{L}_1 and \mathcal{L}_{-1} are, when acting on $\widehat{\mathcal{D}}$, inverses of each other. These facts are pointed out by Hermann [7] who also reviews the way in which Darboux used them to study classes of nonlinear Klein-Gordon equations now known to be completely integrable. All these results may be compared with those of Weiss [13]

We end this section on the Laplace transformation by recording the Laplace invariants of L_1 and L_{-1} since they will make an appearance in later computations :

$$\begin{aligned} h(L_1) &= 2h(L) - k(L) - \partial_{x_1 x_2}^2 \ln |h(L)|, \\ k(L_1) &= h(L), \\ h(L_{-1}) &= k(L), \\ k(L_{-1}) &= 2k(L) - h(L) - \partial_{x_1 x_2}^2 \ln |k(L)|. \end{aligned}$$

3. DARBOUX INTEGRABILITY OF NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE

The field of integrability of nonlinear partial differential equations is still relatively young having begun, properly speaking, in 1967 when Gardner, Greene, Kruskal and Miura published the solution of the initial value problem of the Korteweg de Vries equation by the inverse scattering transform.

In the intervening years, it has been realised that an enormous inheritance has been left to the current generation by the 19th century in the field of integrability; particularly by geometers like Lie, Darboux and Goursat to mention but a few. As a consequence, a great deal of work needs to be done to codify and clarify this body of results and particularly its later developments by Cartan and Vessiot and to work out its consequences and limitations for integrability and solvability questions associated with differential equations. Already in this direction a number of workers have made important contributions which have covered topics such as Lie equations, symmetry groups, prolongation structures, differential Galois theory to mention but a few.

In this section we wish to review briefly another classical construction due to Darboux and extending well known results of Monge. Darboux's construction is based on the notion of *Monge characteristics* associated with any nonelliptic partial differential equation and which generalises the notion of *Cauchy characteristics*.

The theory of characteristics is most clearly expressed by making use of Cartan’s geometric results [1]. In the hyperbolic case, this has been most lucidly expounded in Gardner and Kamran [4] where aspects of the Cartan equivalence problem for hyperbolic partial differential equations in two independent variables is also covered. In this paper we shall not present a full geometric exposition of Darboux’s idea and the reader is referred to Vassiliou [11] for this. We shall instead be content here to present enough details in order to prepare the reader for an application of Darboux’s construction to the A_2 Toda field theory.

Consider the second order quasilinear partial differential equation

$$(3.1) \quad A \frac{\partial^2 u}{\partial x_1^2} + 2H \frac{\partial^2 u}{\partial x_1 \partial x_2} + B \frac{\partial^2 u}{\partial x_2^2} = 0$$

where A, H, B and C are C^1 functions of the variables $x_1, x_2, u, \frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$.

It has been known since the time of Monge that the equation (3.1) may be associated with a pair of modules of differential 1-forms defined on $J^1(\mathbb{R}^2, \mathbb{R})$, the manifold of 1-jets of maps $\mathbb{R}^2 \rightarrow \mathbb{R}$. Let us denote these modules by ${}_1\tilde{\Omega}^{(1)}$ and ${}_2\tilde{\Omega}^{(2)}$ and let $\omega = H + \sqrt{H^2 - AB}$. Then whenever $H^2 - AB \geq 0$ we have that ${}_1\tilde{\Omega}^{(1)}$ and ${}_2\tilde{\Omega}^{(1)}$ are generated as follows

$$\begin{aligned} {}_1\tilde{\Omega}^{(1)} : \{ \theta_1^1 &= \omega dx_1 - A dx_2, \theta_2^1 = \omega dz_1 + B dz_2 - C dx_2, \theta_3^1 = dz - z_1 dx_1 - z_2 dx_2 \} \\ {}_2\tilde{\Omega}^{(1)} : \{ \theta_1^2 &= B dx_1 - \omega dx_2, \theta_2^2 = A dz_1 + \omega dz_2 - C dx_1, \theta_3^2 = dz - z_1 dx_1 - z_2 dx_2 \} \end{aligned}$$

where here as elsewhere curly brackets denote module closure over (in this case) $C^\infty(J^1(\mathbb{R}^2, \mathbb{R}), \mathbb{R})$.

I will not go into the genesis of these modules which were first obtained by Monge in order to integrate equations like (3.1). Neither will I in this paper attempt to connect the modules ${}_1\tilde{\Omega}^{(1)}$ and ${}_2\tilde{\Omega}^{(1)}$ to the work of Cartan as expounded by Gardner and Kamran [4]. Suffice it to say for the present that the maximal dimension regular integral submanifolds of these two modules are *one* in each case and represent what are classically known as the (order 1) Monge characteristics of the partial differential equation (3.1). For our immediate purposes, the significance of ${}_1\tilde{\Omega}^{(1)}$ and ${}_2\tilde{\Omega}^{(1)}$ is the following.

THEOREM 3.1. *Suppose there exists functions $\lambda_1, \lambda_2, \lambda_3 : J^1(\mathbb{R}^2, \mathbb{R}) \rightarrow \mathbb{R}$ such that*

$$\lambda_1 \theta_1^i + \lambda_2 \theta_2^i + \lambda_3 \theta_3^i = dF$$

for $i = 1$ or 2 and for some function $F : J^1(\mathbb{R}^2, \mathbb{R}) \rightarrow \mathbb{R}$. If $f \in C^2(\mathbb{R}^2, \mathbb{R})$ is such that

$$(j^1 f)^* F = 0,$$

then f is a solution of the second order partial differential equation (3.1) where $j^1 f : \mathbb{R}^2 \rightarrow J^1(\mathbb{R}^2, \mathbb{R})$ is the 1-graph of f .

PROOF: See Vassiliou [11]. □

REMARK. F , when it exists, is called a *first integral* of (3.1). Evidently this is a very advantageous situation since it means that solutions of a second order partial differential equation (3.1) may be obtained by solving a *first order* partial differential equation (3.2). Because of the existence of Cauchy vector fields in the latter case the integration of (3.1) is thereby reduced to ordinary differential equations.

An equation (3.1) for which F exists (as in Theorem 3.1) for $i = 1$ or 2 will be called *Monge integrable* or, alternatively, *Darboux integrable at order 1*. If (3.1) is Monge integrable and $i = 1$, it will be called *Monge integrable on system 1*; if $i = 2$ then (3.1) will be called *Monge integrable on system 2*. We wish to use all these results to study equations like (1.2). The proof of the following can be found in Grundland and Vassiliou [5]

THEOREM 3.2. *The linear hyperbolic partial differential equation*

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} + \alpha(x_1, x_2) \frac{\partial u}{\partial x_1} + \beta(x_1, x_2) \frac{\partial u}{\partial x_2} + \gamma(x_1, x_2)u = 0$$

is Monge integrable on system 1 if and only if

$$h = \frac{\partial \alpha}{\partial x_1} + \alpha\beta - \gamma = 0.$$

and Monge integrable on system 2 if and only if

$$k = \frac{\partial \beta}{\partial x_2} + \alpha\beta - \gamma = 0.$$

There can be only one such functionally independent first integral.

A remarkable fact about the modules ${}_1\tilde{\Omega}^{(1)}$ and ${}_2\tilde{\Omega}^{(1)}$ and Theorem 3.1 is that the whole construction can be lifted to higher order jet bundles where new results are to be found. This lift is known as *Darboux's method*. We shall formulate the results for the equation

$$(3.3) \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = f\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right),$$

however we note that this restriction is unnecessary and Darboux's method applies to a larger set of equations than that represented by (3.3). It is possible to describe Darboux's method by defining an appropriate pair of differential 1-forms on the k^{th}

order jet bundle. For the purposes of later calculation however we shall instead express everything in terms of vector fields.

Let Δ_k denote the subset of $J^k(\mathbb{R}^2, \mathbb{R})$ determined by the $(k - 2)$ prolongation of equation (3.3). Thus $\Delta_3 \subset J^3(\mathbb{R}^2, \mathbb{R})$ is defined by

$$z_{12} = f(x_1, x_2, z, z_1, z_2), z_{112} = \frac{\partial f}{\partial x_1} + z_1 \frac{\partial f}{\partial z} + z_{11} \frac{\partial f}{\partial z_1} + z_{12} \frac{\partial f}{\partial z_2},$$

$$z_{122} = \frac{\partial f}{\partial x_2} + z_2 \frac{\partial f}{\partial z} + z_{12} \frac{\partial f}{\partial z_1} + z_{22} \frac{\partial f}{\partial z_2}.$$

Let $D_1^{(k)}$ and $D_2^{(k)}$ be total differential operators in $J^k(\mathbb{R}^2, \mathbb{R})$ restricted to the subset Δ_k .

DEFINITION 3.1: The vector field systems

$${}_1\Omega^{(k)}(f) : \left\{ D_1^{(k)} + \left((D_2^{(k)})^{k-1} f \right) \frac{\partial}{\partial z_{22\dots 2}}, \frac{\partial}{\partial z_{21\dots 1}} \right\}$$

$${}_2\Omega^{(k)}(f) : \left\{ D_2^{(k)} + \left((D_1^{(k)})^{k-1} f \right) \frac{\partial}{\partial z_{11\dots 1}}, \frac{\partial}{\partial z_{22\dots 2}} \right\}$$

which are modules of vector fields on Δ_k will be called the k^{th} order Monge characteristic vector field systems associated with equation (3.3).

REMARK. The vector field systems ${}_1\Omega^{(k)}$ and ${}_2\Omega^{(k)}$ generalise ${}_1\tilde{\Omega}^{(1)}$ and ${}_2\tilde{\Omega}^{(1)}$ in that by setting $k = 1$ in ${}_1\Omega^{(k)}$ and ${}_2\Omega^{(k)}$ and computing their duals we obtain precisely the modules ${}_1\tilde{\Omega}^{(1)}$ and ${}_2\tilde{\Omega}^{(1)}$. It may be checked that the vector field systems of Definition 3.1 each have genus $g = 1$. Furthermore, their 1-dimensional integral submanifolds are classically known as the k^{th} order Monge characteristics. Full details will be found in Vassiliou [11]

Our interest for the present is the generalisation of Theorem 3.1 which is the following.

THEOREM 3.3. Let $\psi(x_1, x_2)$ be any C^{k+1} solution of

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = f\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right).$$

Suppose $\pi : \Delta_k \rightarrow \mathbb{R}$ is an invariant of ${}_i\Omega^{(k)}$, for $i = 1, 2$. Then

$$\frac{\partial}{\partial x_i} (\pi \circ j^k \psi) = 0, \quad i = 1, 2$$

where $j^k\psi$ is the k -graph of ψ .

PROOF: For each $i = 1, 2$, let M_1^i and M_2^i be the two vector fields generating ${}_i\Omega^{(k)}$. Then if ψ is the stated solution we have

$$(j^k\psi)_* \left(\frac{\partial}{\partial x_i} \right) (\pi) = \frac{\partial}{\partial x_i} (\pi \circ j^k\psi) = (M_1^i\pi) \circ j^k\psi + \frac{\partial^{k+1}\psi}{\partial x_i^{k+1}} (M_2^i\pi) \circ j^k\psi = 0,$$

by definition of π . □

Theorem 3.3 makes clear the fact that from equation (3.3) we have constructed a compatible system of partial differential equations, assuming that we are in fact able to construct the invariants π_1 of ${}_1\Omega^{(k)}$ and π_2 of ${}_2\Omega^{(k)}$. These equations are as follows.

$$(3.4) \quad \frac{\partial^2 \psi}{\partial x_1 \partial x_2} = f \left(x_1, x_2, \psi, \frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2} \right),$$

$$(3.5) \quad \frac{\partial}{\partial x_1} (\pi_1 \circ j^k\psi) = 0,$$

$$(3.6) \quad \frac{\partial}{\partial x_2} (\pi_2 \circ j^k\psi) = 0.$$

This system is, by Theorem 3.3, guaranteed to be compatible whenever equation (3.4) has nontrivial C^{k+1} solutions. At this point it may be argued that system (3.4) to (3.6) is a worse problem than solving (3.4) by itself. In the first place the system is over determined and the possibility of nontrivial integrability conditions is not excluded in fact by Theorem 3.3. In the second place, the order of system (3.4) to (3.6) is in general higher than 2. So what advantage is provided by this Theorem?

A close inspection of equations (3.5) and (3.6) shows in fact that even though ψ is a function of the two variables x_1 and x_2 , the equation (3.5) contains no differentiations with respect to x_1 while (3.6) contains no differentiations with respect to x_2 . This is because x_1 and x_2 are themselves invariants of ${}_2\Omega^{(k)}$ and ${}_1\Omega^{(k)}$ respectively and hence are what are commonly referred to as *characteristic variables*. The effect of this is that equations (3.5) and (3.6) are not really partial differential equations at all but parametrised families of ordinary differential equations. In order to integrate equation (3.4) we turn Theorem 3.3 on its head and study instead of (3.4) either or both of equations (3.5) and (3.6). If we succeed in solving these ordinary differential equations then by Theorem 3.3, we also obtain solutions of the original second order partial differential equation (3.4). This makes clear the very interesting fact that even though second order partial differential equations have no Cauchy vectors associated to them in general, if their Monge fields carry sufficient invariants, the construction of their *general solutions* can be reduced to solving ordinary differential equations. To see

explicit examples of this the reader is referred Grundland and Vassiliou [5, 6] and to Vassiliou [11].

In order to study the A_2 Toda field theory we need to generalise Theorem 3.2 to the second order jet bundle $J^2(\mathbb{R}^2, \mathbb{R})$ and to study the integrability via Darboux’s method of equation (1.2).

We begin by letting Γ_k denote the set of direct sums of modules ${}_1\Omega^{(k)}(f) \oplus {}_2\Omega^{(k)}(f)$ as f ranges over all partial differential equations (3.3). Let $\tilde{\mu}$ be the map which assigns to each f its direct sum of Monge characteristic modules in Γ_k . Let μ be the restriction of $\tilde{\mu}$ to the partial differential equations $Lu = 0$, $L \in \mathcal{D}$. Specifically, $\mu : \mathcal{D} \rightarrow \Gamma_2$ is defined by $\mu(L) = {}_1\Omega^{(2)}(L) \oplus {}_2\Omega^{(2)}(L)$, where ${}_i\Omega^{(2)}(L)$, $i = 1, 2$, are the Monge characteristic vector field systems associated to the partial differential equation $Lu = 0$. Let $\Gamma_2(2, 2)$ denote the subset of Γ_2 for which ${}_i\Omega^{(2)}(L)$, $i = 1, 2$, each possess precisely two functionally independent invariants and let $\gamma_2(2, 2)$ be the restriction of $\Gamma_2(2, 2)$ to the partial differential equation $Lu = 0$, $L \in \mathcal{D}$.

A partial differential equation (3.3) will be called (2,2)-Darboux integrable at order 2 if $\tilde{\mu}(f) \in \Gamma_2(2, 2)$. Similarly an operator $L \in \mathcal{D}$ will be called (2,2)-Darboux integrable at order 2 or, simply, (2,2)-Darboux integrable if $\mu(L) \in \gamma_2(2, 2)$. The following theorem characterises the (2,2)-Darboux integrable operators in \mathcal{D} .

THEOREM 3.4. *Let*

$$L = \partial_{x_1 x_2} + \alpha \partial_{x_1} + \beta \partial_{x_2} + \gamma$$

be a linear second order partial differential operator belonging to \mathcal{D} . Then the partial differential equation

$$Lu = 0$$

has precisely two invariants on system ${}_1\Omega^{(2)}$ if and only if $h(L)$ and $k(L)$ satisfy

$$(3.7) \quad \frac{\partial^2}{\partial x_1 \partial x_2} \ln |h(L)| = 2h(L) - k(L), \quad h(L) \neq 0,$$

and precisely two invariants on system ${}_2\Omega^{(2)}$ if and only if

$$(3.8) \quad \frac{\partial^2}{\partial x_1 \partial x_2} \ln |k(L)| = -h(L) + 2k(L), \quad k(L) \neq 0.$$

PROOF: In Grundland and Vassiliou [5], it is shown that $Lu = 0$ has precisely two invariants on system ${}_1\Omega^{(2)}$ if and only if the determinant of the matrix

$$\begin{pmatrix} h & K \\ \frac{\partial h}{\partial x_1} + \alpha h & \frac{\partial K}{\partial x_1} + hB + \alpha K \end{pmatrix}$$

is zero while $h(L) \neq 0$, where

$$K = \frac{\partial A}{\partial x_1} + \alpha B + \beta A - C, \quad A = \frac{\partial \alpha}{\partial x_2} + \alpha^2,$$

$$B = \frac{\partial \beta}{\partial x_2} - \alpha\beta + \gamma, \quad C = \frac{\partial \gamma}{\partial x_2} - \alpha\gamma.$$

Calculation in fact shows that

$$K = \frac{\partial h}{\partial x_1} - 2\alpha h.$$

Hence the determinantal condition becomes

$$\begin{aligned} 0 &= \frac{\partial K}{\partial x_1} + hB - \frac{K}{h} \frac{\partial h}{\partial x_1} = \frac{\partial^2 h}{\partial x_1 \partial x_2} - 2\alpha \frac{\partial h}{\partial x_1} - 2h \frac{\partial \alpha}{\partial x_1} \\ &\quad + h(k - 2\alpha\beta + 2\gamma) - h^{-1} \left(\frac{\partial h}{\partial x_2} - 2\alpha h \right) \frac{\partial h}{\partial x_1} \\ &= \frac{\partial^2 h}{\partial x_1 \partial x_2} - 2h \left(\frac{\partial \alpha}{\partial x_1} + \alpha\beta - \gamma \right) - h^{-1} \frac{\partial h}{\partial x_1} \frac{\partial h}{\partial x_2} + hk \\ &= h^{-1} \frac{\partial^2 h}{\partial x_1 \partial x_2} - h^{-2} \frac{\partial h}{\partial x_1} \frac{\partial h}{\partial x_2} - 2h + k \\ &= \frac{\partial^2}{\partial x_1 \partial x_2} \ln |h| - 2h + k \end{aligned}$$

which is condition (3.7).

A similar calculation associated with the module ${}_2\Omega^{(2)}$ leads to the equation (3.8). Let us note that the equations (3.7) and (3.8) correspond precisely to the vanishing of the Laplace invariants $h(L_1)$ and $k(L_{-1})$; see Section 2. \square

It follows from Theorem 3.4 that $\mu(L) \in \gamma_2(2, 2)$ if and only if $h(L)$ and $k(L)$ satisfy the coupled system (3.7) and (3.8). In the next section we shall show how the class $\mu^{-1}(\gamma_2(2, 2))$ can be explicitly constructed. Indeed we give a parametrisation of $\mu^{-1}(\gamma_2(2, 2))$ in terms of *four* arbitrary functions. This parametrisation was first obtained by Vessiot [12] who however does not seem to have written out a proof of the correctness of his algorithm. This is provided by Theorem 4.2 which is a general proof of correctness and supplies the above mentioned parametrisation as a corollary. First however, let us notice that the system consisting of equations (3.7) and (3.8) do indeed constitute the A_2 Toda field theory discussed in Leznov and Saveliev [8].

The equations (3.7) and (3.8) have the form

$$(3.10) \quad \begin{aligned} \frac{\partial^2}{\partial x_1 \partial x_2} \ln |h| &= 2h - k, \\ \frac{\partial^2}{\partial x_1 \partial x_2} \ln |k| &= -h + 2k, \end{aligned}$$

where we have abbreviated $h(L)$ to h and $k(L)$ to k . Now suppose there exists an open subset of \mathbb{R}^2 such that $hk > 0$. Then in this subset there exists functions $u(x_1, x_2)$ and $v(x_1, x_2)$ such that u and v satisfy the system

$$(3.11) \quad \begin{aligned} \frac{\partial^2 u}{\partial x_1 \partial x_2} &= 2e^u - e^v, \\ \frac{\partial^2 v}{\partial x_1 \partial x_2} &= -e^u + 2e^v. \end{aligned}$$

System (3.11) is precisely the Toda field theory for A_2 and corresponds to the Leznov-Saveliev system (1.1) when

$$k = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

which is the Cartan matrix for the Lie algebra A_2 . A further system is obtained according to the existence of open subsets of \mathbb{R}^2 in which $hk < 0$. In this case functions u and v exist such that

$$(3.12) \quad \begin{aligned} \frac{\partial^2 u}{\partial x_1 \partial x_2} &= -2e^u - e^v, \\ \frac{\partial^2 v}{\partial x_1 \partial x_2} &= e^u + 2e^v. \end{aligned}$$

Clearly (3.12) is equivalent to (3.11) with the change of variable $v \rightarrow v + i\pi$ but it is not clear that there exists a real local diffeomorphism relating these systems. The system (3.12) is not related to the Cartan matrix of any simple Lie algebra but is nevertheless an explicitly integrable system as we shall see.

4. A LOCAL EQUIVALENCE PROBLEM FOR VECTOR FIELD SYSTEMS AND INTEGRATION OF THE A_2 TODA FIELD THEORY

In the previous section we showed that when the conditions defining the (2,2)-Darboux integrability of the operators $L \in \mathcal{D}$ are expressed in terms of the Laplace invariants of L one recovers the A_2 Toda field theory. By combining this result with the solution of a local equivalence problem for vector field systems (Theorem 4.2) we obtain in this section an explicit solution of the Toda field theory containing four (almost) arbitrary functions. (The word “almost” in the previous sentence will be explained in the sequel.) Here we understand by a “solution of an equivalence problem”, the construction of an *explicit diffeomorphism* which maps one vector field system to another. This should be contrasted with the solution of a *Cartan equivalence problem* which employs powerful techniques to find necessary and sufficient conditions for equivalence in an algorithmic way in the form of a complete set of invariants [3].

Recall that $\Gamma_2(2,2)$ is the set of modules of Monge characteristic vector fields associated with the (2,2)–Darboux integrable partial differential equations

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = f\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right).$$

Call this set of equations $E(2,2)$. That is, $f \in E(2,2)$ if and only if $\tilde{\mu}(f) \in \Gamma_2(2,2)$. Vessiot [12], has proved the following remarkable theorem regarding the set of equations $E(2,2)$.

THEOREM 4.1 (VESSIOT). *Let $\Delta_2(f) \subset J^2(\mathbb{R}^2, \mathbb{R})$ denote the submanifold in the second order jet bundle defined by a partial differential equation $f \in E(2,2)$.*

Then there exist

1. *functions $\phi_\alpha, \psi_\alpha \in C^\infty(\mathbb{R}, \mathbb{R})$, $\alpha = 1, 2, 3$;*
2. *a Lie group G , with $\dim G = 3$;*
3. *a local diffeomorphism $\pi : \Delta_2(f) \rightarrow G \times \mathbb{R}^4$ such that*

$$\begin{aligned} \pi_*\left({}_1\Omega^{(2)}(f)\right) &= \left\{ \frac{\partial}{\partial u} + u_1\phi_\alpha(u)L_\alpha, \frac{\partial}{\partial u_1} \right\}, \\ \pi_*\left({}_2\Omega^{(2)}(f)\right) &= \left\{ \frac{\partial}{\partial v} + v_1\psi(v)R_\alpha, \frac{\partial}{\partial v_1} \right\}, \end{aligned}$$

where $\{L_\alpha\}_{\alpha=1}^3$ and $\{R_\alpha\}_{\alpha=1}^3$ form bases for the left and right invariant vector fields on G respectively. Finally, $\{u, u_1\}$ are any two functionally independent invariants of ${}_2\Omega^{(2)}(f)$ while $\{v, v_1\}$ are any two functionally independent invariants of ${}_1\Omega^{(2)}(f)$.

Vessiot’s theorem gives us a canonical form for the collection of modules

$$\tilde{\mu}(E(2,2)) = {}_1\Omega^{(2)}(f) \oplus {}_2\Omega^{(2)}(f), \quad f \in E(2,2).$$

in terms of the three dimensional Lie groups. However the problem that we are interested in here may be thought of as the “converse” of Vessiot’s theorem. That is, given a module of vector fields on $\mathbb{R}^4 \times G$ of the form

$$(4.1) \quad \left\{ \frac{\partial}{\partial u} + u_1\phi_\alpha(u)L_\alpha, \frac{\partial}{\partial u_1} \right\} \oplus \left\{ \frac{\partial}{\partial v} + v_1\psi_\alpha(v)R_\alpha, \frac{\partial}{\partial v_1} \right\},$$

find the inverse of Vessiot’s local diffeomorphism π . Once π^{-1} has been found one can recover the associated partial differential equation $f \in E(2,2)$. As a corollary of this we integrate the Toda equations by a suitable choice of Lie group G . We note in passing that Vessiot’s theorem is an *existence* result and the theorem itself gives no

clue as to how π or its inverse can be constructed. Nevertheless Vessiot’s paper does contain clues and explicit computations; however there is no proof. This is provided by the following theorem.

Let us denote the collection of modules of vector fields of the form (4.1) by $\Sigma(\phi_\alpha; \psi_\alpha; G)$, which is shorthand for $\Sigma(\{\phi_\alpha\}_{\alpha=1}^3; \{\psi_\alpha\}_{\alpha=1}^3; G)$.

THEOREM 4.2. *Suppose $\sigma \in \Sigma(\phi_\alpha; \psi_\alpha; G)$ is a module of vector fields on $\mathbb{R}^4 \times G$ generated by*

$$\begin{aligned} \sigma &= \left\{ \frac{\partial}{\partial u} + u_1 \phi_\alpha(u) L_\alpha, \frac{\partial}{\partial u_1} \right\} \oplus \left\{ \frac{\partial}{\partial v} + v_1 \psi_\alpha R_\alpha, \frac{\partial}{\partial v_1} \right\} \\ &= \left\{ \mathbf{L}, \frac{\partial}{\partial u_1} \right\} \oplus \left\{ \mathbf{R}, \frac{\partial}{\partial v_1} \right\}. \end{aligned}$$

Suppose the vector field system \mathcal{A} generated by

$$\mathcal{A} = \left\{ \left[\frac{\partial}{\partial u_1}, \mathbf{L} \right], \left[\frac{\partial}{\partial v_1}, \mathbf{R} \right], \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1} \right\} \subset \sigma \oplus [\sigma, \sigma],$$

has three functionally independent invariants x_1, x_2 , and z such that $\mathbf{L}x_2 = \mathbf{R}x_1 = 0$ and $\mathbf{L}x_1 \mathbf{R}x_2 \neq 0$.

Define the functions

$$z_1 = \frac{\mathbf{L}z}{\mathbf{L}x_1}, \quad z_2 = \frac{\mathbf{R}z}{\mathbf{R}x_1}, \quad z_{11} = \frac{\mathbf{L}z_1}{\mathbf{L}x_1}, \quad z_{22} = \frac{\mathbf{R}z_2}{\mathbf{R}x_2}.$$

Then whenever the functions $(x_1, x_2, z, z_1, z_2, z_{11}, z_{22})$ are functionally independent they define a local diffeomorphism

$$\mathcal{P} : \mathbb{R}^4 \times G \rightarrow \Delta_2(f),$$

such that,

$$\mathcal{P}_*(\sigma) = {}_1\Omega^{(2)}(f) \oplus {}_2\Omega^{(2)}(f),$$

where f is defined by

$$z_{12} = \frac{\mathbf{L}z_2}{\mathbf{L}x_1} = \frac{\mathbf{R}z_1}{\mathbf{R}x_2}.$$

Furthermore, $f \in E(2, 2)$ whenever $\frac{\partial f}{\partial z_{11}} = \frac{\partial f}{\partial z_{22}} = 0$.

This theorem says that if the stated conditions are satisfied for $\sigma \in \Sigma(\phi_\alpha; \psi_\alpha; G)$ then a local diffeomorphism $\mathcal{P} : \mathbb{R}^4 \times G \rightarrow \Delta_2(f)$ can be explicitly constructed such that

$$\mathcal{P}_*(\sigma) \in \Gamma_2(2, 2).$$

It then follows that \mathcal{P} is the sought after inverse of Vessiot's diffeomorphism π . Moreover, we shall see that if we choose σ from the collection $\Sigma(\phi_\alpha; \psi_\alpha; \mathbb{R}^3)$ of modules of vector field systems then $\mathcal{P}_*(\sigma) \in \gamma_2(2, 2)$, the image under μ of the (2,2)-Darboux integrable linear differential operators in \mathcal{D} .

PROOF OF THEOREM 4.2: The proof proceeds as in [10] by finding a particular solution of the partial differential equations which define the equivalence map:

$$\mathcal{P} : \mathbb{R}^4 \times G \rightarrow \Delta_2(f),$$

such that

$$\mathcal{P}_*(\sigma) = {}_1\Omega^{(2)}(f) \oplus {}_2\Omega^{(2)}(f).$$

Such an equivalence implies the existence of a function $g : \mathbb{R}^4 \times G \rightarrow GL(4, \mathbb{R})$ such that

$$\mathcal{P}_* \begin{pmatrix} \mathbf{L} \\ \mathbf{R} \\ \frac{\partial}{\partial u_1} \\ \frac{\partial}{\partial v_1} \end{pmatrix} = g \begin{pmatrix} D_1^{(2)} \\ D_2^{(2)} \\ \frac{\partial}{\partial z_{11}} \\ \frac{\partial}{\partial z_{22}} \end{pmatrix},$$

where $D_1^{(2)}, D_2^{(2)}, \frac{\partial}{\partial z_{11}}$ and $\frac{\partial}{\partial z_{22}}$ are generators of ${}_1\Omega^{(2)}(f) \oplus {}_2\Omega^{(2)}(f)$ for some partial differential equation

$$\frac{\partial^2 y}{\partial x_1 \partial x_2} = f \left(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2} \right).$$

Since it is enough to find a particular solution of the overdetermined system of first order partial differential equations defining the equivalence map \mathcal{P} , we specialise g to lie in the subgroup of $GL(4, \mathbb{R})$ described by the matrices

$$g = \begin{pmatrix} \omega_{11} & 0 & \omega_{13} & \omega_{14} \\ 0 & \omega_{22} & \omega_{23} & \omega_{24} \\ 0 & 0 & \omega_{33} & \omega_{34} \\ 0 & 0 & \omega_{43} & \omega_{44} \end{pmatrix}.$$

This choice of subgroup is based purely on heuristic arguments. Hence,

$$\begin{aligned} \mathcal{P}_*(\mathbf{L}) &= \omega_{11} D_1^{(2)} + \omega_{13} \frac{\partial}{\partial z_{11}} + \omega_{14} \frac{\partial}{\partial z_{22}}, \quad \mathcal{P}_*(\mathbf{R}) = \omega_{22} D_2^{(2)} + \omega_{23} \frac{\partial}{\partial z_{11}} + \omega_{24} \frac{\partial}{\partial z_{22}}, \\ \mathcal{P}_* \left(\frac{\partial}{\partial u_1} \right) &= \omega_{33} \frac{\partial}{\partial z_{11}} + \omega_{34} \frac{\partial}{\partial z_{22}}, \quad \mathcal{P}_* \left(\frac{\partial}{\partial v_1} \right) = \omega_{43} \frac{\partial}{\partial z_{11}} + \omega_{44} \frac{\partial}{\partial z_{22}}. \end{aligned}$$

Note that $\Delta_2(f)$ has local coordinates $x_1, x_2, z, z_1, z_2, z_{11}, z_{22}$, that is, all the local coordinates of $J^2(\mathbb{R}^2, \mathbb{R})$ except for z_{12} since this coordinate is specified by the partial differential equation f .

The local coordinate expressions for $D_1^{(2)}$ and $D_2^{(2)}$ when combined with equations (4.2) give the following overdetermined system for \mathcal{P} :

$$(4.3)_1 \quad \mathbf{L}x_1 = \omega_{11}, \quad \frac{\partial x_1}{\partial u_1} = 0, \quad \frac{\partial x_1}{\partial v_1} = 0, \quad \mathbf{R}x_1 = 0,$$

$$(4.3)_2 \quad \mathbf{R}x_2 = \omega_{22}, \quad \frac{\partial x_2}{\partial u_1} = 0, \quad \frac{\partial x_2}{\partial v_1} = 0, \quad \mathbf{L}x_2 = 0,$$

$$(4.3)_3 \quad \mathbf{L}z = \omega_{11}z_1, \quad \frac{\partial z}{\partial u_1} = 0, \quad \frac{\partial z}{\partial v_1} = 0, \quad \mathbf{R}z = \omega_{22}z_2,$$

$$(4.3)_4 \quad \mathbf{L}z_1 = \omega_{11}z_{11}, \quad \mathbf{R}z_1 = \omega_{22}f, \quad \frac{\partial z_1}{\partial u_1} = 0, \quad \frac{\partial z_1}{\partial v_1} = 0,$$

$$(4.3)_5 \quad \mathbf{L}z_2 = \omega_{11}f, \quad \mathbf{R}z_2 = \omega_{22}z_{22}, \quad \frac{\partial z_2}{\partial u_1} = 0, \quad \frac{\partial z_2}{\partial v_1} = 0,$$

$$(4.3)_6 \quad \mathbf{L}z_{11} = \omega_{13}, \quad \mathbf{R}z_{11} = \omega_{23}, \quad \frac{\partial z_{11}}{\partial u_1} = 0, \quad \frac{\partial z_{11}}{\partial v_1} = 0,$$

$$(4.3)_7 \quad \mathbf{L}z_{22} = \omega_{14}, \quad \mathbf{R}z_{22} = \omega_{24}, \quad \frac{\partial z_{22}}{\partial u_1} = \omega_{34}, \quad \frac{\partial z_{22}}{\partial v_1} = \omega_{44}.$$

Suppose for the present that we can find x_1 and x_2 which satisfy (4.3)₁ and (4.3)₂. Let us consider the equation (4.3)₃. From the first two equations we have

$$(4.4) \quad \left[\frac{\partial}{\partial u_1}, \mathbf{L} \right] z = \frac{\partial}{\partial u_1}(\mathbf{L}z) - \mathbf{L} \left(\frac{\partial z}{\partial u_1} \right) = \frac{\partial}{\partial u_1}(\omega_{11}z_1) = \frac{\partial \omega_{11}}{\partial u_1} z_1 + \omega_{11} \frac{\partial z_1}{\partial u_1},$$

and from the second two equations we obtain

$$(4.5) \quad \left[\frac{\partial}{\partial v_1}, \mathbf{R} \right] z = \frac{\partial \omega_{22}}{\partial v_1} z_2 + \omega_{22} \frac{\partial z_2}{\partial v_1}.$$

But from (4.3)₁ and (4.3)₂ we find that

$$(4.6) \quad \frac{\partial \omega_{11}}{\partial u_1} = \left[\frac{\partial}{\partial u_1}, \mathbf{L} \right] x_1, \quad \frac{\partial \omega_{22}}{\partial v_1} = \left[\frac{\partial}{\partial v_1}, \mathbf{R} \right] x_2.$$

Suppose then that at this point, in addition to $\mathbf{R}x_1 = \mathbf{L}x_2 = 0$ we require that x_1 and x_2 be invariants of the vector field system

$$(4.7) \quad \left\{ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \left[\frac{\partial}{\partial u_1}, \mathbf{L} \right], \left[\frac{\partial}{\partial v_1}, \mathbf{R} \right] \right\},$$

such that $(\mathbf{L}x_1)(\mathbf{R}x_2) \neq 0$.

Matters being so, equations (4.4) and (4.5) become

$$(4.8) \quad \frac{\partial z_1}{\partial u_1} = (\omega_{11})^{-1} \left[\frac{\partial}{\partial u_1}, \mathbf{L} \right] z, \quad \frac{\partial z_2}{\partial v_1} = (\omega_{22})^{-1} \left[\frac{\partial}{\partial v_1}, \mathbf{R} \right] z.$$

In addition we have

$$(4.9) \quad \frac{\partial z_1}{\partial v_1} = (\omega_{11})^{-1} \left[\frac{\partial}{\partial v_1}, \mathbf{L} \right] z, \quad \frac{\partial z_2}{\partial u_1} = (\omega_{22})^{-1} \left[\frac{\partial}{\partial u_1}, \mathbf{R} \right].$$

Thus in order to satisfy the middle two equations of (4.3)₃, the third equation in (4.3)₄ and the last equation in (4.3)₅ we need to require that z be an invariant of (4.7). From (4.9) we see that $\frac{\partial z_1}{\partial u_1} = \frac{\partial z_2}{\partial v_1} = 0$ are satisfied by virtue of the fact that $\left[\frac{\partial}{\partial v_1}, \mathbf{L} \right] = \left[\frac{\partial}{\partial u_1}, \mathbf{R} \right] = 0$. This effectively solves the partial differential equations defining the equivalence map \mathcal{P} .

Summarising, we have shown that if x_1, x_2 and z are chosen to be any functionally independent invariants of

$$\left\{ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \left[\frac{\partial}{\partial u_1}, \mathbf{L} \right], \left[\frac{\partial}{\partial v_1}, \mathbf{R} \right] \right\},$$

such that $(\mathbf{L}x_1)(\mathbf{R}x_2) \neq 0$ and $\mathbf{R}x_1 = \mathbf{L}x_2 = 0$ then the remaining variables are obtained from the system (4.3)₃ to (4.3)₅ as in the statement of the theorem. In particular the partial differential equation f is defined by

$$f = \frac{\mathbf{L}z_2}{\mathbf{L}x_2} = \frac{\mathbf{R}z_1}{\mathbf{R}x_2},$$

and justifies our assertion that \mathcal{P} satisfies

$$\mathcal{P}_*(\sigma) = {}_1\Omega^{(2)}(f) \oplus {}_2\Omega^{(2)}(f),$$

for some partial differential equation

$$\frac{\partial^2 y}{\partial x_1 \partial x_2} = f \left(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial^2 y}{\partial x_1^2}, \frac{\partial^2 y}{\partial x_2^2} \right).$$

We must now show that if $\frac{\partial f}{\partial z_{11}} = \frac{\partial f}{\partial z_{22}} = 0$ then $\tilde{\mu}(f) \in E(2, 2)$. First of all we show that with \mathcal{P} defined as in the statement of the Theorem, then \mathcal{P}_* maps $\left\{ \mathbf{L}, \frac{\partial}{\partial u_1} \right\}$ to ${}_1\Omega^{(2)}(f)$ and $\left\{ \mathbf{R}, \frac{\partial}{\partial v_1} \right\}$ to ${}_2\Omega^{(2)}(f)$. To do this we need to first show that $\omega_{34} = \omega_{43} = 0$, since then we have

$$\mathcal{P}_* \left(\frac{\partial}{\partial u_1} \right) = \omega_{33} \frac{\partial}{\partial z_{11}}, \quad \mathcal{P}_* \left(\frac{\partial}{\partial v_1} \right) = \omega_{44} \frac{\partial}{\partial z_{22}}.$$

Now $\omega_{34} = \frac{\partial z_{22}}{\partial u_1}$ and from the equations $\mathbf{R}z_2 = \omega_{22}z_{22}$ and $\frac{\partial z_2}{\partial u_1} = 0$ we have

$$\frac{\partial}{\partial u_1}(\mathbf{R}z_2) - \mathbf{R} \frac{\partial z_2}{\partial u_1} = \left[\frac{\partial}{\partial u_1}, \mathbf{R} \right] z_2 = \frac{\partial \omega_{22}}{\partial u_1} z_{22} + \omega_{22} \frac{\partial z_{22}}{\partial u_1}.$$

However with \mathcal{P} defined as above we have $\frac{\partial \omega_{22}}{\partial u_1} = 0$, whence

$$\frac{\partial z_{22}}{\partial u_1} = \omega_{22}^{-1} \left[\frac{\partial}{\partial u_1}, \mathbf{R} \right] z_2.$$

Noting that $\left[\frac{\partial}{\partial u_1}, \mathbf{R} \right] = 0$ gives the required result. A similar argument shows that

$$\frac{\partial z_{11}}{\partial v_1} = \omega_{11}^{-1} \left[\frac{\partial}{\partial v_1}, \mathbf{L} \right] z_1,$$

and since $\left[\frac{\partial}{\partial v_1}, \mathbf{L} \right] = 0$ the result follows.

So far we have shown that

$$\begin{aligned} \mathcal{P}_* \left\{ \mathbf{L}, \frac{\partial}{\partial u_1} \right\} &= \left\{ D_1^{(2)} + \frac{\omega_{14}}{\omega_{11}} \frac{\partial}{\partial z_{22}}, \frac{\partial}{\partial z_{11}} \right\}, \\ \mathcal{P}_* \left\{ \mathbf{R}, \frac{\partial}{\partial v_1} \right\} &= \left\{ D_2^{(2)} + \frac{\omega_{23}}{\omega_{22}} \frac{\partial}{\partial z_{11}}, \frac{\partial}{\partial z_{22}} \right\}. \end{aligned}$$

Our next job is to show that

$$\frac{\omega_{14}}{\omega_{11}} = D_2^{(2)}(f) \quad \text{and} \quad \frac{\omega_{23}}{\omega_{22}} = D_1^{(2)}(f).$$

To do this we first generalise the well known fact that if $F = \{X_1, X_2, \dots, X_m\}$ is a $C^\infty(\mathbb{R}^n, \mathbb{R})$ module of vector fields of constant dimension on an open subset U of \mathbb{R}^n and if $\phi : U \rightarrow U$ is a local diffeomorphism then $\phi_*(\text{char } F) = \text{char}(\phi_*F)$. That is, characteristic submodules are mapped to characteristic submodules. Indeed for fixed $X \in F$, write

$$F_X^{(1)} = F \oplus [X, F].$$

We assume that here as elsewhere we are in the nonsingular case where the dimension of all vector field systems “derived” from F does not change on U . Following Stormark [9], we define a function

$$r_F : F \rightarrow \mathbb{R}$$

by

$$r_F(X) = \dim \left(F_X^{(1)} / F \right).$$

Let

$$r_1 = \max_{X \in F} r_F(X).$$

Call X *regular in F* if $r_F(X) = r_1$; *singular in F* if $r_F(X) < r_1$ and *characteristic in F* if $r_F(X) = 0$. The following is easy to establish.

LEMMA. r_F is a $\text{Diff}_{\text{loc}}(\mathbb{R}^n)$ invariant.

This result generalises the well known fact that if X is characteristic in F then ϕ_*X is characteristic in ϕ_*F . So if we give the number $r_F(X)$ the obvious name of the rank of X in F then we see that it is a $\text{Diff}_{\text{loc}}(\mathbb{R}^n)$ invariant so that X being characteristic corresponds to the special case $r_F(X) = 0$. More generally, regular vector fields are mapped to regular vector fields and singular to singular.

LEMMA.

The local diffeomorphism \mathcal{P} , when acting on $\left\{ \mathbf{L}, \frac{\partial}{\partial u_1} \right\}$ and $\left\{ \mathbf{R}, \frac{\partial}{\partial v_1} \right\}$ maps these to the Monge characteristic vector field systems. That is,

$$\mathcal{P}_* \left\{ \mathbf{L}, \frac{\partial}{\partial u_1} \right\} = {}_1\Omega^2(f),$$

and

$$\mathcal{P}_* \left\{ \mathbf{R}, \frac{\partial}{\partial v_1} \right\} = {}_2\Omega^2(f),$$

for some $f \in E(2,2)$.

PROOF: We prove the result for $\left\{ \mathbf{L}, \frac{\partial}{\partial u_1} \right\}$; the other being similar. Firstly, we note that

$$\sigma_{\mathbf{L}}^{(1)}/\sigma = \phi_\alpha L_\alpha.$$

Assuming that not all the ϕ_α are zero, we have $r_\sigma(\mathbf{L}) = 1$. Let us compute

$$\mathcal{H} = [\mathcal{P}_*\mathbf{L}, \mathcal{P}_*\sigma] / \mathcal{P}_*\sigma.$$

We have already shown that

$$\mathcal{P}_*\mathbf{L} = \omega_{11}D_1^{(2)} + \omega_{13}\frac{\partial}{\partial z_{11}} + \omega_{14}\frac{\partial}{\partial z_{14}},$$

and by construction that,

$$\mathcal{P}_*\sigma = \left\{ D_1^{(2)}, D_2^{(2)}, \frac{\partial}{\partial z_{11}}, \frac{\partial}{\partial z_{22}} \right\},$$

where,

$$D_1^{(2)} = \frac{\partial}{\partial x_1} + z_1 \frac{\partial}{\partial z} + z_{11} \frac{\partial}{\partial z_1} + f \frac{\partial}{\partial z_2},$$

$$D_2^{(2)} = \frac{\partial}{\partial x_2} + z_2 \frac{\partial}{\partial z} + f \frac{\partial}{\partial z_1} + z_{22} \frac{\partial}{\partial z_2},$$

are the total differential operators restricted to $\Delta_2(f)$.

Calculation then gives

$$\begin{aligned} [D_1^{(2)}, \mathcal{P}_*\mathbf{L}] &\equiv \omega_{14} \frac{\partial}{\partial z_1}, \quad \left[\frac{\partial}{\partial z_{11}}, \mathcal{P}_*\mathbf{L} \right] \equiv \omega_{11} \frac{\partial}{\partial z_1}, \\ [D_2^{(2)}, \mathcal{P}_*\mathbf{L}] &\equiv \omega_{11} \left((D_1^{(2)}f) \frac{\partial}{\partial z_1} - (D_2^{(2)}f) \frac{\partial}{\partial z_2} \right) + \omega_{14} \frac{\partial}{\partial z_2}, \quad \left[\frac{\partial}{\partial z_{22}}, \mathcal{P}_*\mathbf{L} \right] \equiv 0. \end{aligned}$$

Hence

$$\mathcal{H} = \left\{ \frac{\partial}{\partial z_1}, \left(\omega_{14} - \omega_{11} D_2^{(2)}f \right) \frac{\partial}{\partial z_2} \right\}.$$

Now by the previous lemma, $r_{\mathcal{P}_*\sigma}(\mathcal{P}_*\mathbf{L}) = 1$ since \mathcal{P} is a local diffeomorphism. Hence \mathcal{H} must have dimension one. This in turn forces $\omega_{14} - \omega_{11} D_2^{(2)}f = 0$.

Now, because

$$\mathcal{P}_* \left(\frac{\partial}{\partial u_1} \right) = \omega_{33} \frac{\partial}{\partial z_{11}},$$

we have shown, upon comparing with Definition 3.1, that

$$\mathcal{P}_* \left\{ \mathbf{L}, \frac{\partial}{\partial u_1} \right\} = {}_1\Omega^{(2)}(f).$$

A similar argument shows that

$$\mathcal{P}_* \left\{ \mathbf{R}, \frac{\partial}{\partial v_1} \right\} = {}_2\Omega^{(2)}(f).$$

Finally, we note that $\left\{ \mathbf{L}, \frac{\partial}{\partial u_1} \right\}$ has two functionally independent invariants (v, v_1) while $\left\{ \mathbf{R}, \frac{\partial}{\partial v_1} \right\}$ has two functionally independent invariants (u, u_1) . It follows that their diffeomorphic images, ${}_1\Omega^{(2)}(f)$ and ${}_2\Omega^{(2)}(f)$, each have two functionally independent invariants. This proves that $f \in E(2, 2)$ and completes the proof of the lemma and the theorem. □

We are now in a position to solve the A_2 Toda field theory by making use of Theorem 4.2 and specialising the Lie group G that appears there to be the vector space \mathbb{R}^3 . That is, we study the collection of vector field systems $\Sigma(\phi_\alpha; \psi_\alpha; \mathbb{R}^3)$. Explicitly these vector field systems have the form

$$\begin{aligned} \sigma = \left\{ \frac{\partial}{\partial u} + u_1 \left(\phi_1(u) \frac{\partial}{\partial w_1} + \phi_2(u) \frac{\partial}{\partial w_2} + \phi_3(u) \frac{\partial}{\partial w_3} \right), \frac{\partial}{\partial u_1}, \right. \\ \left. \frac{\partial}{\partial v} + v_1 \left(\psi_1(v) \frac{\partial}{\partial w_1} + \psi_2(v) \frac{\partial}{\partial w_2} + \psi_3(v) \frac{\partial}{\partial w_3} \right), \frac{\partial}{\partial v_1} \right\}, \end{aligned}$$

where w_1, w_2 and w_3 are coordinates on \mathbb{R}^3 . Now σ is defined up to local diffeomorphisms so we may use this freedom to simplify σ . Indeed, by re-parametrising u and v we can transform ϕ_1 and ψ_1 (say) to 1. This means that we can without loss of generality study the family of modules $\Sigma(1, \phi_2, \phi_3; 1, \psi_2, \psi_3; \mathbb{R}^3)$. Now it transpires that the condition necessary and sufficient in order that the submodules $\left\{ \mathbb{L}, \frac{\partial}{\partial u_1} \right\}$ and $\left\{ \mathbb{R}, \frac{\partial}{\partial v_1} \right\}$ each have two functionally independent invariants is that the Wronskians of $(1, \phi_2, \phi_3)$ and $(1, \psi_2, \psi_3)$ are each not identically zero. That is, the ϕ 's and ψ 's must be linearly independent.

Matters being so, application of Theorem 4.2 suggests that we compute the invariants of the submodule \mathcal{A} of $\sigma^{(1)} = \sigma \oplus [\sigma, \sigma]$. In this case \mathcal{A} is given by

$$\left\{ \frac{\partial}{\partial w_1} + \phi_2 \frac{\partial}{\partial w_2} + \phi_3 \frac{\partial}{\partial w_3}, \frac{\partial}{\partial w_1} + \psi_2 \frac{\partial}{\partial w_2} + \psi_3 \frac{\partial}{\partial w_3}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1} \right\}.$$

Under the stated nondegeneracy conditions on the ϕ 's and ψ 's, \mathcal{A} will have three functionally independent invariants which are easily computed:

$$(4.10) \quad x_1 = u, \quad x_2 = v, \quad z = w_1\theta_1 + w_2\theta_2 + w_3\theta_3,$$

where
$$\theta_1 = \phi_2\psi_3 - \phi_3\psi_2, \quad \theta_2 = \phi_3 - \psi_3, \quad \theta_3 = \psi_2 - \phi_3.$$

These invariants satisfy the conditions of Theorem 4.2 and define a local diffeomorphism, as may easily be checked directly. Now in order to construct the partial differential equations associated with any $\sigma \in \Sigma(1, \phi_2, \phi_3; 1, \psi_2, \psi_3; \mathbb{R}^3)$ we need to compute z_1, z_2 and z_{12} . By applying Theorem 4.2 once again we have (only by carrying out differentiations)

$$(4.11) \quad z_1 = \sum_{\alpha=1}^3 w_\alpha \frac{\partial \theta_\alpha}{\partial u}, \quad z_2 = \sum_{\alpha=1}^3 w_\alpha \frac{\partial \theta_\alpha}{\partial v},$$

$$(4.12) \quad z_{12} = \sum_{\alpha=1}^3 w_\alpha \frac{\partial^2 \theta_\alpha}{\partial u \partial v} = w_1 \frac{\partial^2 \theta_1}{\partial u \partial v}.$$

We are therefore led naturally to express w_1 as a function of the local co-ordinates on $\Delta_2(f)$. Combining (4.10) with (4.11) we find that w_1, w_2 and w_3 satisfy the linear algebraic system

$$(4.13) \quad \begin{aligned} z &= w_1\theta_1 + w_2\theta_2 + w_3\theta_3, \\ z_1 &= w_1 \frac{\partial \theta_1}{\partial u} + w_2 \frac{\partial \theta_2}{\partial u} + w_3 \frac{\partial \theta_3}{\partial u}, \\ z_2 &= w_1 \frac{\partial \theta_1}{\partial v} + w_2 \frac{\partial \theta_2}{\partial v} + w_3 \frac{\partial \theta_3}{\partial v}. \end{aligned}$$

Because of the form of the ϕ 's and ψ 's the above system can always be solved to express w_1, w_2 and w_3 as linear combinations of z, z_1 and z_2 . Suppose we write the solution w_1 of (4.13) in the form

$$w_1 = A(u, v)z + B(u, v)z_1 + C(u, v)z_2.$$

This then gives $\Delta_2(f)$ as a submanifold of $J^2(\mathbb{R}^2, \mathbb{R})$ defined by

$$z_{12} = \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} (A(x_1, x_2)z + B(x_1, x_2)z_1 + C(x_1, x_2)z_2),$$

and shows that the class of partial differential equations associated with $\Sigma(1, \phi_2, \phi_3; 1, \psi_2, \psi_3; \mathbb{R}^3)$ is parametrised by

$$\frac{\partial^2 y}{\partial x_1 \partial x_2} = \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} \left(A(x_1, x_2)y + B(x_1, x_2) \frac{\partial y}{\partial x_1} + C(x_1, x_2) \frac{\partial y}{\partial x_2} \right), \tag{4.14}$$

where $\theta_1, A, B,$ and C are defined as above. Since the partial differential equation (4.14) is independent of the higher order derivatives $\frac{\partial^2 y}{\partial x_1^2}$ and $\frac{\partial^2 y}{\partial x_2^2}$ it follows once again by Theorem 4.2 that the class of partial differential equations defined by (4.14) satisfies the condition that each member is (2,2)-Darboux integrable. In other words, we have explicitly constructed Vessiot's class of second order linear differential operators $\mu^{-1}(\gamma_2(2, 2))$, namely,

$$L = \partial_{x_1 x_2}^2 + \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} (B \partial_{x_1} + C \partial_{x_2} + A)$$

and hence $h(L)$ and $k(L)$ are by Theorem 3.4 solutions of equations (3.10). We thus obtain solutions of the Toda field equations containing four arbitrary linearly independent, twice differentiable functions each of one variable in the case when $h(L)k(L) > 0$ and of equations (3.12) when $h(L)k(L) < 0$.

EXAMPLE. We end by giving an elementary example to illustrate the theory. We construct the partial differential equation associated with the vector field system $\Sigma(1, u, u^{-1}; 1, v, v^{-1}; \mathbb{R}^3)$. In this case $\theta_1 = u/v - v/u, \theta_2 = 1/u - 1/v, \theta_3 = v - u$ and

we find that solving the linear algebraic system (4.13) leads to

$$w_1 = \frac{(uv)^2(u^2 - v^2)z - u^3v^2(u - v)^2z_1 + u^2v^3(u - v)^2z_2}{-uv(u - v)^4}.$$

Then from

$$z_{12} = \frac{\partial^2 \theta_1}{\partial u \partial v} w_1,$$

one obtains the associated partial differential equation in the form

$$(4.15) \quad \frac{\partial^2 y}{\partial x_1 \partial x_2} + \frac{(x_1 + x_2)}{x_2(x_1 - x_2)} \frac{\partial y}{\partial x_1} - \frac{(x_1 + x_2)}{x_1(x_1 - x_2)} \frac{\partial y}{\partial x_2} - \frac{(x_1 + x_2)^2}{x_1 x_2 (x_1 - x_2)^2} y = 0.$$

The Laplace invariants of the operator defined by (4.15) are

$$(4.16) \quad \begin{aligned} h(L) &= \frac{\partial}{\partial x_1} \left(\frac{x_1 + x_2}{x_2(x_1 - x_2)} \right) = \frac{-2}{(x_1 - x_2)^2}, \\ k(L) &= -\frac{\partial}{\partial x_2} \left(\frac{x_1 + x_2}{x_1(x_1 - x_2)} \right) = \frac{-2}{(x_1 - x_2)^2}. \end{aligned}$$

Here we are in the “equal invariants” case because of the symmetric form we chose for the ϕ 's and ψ 's. In this case the Toda equations degenerate to the Liouville equation for which (4.16) defines a solution. Obviously the choice of “nonsymmetric” forms for the ϕ 's and ψ 's lead to other solutions of the Toda equations which are not solutions of the Liouville equation.

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