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On semisubtractive halfrings

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Analogues of Artin-Wedderburn and Goldie structure theorems are obtained for a class of halfrings which includes the semisubtractive ones. In the semisubtractive case, precise results are obtained which show that non-ring examples of these structures are relatively limited.

1. Introduction

In recent articles Dulin [2] and Mosher [3, 4] have studied the structure of hemirings under a strong hypothesis called *semisubtractivity*: for every a, b in H at least one of the equations $a + x = b$, $a = b + x$ is solvable in H. Several unpublished doctoral theses known to the authors have also been based on this condition.

The purpose of this note is two-fold. First, we show that a natural generalization of semisubtractivity allows us to obtain analogues of the classical structural results of ring theory with very little effort. Secondly, in the presence of semisubtractivity we obtain structure theorems so tight that they are essentially negative in import: there are too few semisubtractive halfrings other than rings to justify the study of the class.

Let us review pertinent definitions and establish notation. A *hemiring* is a triple $(H, +, \cdot)$ where $(H, +)$ is a commutative semigroup with an identity 0 which is the zero of the semigroup $(H, \cdot)$ whose multiplication distributes over the addition. A *right ideal* I of H is

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a subsemiring such that $HI \subseteq I$. An additive subsemigroup S is *subtractive* if $s + h \in S$ implies $h \in S$ for all $s \in S$, all $h \in H$; the kernels of homomorphisms are the subtractive two-sided ideals. The set $H^* = \{h \in H : 0 \in h+H\}$ of all additively invertible elements of H is a subtractive ideal of H . A subsemiring S is *differential* in H if for each $h \in H$ there exist $s, t \in S$ with $s + h = t$; in other words, the subtractive subsemiring generated by S is all of H . If H is differential in a ring, it is a *halfring*; a necessary and sufficient condition for this is that the cancellation law hold for addition, in which case there is a ring of differences $\overline{H}$ determined uniquely up to isomorphism. Stone [9] defined the *right ideal type* of a subtractive right ideal K to be the collection of all ring right ideals I of $\overline{H}$ such that $I \cap H = K$. If this collection consists of $\overline{K}$ only, we say that K is *right monotypic* in H or H is *right monotypic* at K . A halfring H is *right monotypic* if it is right monotypic at each subtractive right ideal.

Stone [9] observed that in a semisubtractive halfring, every subtractive subsemigroup of $(H, +)$ is "subsemigroup monotypic". Hence the class of right monotypic halfrings includes the semisubtractive halfrings. The following example shows that the inclusion is proper.

EXAMPLE A. Let F be the halfring of non-negative rationals, and let $H = \{0\} \cup \{r \in F : r \geq 2\}$. Then H is trivially right monotypic, but is not semisubtractive.

2. Artinian halfrings

Although the initial definitions are more general, it is easy to see that the semirings for which an Artin-Wedderburn Theorem is obtained in [4] are semisubtractive halfrings. For right monotypic halfrings H , it is immediate that chain conditions on the subtractive right ideals of H go over into chain conditions in $\overline{H}$. Although it is possible to proceed by considering the radical in H , we prefer to go directly to a characterization based on complete reducibility. As in ring theory, we say that H is a *direct sum* of subtractive right ideals S and T if $S + T = H$ and $S \cap T = 0$.

THEOREM 1. *For a halfring H , the following are equivalent:*

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(a) *every subtractive right ideal of* H *is a summand, and* H *is right monotypic at* 0 ;

(b) H *is right monotypic and* $\overline{H}$ *is a semisimple artinian ring.*

Proof. It is clear that *(b)* implies *(a)*. Let K and C be subtractive right ideals of H such that $K \cap C = 0$, $K + C = H$. Then $\overline{K} + \overline{C} = \overline{H}$, and $(\overline{K} \cap \overline{C}) \cap H = K \cap C = 0$. Since H is right monotypic at 0, $\overline{K} \cap \overline{C} = 0$ and $\overline{H}$ is the direct sum of $\overline{K}$ and $\overline{C}$. Now let I be a right ideal of $\overline{H}$ with $I \cap H = K$. Then $\overline{K} \subseteq I$. Now $(I \cap \overline{C}) \cap H = K \cap C = 0$, whence $I \cap \overline{C} = 0$. For $i \in I$ there are $c_1, c_2 \in C$ and $k_1, k_2 \in K$ with $i = (k_2+c_2) - (k_1+c_1)$; hence $c_2 - c_1 = i - (k_2-k_1) \in I \cap \overline{C} = 0$. Therefore $i \in \overline{K}$, so that $I = \overline{K}$ and K is right monotypic in H. It now follows that every right ideal of $\overline{H}$ is a summand, whence $\overline{H}$ is semisimple artinian.

We now know that $\overline{H}$ is a direct sum of matrix rings over (skew) fields, and is a direct sum of minimal right ideals. Since H is right monotypic, each minimal right ideal has nonzero intersection with H. Nevertheless, H need not be the direct sum of these intersections, and even if $\overline{H}$ is a matrix ring it does not follow that H is a matrix halfring over some halfring P. We present a simple example of this. The notation $M_n(H)$ refers to the halfring of $n \times n$ matrices with entries from H.

EXAMPLE B. Let H be the halfring of Example A, and let S be the direct sum of $M_2(H)$ with itself, together with the elements (Z, I), (I, I), and (I, Z), where Z and I are the zero and identity of $M_2(\overline{H})$ respectively. Then S is a halfring, $\overline{S}$ is the direct sum of $M_2(\overline{H})$ with itself, and S is not the direct sum of its intersections with the two-sided or one-sided ideals of $\overline{S}$. However $\overline{S}$ is not right monotypic.

We do not know what happens if such a halfring is right monotypic. In case the halfring is a direct sum or is a matrix halfring, we can analyze the situation more closely. A halfring H is *unital* if $\overline{H}$ has an identity.

THEOREM 2. *Let* H *be a unital halfring which is a complete direct*

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sum of a family of ideals $\{H_\alpha : \alpha \in A\}$. *Then* H *is right monotypic if and only if each* H_α *is right monotypic.*

Proof. Clearly $\overline{H} = \sum \overline{H}_\alpha$, and if each H_α is right monotypic the sum is direct. Let I be a right ideal of $\overline{H}$ with $I \cap H = K$. Now each $\overline{H}_\alpha$ has a two-sided identity 1_α , and $H1_\alpha$ is isomorphic to H_α . Since $K1_\alpha$ is a subtractive right ideal of $H1_\alpha$ and $I1_\alpha$ is a right ideal of $\overline{H}_\alpha$ such that $(I1_\alpha) \cap (H1_\alpha) = K1_\alpha$, by hypothesis $I1_\alpha = (K1_\alpha)^-$. Hence $I = \sum (I1_\alpha) = \sum (K1_\alpha)^- = \overline{K}$, and H is right monotypic. Conversely, if H is right monotypic, it is again a direct sum of the $\overline{H}_\alpha$. Since each subtractive right ideal of H_α is a subtractive right ideal of H , it follows immediately that each H_α is right monotypic.

THEOREM 3. *Let* $M_n(H)$ *be the halfring of all* $n \times n$ *matrices with entries in halfring* H . *If* $M_n(H)$ *is right monotypic, then either* H *is a ring, or* H *is right monotypic and* $n = 1$.

Proof. Suppose that $H^* \neq H$ and $n > 1$. Let

$$K = \{a \in M_n(\overline{H}) : a(i, 1) = -a(i, 2),\ a(i, j) = 0 \text{ for } 2 < j \leq n\} .$$

Then K is a right ideal of $M_n(\overline{H}) = M_n(H)^-$, and

$$K \cap M_n(H) = M_n(H^*) = M_n(H^*)^- .$$

Hence $M_n(H^*)$ is not right monotypic in $M_n(H)$.

3. Goldie halfrings

The theory here is unsatisfactory for right monotypic halfrings, although a complete theory is available in the semisubtractive case. The following result is immediate.

THEOREM 4. *Let* H *be a right monotypic right Goldie halfring. Then* $\overline{H}$ *is a right Goldie ring, and* $\overline{H}$ *is semiprime if and only if* H *is semiprime.*

Then $\overline{H}$ has a left quotient ring $Q(\overline{H})$ which is semisimple artinian. We refer to David A. Smith [5] for an exposition of the classical quotient construction with special attention to semirings. A *left divisor set* is a multiplicatively closed subset of multiplicatively cancellable elements satisfying the common left multiple property: for each $h \in H$, each $d \in D$, $Dh = Hd$. We denote by $Q_D(H)$ the left quotient hemiring of H by D.

THEOREM 5. *If H is a halfring, then $Q_D(H)$ is a halfring, and $Q_D(H)^- = Q_D(\overline{H})$. If K is a subtractive right ideal of H, then $D^{-1}K = \{d^{-1}k : d \in D, k \in K\}$ is the subtractive right ideal of $Q_D(H)$ generated by K, and if K is right monotypic in H then $D^{-1}K$ is right monotypic in $Q_D(H)$. If $H_D = \{h \in H : Dh \cap H^* \neq \emptyset\}$, then $Q_D(H)^* = D^{-1}H_D$.*

Proof. Most of the proof is straightforward. Suppose that I is a right ideal in $Q_D(H)^-$ such that $I \cap H = K$. Then $I \cap Q_D(H) = D^{-1}[I \cap Q_D(H) \cap H] = D^{-1}K$. Since $Q_D(H)^- = Q_D(\overline{H})$, $I = D^{-1}[I \cap \overline{H}]$. But $(I \cap \overline{H}) \cap H = I \cap H = K$, and since K is right monotypic in H, $I \cap \overline{H} = \overline{K}$. Then $I = D^{-1}\overline{K} = (D^{-1}K)^- = [I \cap Q_D(H)]^-$, so that $D^{-1}K$ is right monotypic in $Q_D(K)$.

The obstacle which prevents nicer results here is that $Q(H)^-$ is in general a proper subring of $Q(\overline{H})$, due to the fact that $\overline{H}$ has many cancellable elements not contained in H, and $Q(H)$ is $Q_D(H)$ for the particular left divisor set D consisting of all cancellable elements of H. In general, $\overline{H}$ is a subring of $Q(H)^-$, which is a subring of $Q(\overline{H})$, when all these exist. We give an example.

EXAMPLE C. Let H be the halfring of polynomials with nonnegative integer coefficients. Then $Q(H)^-$ is a proper subring of $Q(\overline{H})$, since $Q(\overline{H})$ is a field, while the polynomial $(x-1)$ in $Q(H)^-$ has no inverse there.

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As Theorem 5 indicates, $Q_D(H)$ may be a ring even though H is not a ring. This exceptionally intimate relationship of H with a ring is an important constraint, but for our purposes its appearance is understood as evidence that the class of halfrings under consideration is not much larger than the rings.

EXAMPLE D. Let H be the halfring of polynomials with integer coefficients, but with nonnegative constant term. Then $H^* = Hx \neq H$, but $Q(H)^* = Q(H) = Q(\overline{H})$.

Note that in this instance we have a semisubtractive halfring. It is this which forces $Q(H) = Q(\overline{H})$. We will conclude our work in this section by relating the non-ring parts of H and $Q(H)$.

THEOREM 6. *Let D be a left divisor set in a halfring H. Then $Q_D(H)/Q_D(H)^*$ contains a semi-isomorphic image of H/H_D. In case H_D is monotypic in H, this is an isomorphic image.*

Proof. From Theorem 5, $H_D = H \cap Q_D(H)^*$. Both assertions then follow from Stone [9], Theorem 4.

4. Semisubtractive halfrings

We now consider the refinements of the preceding results which are available in the semisubtractive case. These refinements rest on the following simple theorem.

THEOREM 7. *Let H be a semisubtractive hemiring, and let A and B be subsemirings such that $AB \subseteq H^*$. Then either $A^2 \subseteq H^*$ or $B^2 \subseteq H^*$.*

Proof. Suppose $A^2 \nsubseteq H^*$. Then there are $a, a' \in A$ with $a', a \notin H^*$. Let $b, b' \in B$. If $a + h = b$ for some $h \in H$, then $a'a + a'h = a'b \in AB \subseteq H^*$, so that $a'a \in H^*$ contrary to assumption. Since H is semisubtractive, $a = b + h$ for some $h \in H$. Then $bb' + hb' = ab' \in AB \subseteq H^*$, so that $bb' \in H^*$. It follows that $B^2 \subseteq H^*$.

We turn immediately to the artinian case. A halfring H is called a *halffield* if $\overline{H}$ is a division ring; Example 1 is a commutative halffield. This result should be compared with that of Wiegandt [10] on semirings which are completely reducible with respect to (not necessarily

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subtractive) right ideals.

THEOREM 8. *Let H be a completely reducible semisubtractive halfring. Then H is the direct sum of a semisimple artinian ring and possibly a single division halffield.*

Proof. By Theorem 1, $\overline{H}$ is semisimple artinian, a direct sum of simple ideals. By Theorem 7, at most one of these is not a ring. By the semisubtractivity, every idempotent of $\overline{H}$ belongs to H, so that H is a direct sum of minimal subtractive right ideals. Thus without loss of generality, $H^* = 0$ and $\overline{H}$ is a matrix ring. Let $a, b \in H$ belong to distinct minimal right ideals. If $a + h = b$ and if h' is the component of h in the same ideal with a, then $a + h' = 0$ by the uniqueness of the representation. Thus $a = h' = 0$. If $a = b + h$, a similar argument shows $b = 0$; hence H is a halffield. It is immediate from semi-subtractivity that H is a division halffield.

In the semisubtractive case we obtain a very precise Goldie Theorem, which describes the structure without recourse to constructing the ring of differences. However, the semirings of this class are still very tightly tied to rings.

THEOREM 9. *Let H be a semisubtractive semiprime right Goldie halfring. Then the set D of all cancellable elements of H is a left divisor set, and $Q(H)$ is the direct sum of $D^{-1}K$ and $Q(H/K)$, where K is an ideal with $H^* \subseteq K \subseteq H$ such that $D^{-1}K$ is a semisimple artinian ring and $Q(H/K)$ is a division halffield.*

Proof. By Theorem 4, $\overline{H}$ is semiprime right Goldie. The cancellable elements of $\overline{H}$ are $\{\pm d : d \in D\}$; it follows that D is a left divisor set in H and $Q(H) = Q_D(H) = Q_D(\overline{H}) = Q(\overline{H})$. It is easy to verify that $Q(H)$ is also semisubtractive; hence, by Theorem 8, $Q(H)$ is the direct sum of $Q(H)^*$ and a division halffield P. Let $K = H \cap Q(H)^*$. By Theorem 5, $D^{-1}K = Q(H)^*$ and by Theorem 6, $Q(H)/Q(H)^* = P$ contains an isomorphic copy of H/K. Evidently $P = Q(H/K)$.

The import of these results is that there is not sufficient scope to justify confining attention to the class of semisubtractive halfrings. The constraints of the class become an advantage, however, when they are used to represent larger classes. For example, F.A. Smith [6, 7, 8] has shown that an interesting class of hemirings can be represented as subdirect sums of semisubtractive hemirings. In results like this, semisubtractivity and similar ideas may play an important role in semiring theory.

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