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Number of Jordan blocks of the maximal size for local monodromies

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Dedicated to Professor Joseph Steenbrink

Abstract

We prove formulas for the number of Jordan blocks of the maximal size for local monodromies of one-parameter degenerations of complex algebraic varieties where the bound of the size comes from the monodromy theorem. In the case when the general fibers are smooth and compact, the proof calculates some part of the weight spectral sequence of the limit mixed Hodge structure of Steenbrink. In the singular case, we can prove a similar formula for the monodromy on the cohomology with compact supports, but not on the usual cohomology. We also show that the number can really depend on the position of singular points in the embedded resolution, even in the isolated singularity case, and hence there are no simple combinatorial formulas using the embedded resolution in general.

Introduction

Let $f : X \to \Delta$ be a proper surjective morphism of a connected complex manifold X to an open disk Δ, which is smooth over Δ^*. Assume that there is a proper surjective morphism from a Kähler manifold to X. We may assume for simplicity that f is a projective morphism. For a divisor D on X, set

$$U := X \backslash D, \quad f_U := f|_U : U \to \Delta, \quad U_t := f_U^{-1}(t), \quad X_t := f^{-1}(t).$$

Shrinking Δ if necessary, we may assume that the $H^j(U_t, \mathbf{Q})(t \in \Delta^*)$ form local systems and, moreover, $H^j(U_t, \mathbf{Q}) = (R^j(f_U)_* \mathbf{Q}_U)_t (t \in \Delta^*)$, for any j. We are interested in their monodromy around the origin. So we may assume that $X_0 \cup D$ is a divisor with simple normal crossings, and all irreducible components D_k of D are dominant over Δ. Then, shrinking Δ if necessary, we may assume, moreover, that any D_k and any intersections of D_k are smooth over Δ^*. Let Y_i be the irreducible components of $Y := X_0 \subset X$, with m_i the multiplicity of Y at the generic point of Y_i. Set $Y_I := \bigcap_{i \in I} Y_i$.

Set $J(\lambda) := \{i \mid \lambda^{m_i} = 1\}$ for a root of unity λ in $\mathbf{C}^*$. For $I \subset J(\lambda)$, let $Y_I^{(\lambda)} \subset Y_I$ be the union of the connected components of Y_I which do not intersect $Y_{i'}$ for any $i' \notin J(\lambda)$. Note that $Y_I^{(1)} = Y_I$ for $\lambda = 1$. We have the complex $C_{f,\lambda}^{\bullet}$, defined by

$$C_{f,\lambda}^j := \bigoplus_{I \subset J(\lambda), |I| = j+1} H^0(Y_I^{(\lambda)}, \mathbf{C}),$$

where the differential is induced by the Cech restriction morphism as in [Ste76]. Similarly, we have $Y_{k,I}$, $Y_{k,I}^{(\lambda)}$, $C_{f_k,\lambda}^{\bullet}$ for each k by replacing $f : X \to \Delta$ with $f_k := f|_{D_k} : D_k \to \Delta$ and Y_i with $Y_{k,i} := D_k \cap Y_i$. There are canonical restriction morphisms

$$r_k : C_{f,\lambda}^{\bullet} \to C_{f_k,\lambda}^{\bullet}.$$

Set $n := \dim X - 1$. Let $\nu_{f_U,\lambda}^j$ (respectively, $\nu_{c,f_U,\lambda}^j$) denote the number of Jordan blocks of the theoretically maximal size $j + 1$ for the eigenvalue λ of the monodromy on $H^j(U_t, \mathbf{C})$ (respectively, $H_c^j(U_t, \mathbf{C})$) for $t \in \Delta^*$ and $j \in [0, n]$. Here, the upper bounds come from the monodromy theorem. This upper bound is $2n - j$ for $j > n$, and the numbers of Jordan blocks of the maximal size for the eigenvalue λ of the monodromy on $H^j(U_t, \mathbf{C})$ and $H_c^j(U_t, \mathbf{C})$ are given, respectively, by $\nu_{c,f_U,\overline{\lambda}}^{2n-j}$ and $\nu_{f_U,\overline{\lambda}}^{2n-j}$ for $j \in [n, 2n]$ by duality. Thus it is enough to consider $\nu_{f_U,\lambda}^j$, $\nu_{c,f_U,\lambda}^j$ for $j \in [0, n]$ in the smooth case. We have the following question.

Question 1. Do the following equalities hold for $j \in [0, n]$?

$$\nu_{f_U,\lambda}^j = \dim H^j C_{f,\lambda}^{\bullet}, \quad \nu_{c,f_U,\lambda}^j = \dim \operatorname{Ker}\left(H^j C_{f,\lambda}^{\bullet} \to \bigoplus_k H^j C_{f_k,\lambda}^{\bullet} \right).$$

These equalities follow from the theory of limit mixed Hodge structures [Ste76, SZ85] if $\lambda = 1$, and they were also expected to hold for $\lambda \neq 1$ if f is obtained by a desingularization of a good compactification of a germ of a holomorphic function g_0 with an isolated singularity, as in Theorem 3 below, where $D = \emptyset$; i.e. $f_U = f$. In fact, we can prove the equality $\nu_{f,\lambda}^n = \nu_{g_0,\lambda}^n = (-1)^n \chi(C_{f,\lambda}^{\bullet})$ for $\lambda \neq 1$ as in Theorem 3 below, for instance, in the case of super-isolated singularities [Lue87] or, more generally, Yomdin singularities [Yom75] with $n = 2$: see Proposition 3.8 below (and also [Art94a, Mar12]). However, it turns out that the answer to Question 1 is negative, and there is no simple combinatorial formula in general, since we have quite recently found the following.

THEOREM 1. *The $\nu_{f,\lambda}^j$ cannot be determined only by the combinatorial data of the embedded resolution, and may really depend on the position of the singular points in the embedded resolution, even in the case when f is obtained by a desingularization of a good compactification of a germ of a holomorphic function with an isolated singularity.*

Here, a good compactification means a compactification having only one singular point, as constructed in [Bri70], and the combinatorial data of an embedded resolution in this paper mean the intersection lattice consisting of the connected components of the $Y_I^{(\lambda)}$ with λ fixed (see also [Art94a]). Theorem 1 will be shown in § 4.3 below. In Theorem 3, it will be shown that the $\nu_{f,\lambda}^j$ are determined by the dimensions of the $C_{f,\lambda}^j$ (i.e. by the numbers of the connected components of the $Y_I^{(\lambda)}$ with $|I| = j + 1$) in the case of a desingularization of a good compactification of a germ of a holomorphic function with an isolated singularity, provided that $B_{f,\lambda}^j = C_{f,\lambda}^j$ for any j in the notation of Theorem 2 below. A sufficient condition for the last equality is given in Theorem 4(i). Note that Theorem 1 is related to certain earlier work in the literature, such as [Art94a, Art94b, AC98, GN96, GLM97, Lib82, Mel00, Zar29], etc.

We now explain an improvement of the above formula in Question 1. Let $H^j(U_\infty)_\lambda$ (respectively, $H_c^j(U_\infty)_\lambda$) denote the λ-eigenspace of the limit mixed Hodge structure with a $\mathbf{C}$-coefficient. Let W be the weight filtration of the limit mixed Hodge structure. We have the following.

Theorem 2. *There are complexes $B^\bullet_{f,\lambda}$, $B^\bullet_{f_k,\lambda}$ and morphisms $r'_k : B^\bullet_{f,\lambda} \to B^\bullet_{f_k,\lambda}$ with $B^j_{f,\lambda}$, $B^j_{f_k,\lambda}$ direct factors of $C^j_{f,\lambda}$, $C^j_{f_k,\lambda}$ respectively, and such that we have for $j \in [0, n]$*

$$\operatorname{Gr}^W_0 H^j(U_\infty)_\lambda = H^j B^\bullet_{f,\lambda}, \quad \operatorname{Gr}^W_0 H^j_c(U_\infty)_\lambda = \operatorname{Ker}\left(H^j B^\bullet_{f,\lambda} \to \bigoplus_k H^j B^\bullet_{f_k,\lambda} \right), \qquad (0.1)$$

$$\nu^j_{f_U,\lambda} = \dim H^j B^\bullet_{f,\lambda}, \quad \nu^j_{c,f_U,\lambda} = \dim \operatorname{Ker}\left(H^j B^\bullet_{f,\lambda} \to \bigoplus_k H^j B^\bullet_{f_k,\lambda} \right). \qquad (0.2)$$

The differentials of $B^\bullet_{f,\lambda}$, $B^\bullet_{f_k,\lambda}$ are induced by the Cech restriction morphisms up to some nonzero constant multiples which may depend on each inclusion of connected components with codimension 1. We have $B^\bullet_{f,1} = C^\bullet_{f,1}$ and $B^\bullet_{f_k,1} = C^\bullet_{f_k,1}$ if $\lambda = 1$.

Here, the problem is the global triviality of the local systems $L_{\lambda,I}$ of rank 1 in (1.1.4) below, which are associated with the nearby cycles, and we get $B^j_{f,\lambda}$ by replacing $Y^{(\lambda)}_I$ in the definition of $C^j_{f,\lambda}$ with a union of the connected components of $Y^{(\lambda)}_I$ on which $L_{\lambda,I}$ is trivial, and choosing a trivialization of $L_{\lambda,I}$ (and similarly for $B^j_{f_k,\lambda}$).

In the proper case (i.e. $D = \emptyset$), Theorem 2 follows from Steenbrink's construction of the limit mixed Hodge structures using V-manifolds [Ste77] together with the theory of bigraded modules of Lefschetz-type [Sai88, § 4] (see also [GN90]). Here, we do not need [Sai88, 4.2.3.1] (i.e. [SZ90, 1.3.8]), since we use the lowest-weight part of the E_1-complex, where only the Cech restriction morphisms appear. The nonproper case then follows by using the limit of weight spectral sequences in [SZ85]. In Theorem 2.2 below, Theorem 2 for $\nu^j_{c,f_U,\lambda}$ will be generalized to the singular case although the assertion for $\nu^j_{f_U,\lambda}$ cannot be extended (see Example 2.3 below).

It is not easy to determine the differential of the complex $B^\bullet_{f,\lambda}$ for $\lambda \neq 1$ in general. This problem can be avoided in the case of good compactifications of isolated singularities as follows.

Theorem 3. *Assume that f is obtained by an embedded resolution of a good compactification $g : X \to \Delta$ of a germ of a holomorphic function $g_0 : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ with an isolated singularity. Define $\nu^n_{g_0,\lambda}$ by using the Milnor monodromy, where the maximal size of Jordan blocks for $\lambda = 1$ is n instead of $n + 1$. Then we have, for any λ,*

$$\nu^n_{f,\lambda} = \nu^n_{g_0,\lambda} = (-1)^n (\chi(B^\bullet_{f,\lambda}) - \delta_{\lambda,1}),$$
$$\nu^j_{f,\lambda} = \delta_{\lambda,1} \delta_{j,0} \quad (j \in [0, n-1]).$$

where $B^\bullet_{f,1} = C^\bullet_{f,1}$ in the case $\lambda = 1$. Here, $\chi(B^\bullet_{f,\lambda})$ is the Euler characteristic of the complex $B^\bullet_{f,\lambda}$, and $\delta_{\alpha,\beta}$ is 1 if $\alpha = \beta$, and 0 otherwise.

It is quite difficult to determine $B^j_{f,\lambda}$, $B^j_{f_k,\lambda}$ for $\lambda \neq 1$ in general. In fact, we may have $\chi(B^\bullet_{f,\lambda}) \neq \chi(C^\bullet_{f,\lambda})$ for $\lambda \neq 1$ and, moreover, the inequality $\dim H^j B^\bullet_{f,\lambda} \leqslant \dim H^j C^\bullet_{f,\lambda}$ does not necessarily hold (see Example 4.1 and § 4.3 below). The following sufficient conditions for the coincidence are quite useful in certain cases.

Theorem 4. *For $\lambda \neq 1$, set $Y^{(\lambda)} = \bigcup_{I \subset J(\lambda)} Y^{(\lambda)}_I$. Let m_λ be the order of λ.*

  (i) *If $H^1(Y^{(\lambda)}_I, \mathbf{Z}/m_\lambda\mathbf{Z}) = 0$ for any $I \subset J(\lambda)$ with $|I| = j + 1$, then $B^j_{f,\lambda} = C^j_{f,\lambda}$.*

  (ii) *If $H^1(Y^{(\lambda)}, \mathbf{Z}/m_\lambda\mathbf{Z}) = 0$, then $B^\bullet_{f,\lambda} = C^\bullet_{f,\lambda}$ as a complex.*

Similar assertions hold for $B^\bullet_{f_k,\lambda}$, $C^\bullet_{f_k,\lambda}$ by replacing, respectively, $Y^{(\lambda)}_I$, $Y^{(\lambda)}$ with $Y^{(\lambda)}_{k,I}$, $Y^{(\lambda)}_k := \bigcup_{I \subset J(\lambda)} Y^{(\lambda)}_{k,I}$.

In (i), the problem is the triviality of certain unramified cyclic covering of $Y_I^{(\lambda)}$ with degree m_λ, and hence the cohomology $H^1(Y_I^{(\lambda)}, \mathbf{Z}/m_\lambda\mathbf{Z})$ appears. This may be replaced with $H_1(Y_I^{(\lambda)}, \mathbf{Z})$ (since the monodromy group of a local system of rank 1 is abelian), but not with $H^1(Y_I^{(\lambda)}, \mathbf{Z})$. Similar assertions hold for (ii), with $Y_I^{(\lambda)}$ replaced by $Y^{(\lambda)}$.

In the case of Theorem 3, Theorem 4(i) is enough, since we do not have to consider the differential in this case. The hypothesis of Theorem 4(ii) is rather strong, and is not often satisfied except for certain special cases; for example, if f is as in Theorem 3 with $n = 1$ (i.e. dim $X = 2$), and the embedded resolution is obtained by repeating point-center blow-ups. In this case, Theorem 3 for $\lambda \neq 1$ means

$$\nu_{g_0,\lambda}^1 = \#\big\{I \subset J(\lambda) \mid |I| = 2, Y_I \neq \emptyset\big\}$$
$$- \#\big\{j \in J(\lambda) \mid Y_j \cap Y_i = \emptyset \text{ for any } i \notin J(\lambda)\big\}. \tag{0.3}$$

For $\lambda = 1$ and $n = 1$, Theorem 3 simply gives a well-known formula $\nu_{g_0,1}^1 = r_{g_0} - 1$, where r_{g_0} is the number of analytic local irreducible components of $g_0^{-1}(0)$.

Theorem 2 improves a result of Matui and Takeuchi [MT12], where the number is bounded by dim $C_{f,\lambda}^j$ in the case of monodromies at infinity of polynomial maps with $\lambda \neq 1$ (since dim $C_{f,\lambda}^j \geqslant$ dim $H^j B_{f,\lambda}^\bullet$). In the case $j = n$, the latter assertion easily follows from a *local* assertion at the level of perverse sheaves in [Sai06, 3.2.2]:

$$\min\big\{k > 0 \mid N^k(\psi_{f,\lambda}\mathbf{C}_X) = 0 \text{ around } x\big\} = \#\big\{i \in J(\lambda) \mid x \in D_i\big\}, \tag{0.4}$$

where N is the nilpotent part of the monodromy T, and $\psi_{f,\lambda}\mathbf{C}_X$ is the λ-eigenspace of the nearby cycles $\psi_f\mathbf{C}_X$, which is a shifted perverse sheaf (see also [DS04, 1.4] for a more precise local structure). This is more or less well known to the specialists in limit mixed Hodge structures who are familiar with the theory of Steenbrink in [Ste77]. A more precise local structure as in [DS04, 1.4] is implicit in the definition of motivic Milnor fibers, and was used in the proof of compatibility with the Hodge realization by Denef and Loeser [Den98].

The rank of the differential of $B_{f,\lambda}^\bullet$, as well as the difference between dim $H^j B_{f,\lambda}^\bullet$ and dim $C_{f,\lambda}^j$, can be quite large, as is seen in the case of Example 4.4 below. This example shows that, even in the nondegenerate Newton boundary case, we have to apply many blow-ups in order to get a divisor with normal crossings (in the usual sense) by taking a suitable subdivision of the dual fan, and the estimate in [MT12] may become rather bad (unless the dual fan is already smooth, i.e. consisting of simplicials generated by integral vectors with determinant 1). The situation seems to be similar in the case of monodromies at infinity.

In §1, we recall some basics of nearby cycles and limit mixed Hodge structures in the nonreduced case, and then prove Theorems 2 and 3. In §2, we partially generalize Theorem 2 to the singular case in Theorem 2.2. In §3, we provide a method to show Theorem 1 in §4, and prove Theorem 4. In §4, we give some interesting examples, and we prove Theorem 1 in §4.3.

1. Nearby cycles and limit mixed Hodge structures

In this section, we recall some basics of nearby cycles and limit mixed Hodge structures in the nonreduced case, and then prove Theorems 2 and 3.

1.1 Local structure of nearby cycle sheaves

Let f be a nonconstant holomorphic function on a complex manifold X of dimension $n + 1$. Let $\psi_f\mathbf{C}_X$ denote the nearby cycle sheaf with monodromy T in [Del73]. It is well known that this is a

shifted perverse sheaf [BBD82] (i.e. $\psi_f \mathbf{C}_X[n]$ is a perverse sheaf). Using the minimal polynomial of T, we have the Jordan decomposition $T = T_s T_u$, where T_s and T_u, respectively, denote the semisimple and the unipotent parts. For $\lambda \in \mathbf{C}^*$, set

$$\psi_{f,\lambda} \mathbf{C}_X := \mathrm{Ker}(T_s - \lambda) \subset \psi_f \mathbf{C}_X,$$

in the abelian category of *shifted* perverse sheaves [BBD82]. Then $\psi_{f,\lambda} \mathbf{C}_X = 0$ except for a finite number of λ which are roots of unity, and

$$\psi_f \mathbf{C}_X = \bigoplus_\lambda \psi_{f,\lambda} \mathbf{C}_X.$$

Set $N = (2\pi i)^{-1} \log T_u$. The weight filtration W on $\psi_f \mathbf{C}_X$ is given by the monodromy filtration with center 0; that is,

$$N^k : \mathrm{Gr}_k^W \psi_f \mathbf{C}_X \xrightarrow{\sim} (\mathrm{Gr}_{-k}^W \psi_f \mathbf{C}_X)(-k) \quad (k > 0), \tag{1.1.1}$$

where $(-k)$ is the Tate twist, which shifts the weights by $2k$. Define the N-primitive part by

$$P\mathrm{Gr}_k^W \psi_{f,\lambda} \mathbf{C}_X := \mathrm{Ker} N^{k+1} \subset \mathrm{Gr}_k^W \psi_{f,\lambda} \mathbf{C}_X \quad (k \geqslant 0),$$

where it is zero for $k < 0$. By (1.1.1), we have the primitive decomposition

$$\mathrm{Gr}_j^W \psi_{f,\lambda} \mathbf{C}_X = \bigoplus_{k \geqslant 0} N^k (P\mathrm{Gr}_{j+2k}^W \psi_{f,\lambda} \mathbf{C}_X)(k). \tag{1.1.2}$$

Assume that $Y := f^{-1}(0)$ is a divisor with simple normal crossings. Let Y_i be the irreducible components of Y with multiplicities m_i. Set $Y_I := \bigcap_{i \in I} Y_i$. For a root of unity λ in $\mathbf{C}^*$, set

$$J(\lambda) := \{i \mid \lambda^{m_i} = 1\}. \tag{1.1.3}$$

By [Sai90a, 3.3] (see also [DS04, 1.4]), we have the decomposition

$$P\mathrm{Gr}_k^W \psi_{f,\lambda} \mathbf{C}_X = \bigoplus_{I \subset J(\lambda), |I| = k+1} (j_{\lambda,I})_{!*} L_{\lambda,I}(-k)[n - k]. \tag{1.1.4}$$

Here, $L_{\lambda,I}$ is a local system of rank 1 underlying a locally constant variation of complex Hodge structure of weight 0 on $U_{\lambda,I} := Y_I \backslash \bigcup_{i \notin J(\lambda)} Y_i$, and $(j_{\lambda,I})_{!*}$ is the intermediate direct image [BBD82] by the natural inclusion $j_{\lambda,I} : U_{\lambda,I} \hookrightarrow Y_I$. Furthermore, the monodromy of $L_{\lambda,I}$ around $Y_j (j \notin J(\lambda))$ is given by the multiplication by λ^{-m_j}, so that

$$(j_{\lambda,I})_{!*} L_{\lambda,I}[n - k] = (j_{\lambda,I})_! L_{\lambda,I}[n - k] = \mathbf{R}(j_{\lambda,I})_* L_{\lambda,I}[n - k]. \tag{1.1.5}$$

Indeed, the last isomorphism follows from the above information of the local monodromies, and the first isomorphism follows from this by the definition of the intermediate direct image $(j_{\lambda,I})_{!*}$ (see [BBD82]).

1.2 Relation with Steenbrink's construction

In the above notation and assumption, let $\widetilde{X}$ be the normalization of the base change of $f : X \to \Delta$ by the totally ramified m-fold covering $\widetilde{\Delta} \to \Delta$ with $m := \mathrm{LCM}(m_i)$ (see [Ste77]). Let $\pi : \widetilde{X} \to X$, $\widetilde{f} : \widetilde{X} \to \widetilde{\Delta}$ be the canonical morphisms. Set $\widetilde{Y} := \pi^{-1}(Y)$, and let $\pi_0 : \widetilde{Y} \to Y$ be the restriction of π over Y. Then we have a canonical isomorphism

$$\psi_f \mathbf{C}_X = (\pi_0)_* \psi_{\widetilde{f}} \mathbf{C}_{\widetilde{X}}, \tag{1.2.1}$$

where the monodromy $\widetilde{T}$ on the right-hand side is identified with the mth power of the monodromy T on the left-hand side, which is unipotent. This follows from the commutative

diagram

$$
\begin{array}{ccccc}
\widetilde{Y} & \stackrel{\tilde{i}}{\hookrightarrow} & \widetilde{X} & \stackrel{\tilde{j}_\infty}{\longleftarrow} & \widetilde{X}_\infty \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_\infty} \\
Y & \stackrel{i}{\hookrightarrow} & X & \stackrel{j_\infty}{\longleftarrow} & X_\infty
\end{array}
\tag{1.2.2}
$$

where X_∞ is the base change of X by the universal covering of Δ^* over Δ, and similarly for $\widetilde{X}_\infty$ with X, Δ^*, Δ replaced by $\widetilde{X}$, $\widetilde{\Delta}^*$, $\widetilde{\Delta}$. Here, π_∞ is an isomorphism. Then (1.2.1) follows from the definition of the nearby cycles $\psi_f \mathbf{C}_X := i^* \mathbf{R}(j_\infty)_* \mathbf{C}_{X_\infty}$ (and similarly for $\psi_{\tilde{f}} \mathbf{C}_{\widetilde{X}}$) by using the diagram (1.2.2) together with the commutativity

$$
(\pi_0)_* \circ \tilde{i}^* = i^* \circ \pi_*,
$$

where π is finite and hence proper. The relation between $\widetilde{T}$ and T^m is clear by the construction of the isomorphism.

By the above construction, the Milnor fiber of $\tilde{f}$ at any point of $\widetilde{Y}$ is connected and we have

$$
\mathcal{H}^0 \psi_{\tilde{f}} \mathbf{C}_{\widetilde{X}} = \mathbf{C}_{\widetilde{Y}}.
$$

Combining this with (1.2.1), we get

$$
\mathcal{H}^0 \psi_f \mathbf{C}_X = (\pi_0)_* \mathbf{C}_{\widetilde{Y}},
$$

since π_0 is finite. The action of T on the left-hand side is semisimple and corresponds to the action of an appropriate generator of the covering transformation group $\mathbf{Z}/m\mathbf{Z}$ of π. So we get

$$
\mathcal{H}^0 \psi_{f,\lambda} \mathbf{C}_X = ((\pi_0)_* \mathbf{C}_{\widetilde{Y}})_\lambda,
\tag{1.2.3}
$$

where the right-hand side denotes the λ-eigenspace.

Set $\widetilde{Y}_I := \pi^{-1}(Y_I)$. We have the Cech resolution

$$
\mathbf{C}_{\widetilde{Y}} \stackrel{\sim}{\longrightarrow} \mathcal{C}_{\widetilde{Y}}^\bullet \quad \text{with } \mathcal{C}_{\widetilde{Y}}^j := \bigoplus_{|I|=j+1} \mathbf{C}_{\widetilde{Y}_I}.
\tag{1.2.4}
$$

Taking the direct image by $(\pi_0)_*$ and the λ-eigenspace, we get the quasi-isomorphism

$$
((\pi_0)_* \mathbf{C}_{\widetilde{Y}})_\lambda \stackrel{\sim}{\longrightarrow} \mathcal{C}_{Y,\lambda}^\bullet := ((\pi_0)_* \mathcal{C}_{\widetilde{Y}}^\bullet)_\lambda.
\tag{1.2.5}
$$

In the notation (1.1.4) we have, moreover,

$$
\mathcal{C}_{Y,\lambda}^j = \bigoplus_{I \in J(\lambda), |I|=j+1} (j_{\lambda,I})_{!*} L_{\lambda,I},
\tag{1.2.6}
$$

since

$$
((\pi_0)_* \mathbf{C}_{\widetilde{Y}_I})_\lambda = \begin{cases} (j_{\lambda,I})_{!*} L_{\lambda,I} & \text{if } I \in J(\lambda), \\ 0 & \text{if } I \notin J(\lambda). \end{cases}
\tag{1.2.7}
$$

This can be reduced to the case $|I|=1$ by choosing any $i \in I$ and using the restriction morphisms

$$
(j_{\lambda,I'})_{!*} L_{\lambda,I'} \to (j_{\lambda,I})_{!*} L_{\lambda,I} \quad \text{for } I' \subset I \in J(\lambda).
\tag{1.2.8}
$$

Moreover, we may restrict to any dense Zariski-open subset of Y_I (e.g. to the smooth points of Y_{red} if $|I|=1$) by using the intermediate direct image by the open inclusion of the Zariski-open subset, since the intermediate direct images commute with the direct image by any finite morphisms (e.g. $(\pi_0)_*$) (see [BBD82]). Then the assertion follows from (1.1.4) and (1.2.3).

1.3 Weight spectral sequences

With the notation of § 1.1, assume that $f : X \to \Delta$ is a projective morphism to an open disk Δ and, moreover, Y is a divisor with simple normal crossings. Then we have the weight spectral sequence

$$E_1^{-k,j+k} = H^j(Y, \mathrm{Gr}_k^W \psi_{f,\lambda} \mathbf{C}_X) \Longrightarrow H^j(X_\infty)_\lambda, \tag{1.3.1}$$

where W on $\psi_{f,\lambda} \mathbf{C}_X$ is the monodromy filtration as in § 1.1, and $H^j(X_\infty)_\lambda$ denotes the λ-eigenspace of the limit mixed Hodge structure with complex coefficients, as in [Ste76, Ste77]. By [Sai88, § 4] (or [GN90]), the filtration on $H^j(X_\infty)$ induced by W on $\psi_{f,\lambda} \mathbf{C}_X$ is also the monodromy filtration with center 0, and we need the *shift* by j to get the weight filtration W on $H^j(X_\infty)$.

From the primitive decomposition (1.1.2) together with (1.1.4, we can deduce the double complex structure of the E_1-complex in [Sai88, § 4] (see also [SZ90, 1.1]) as follows:

$$E_1^{-k,j+k} = \bigoplus_{a-b=k} C_{\lambda,a,b}^j \quad \text{with } C_{\lambda,a,b}^j := \begin{cases} \displaystyle\bigoplus_{|I|=a+b+1} \mathrm{IH}^{j-a-b}(Y_I, L_{\lambda,I})(-a) & \text{if } a, b \geqslant 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3.2}$$

where $\mathrm{IH}^{j-a-b}(Y_I, L_{\lambda,I})$ is the intersection cohomology [BBD82], the action of N is induced by

$$id : C_{\lambda,a,b}^j \to C_{\lambda,a-1,b+1}^j(-1) \quad \text{if } a-1, b \in \mathbf{Z}_{\geqslant 0}, \tag{1.3.3}$$

and the E_1-differential is the sum of

$$d' : C_{\lambda,a,b}^j \to C_{\lambda,a-1,b}^{j+1} \quad \text{and} \quad d'' : C_{\lambda,a,b}^j \to C_{\lambda,a,b+1}^{j+1}, \tag{1.3.4}$$

which are identified up to a certain sign with the morphisms induced, respectively, by the Cech–Gysin morphisms $\widetilde{\gamma}$ and the Cech restriction morphisms $\widetilde{\rho}$ between the $\widetilde{Y}_I$ using the isomorphism (1.2.7). In other words, (1.3.2) is obtained by taking the λ-eigenspace of the direct image by π_0 of the double complex structure of the E_1-complex for $\tilde{f}$ in [Ste77]. (Note that the kernel and image filtrations K_i and I^k in [SZ90] are defined, respectively, by the conditions $a \leqslant i$ and $b \geqslant k$.)

Now consider the *lowest*-weight part of the E_1-complex. Its weight $j + k$ is zero, with

$$a = j - b = 0 \quad \text{in } (1.3.2),$$

since the weight of $\mathrm{IH}^{j-a-b}(Y_I, L_{\lambda,I})(-a)$ is $(j - a - b) + 2a$, with

$$j - a - b \geqslant 0, \quad a \geqslant 0.$$

So the lowest-weight part is the complex with jth component given by

$$C_{\lambda,0,j}^j = \mathcal{H}^0(Y, \mathcal{C}_{Y,\lambda}^j), \tag{1.3.5}$$

where the last isomorphism comes from (1.2.6).

1.4 Limits of weight spectral sequences

With the notation of § 1.1, let D be a divisor with simple normal crossings on X such that all the irreducible components D_j of D are dominant over Δ. Set $U := X \backslash D$, and $D_J := \bigcap_{j \in J} D_j$ (where $D_\emptyset = X$). We have the spectral sequences of mixed $\mathbf{Q}$-Hodge structures compatible with

the action of the semisimple part T_s of the monodromy:

$$
\begin{aligned}
{}_\infty E_1^{-i,j+i} &= \bigoplus_{|J|=i} H^{j-i}(D_{J,\infty})(-i) \Longrightarrow H^j(U_\infty), \\
{}_\infty^c E_1^{i,j-i} &= \bigoplus_{|J|=i} H^{j-i}(D_{J,\infty}) \Longrightarrow H_c^j(U_\infty),
\end{aligned}
\tag{1.4.1}
$$

which are duals of each other. They degenerate at E_2, since they are the 'limit' by $t \to 0$ of the weight spectral sequences

$$
\begin{aligned}
{}_t E_1^{-i,j+i} &= \bigoplus_{|J|=i} H^{j-i}(D_{J,t})(-i) \Longrightarrow H^j(U_t), \\
{}_t^c E_1^{i,j-i} &= \bigoplus_{|J|=i} H^{j-i}(D_{J,t}) \Longrightarrow H_c^j(U_t),
\end{aligned}
\tag{1.4.2}
$$

where the nearby cycle functor ψ of mixed Hodge modules can be used to define the 'limit'. Here, $D_{J,t} := D_J \cap X_t$ for $t \in \Delta^*$. The first spectral sequence in (1.4.1) was obtained in [SZ85] in the unipotent monodromy case, and it can be generalized to the nonunipotent case by [Ste77]. Here, the 'limit' can also be defined by using the nearby cycle functor ψ of mixed Hodge modules, and the spectral sequences are defined by the weight filtration on the shifted perverse sheaves $(j_U)_* \mathbf{Q}_U$ or $(j_U)_! \mathbf{Q}_U$, with $j_U : U \hookrightarrow X$ the natural inclusion. This implies, for instance,

$$
\mathrm{Gr}_0^W (j_U)_* \mathbf{Q}_U = \mathrm{Gr}_0^W (j_U)_! \mathbf{Q}_U = \mathbf{Q}_X.
$$

The E_1-differentials of the spectral sequences are induced by the Cech–Gysin and Cech restriction morphisms.

PROPOSITION 1.5. *Let $H^j(U_\infty)_\lambda$ denote the λ-eigenspace of $H^j(U_\infty, \mathbf{C})$, and similarly for $H_c^j(U_\infty)_\lambda$, etc. Let $\nu_{f_U,\lambda}^j$, $\nu_{c,f_U,\lambda}^j$ be as in the introduction. Then we have, for $j \in [0,n]$,*

$$
\nu_{f_U,\lambda}^j = \dim \mathrm{Gr}_0^W H^j(U_\infty)_\lambda = \dim \mathrm{Gr}_0^W H^j(X_\infty)_\lambda,
\tag{1.5.1}
$$

$$
\begin{aligned}
\nu_{c,f_U,\lambda}^j &= \dim \mathrm{Gr}_{2j}^W H_c^j(U_\infty)_\lambda \\
&= \dim \mathrm{Ker}\left(\mathrm{Gr}_0^W H^j(X_\infty)_\lambda \to \bigoplus_k \mathrm{Gr}_0^W H^j(D_{k,\infty})_\lambda \right),
\end{aligned}
\tag{1.5.2}
$$

where the last morphisms are induced by the restriction morphisms for $D_k \hookrightarrow X$.

Proof. This follows from the spectral sequences in (1.4.1). Let L denote the increasing filtration on $H^j(U_\infty)$, $H_c^j(U_\infty)$ associated with the spectral sequences and shifted by j so that L is the limit of the weight filtration on $H^j(U_t)$, $H_c^j(U_t)$ for $t \in \Delta^*$, and

$$
{}_\infty E_2^{-i,j+i} = \mathrm{Gr}_{j+i}^L H^j(U_\infty), \quad {}_\infty^c E_2^{i,j-i} = \mathrm{Gr}_{j-i}^L H^j(U_\infty).
\tag{1.5.3}
$$

The E_1-differentials are induced by the Gysin or restriction morphisms, and are limits of morphisms of pure Hodge structures of the same weight. Hence they preserve the center of symmetry of the action of N, which coincides with the weight of the pure Hodge structure before taking the limit. Set $d_J := n - |J| = \dim D_J$. It is well known that

$$
wt(H^j(D_{J,\infty})) \subset \begin{cases} [0, 2j] & \text{if } j \in [0, d_J], \\ [2j - 2d_J, 2d_J] & \text{if } j \in [d_J, 2d_J], \end{cases}
\tag{1.5.4}
$$

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where the left-hand side is the set of weights of $H^j(D_{J,\infty})$. This can be shown by using the invariance of the dimension of the graded pieces of the Hodge filtration by passing to the limit mixed Hodge filtration F, since the latter, together with its conjugate Hodge filtration $\overline{F}$, determines the limit mixed Hodge numbers (see [Del71]).

We first show (1.5.2). Using (1.4.1), (1.5.3) and (1.5.4), we get the first equality of (1.5.2), since

$$\nu^j_{c,f_U,\lambda} \leqslant \dim \mathrm{Gr}^W_{2j} H^j_c(U_\infty)_\lambda = \dim \mathrm{Gr}^W_{2j} \mathrm{Gr}^L_j H^j_c(U_\infty)_\lambda \leqslant \nu^j_{c,f_U,\lambda}.$$

Here, the first inequality follows from

$$wt(H^j_c(U_\infty)) \subset [0, 2j],$$

the middle equality follows from

$$\mathrm{Gr}^W_{2j} \mathrm{Gr}^L_i H^j_c(U_\infty) = 0 \quad \text{for } i \neq j,$$

and the last inequality follows from the fact that the E_1-differential preserves the center of symmetry of the action of N. Moreover, the E_1-differential $_\infty^c E_1^{0,j} \to _\infty^c E_1^{1,j}$ is given by the restriction morphism

$$H^j(X_\infty) \to \bigoplus_k H^j(D_{k,\infty}).$$

So we also get the second equality of (1.5.2).

The argument is similar for (1.5.1), and is simpler since in this case we use the Gysin morphism

$$\bigoplus_k H^{j-2}(D_{k,\infty})(-1) \to H^j(X_\infty),$$

where the image has weights in $[2, 2j-2]$ so that it can be neglected for the calculation of $\nu^j_{f_U,\lambda}$. This finishes the proof of Proposition 1.5. $\square$

Remark 1.6. If we replace the complex manifold X with a Kähler manifold X' that has a bimeromorphic proper morphism $X' \to X$, then $\nu_{f,\lambda}$ does not change. Indeed, $H^j(X_t, \mathbf{Q})$ is a direct factor of $H^j(X'_t, \mathbf{Q})$ for $t \in \Delta^*$, and the level of its complement is strictly less than $\min(j, 2\dim X_t - j)$. Here, the level of a mixed Hodge structure H is the difference between the maximal and minimal integers p, with $\mathrm{Gr}^p_F H_\mathbf{C} \neq 0$.

1.7 Proof of Theorem 2

We can define the spectral sequence (1.3.1) together with the decomposition (1.3.2) without assuming X Kähler. We have to show its E_2-degeneration together with the symmetry of the E_2-term by the action of N (i.e. the induced filtration on $H^j(X_\infty)$ is the monodromy filtration with center 0). By hypothesis, there is a proper surjective morphism from a Kähler manifold X' to X. Then, using the decomposition theorem for $X' \to X$ (see [Sai90b]), the above properties are reduced to the Kähler case, and then follow from [Ste76, §4] (or [GN90]). So the assertion in the case $D = \emptyset$ follows from (1.3.5) by setting

$$B^j_{f,\lambda} := H^0(Y, \mathcal{C}^j_{Y,\lambda}). \tag{1.7.1}$$

The general case is then reduced to the case $D = \emptyset$ by Proposition 1.5. This completes the proof of Theorem 2.

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1.8 Proof of Theorem 3

The nearby and vanishing cycle functors commute with the direct image by the proper morphism $f' : X' \to \Delta$. So we get

$$\varphi_t \mathbf{R} f'_* \mathbf{C}_{X'}[n] = (\varphi_{f'} \mathbf{C}_{X'}[n])_0, \tag{1.8.1}$$

where the right-hand side is identified with the reduced Milnor cohomology at $0 \in X'$ (which is the only singular point of f'). Furthermore, we have

$$\varphi_{t,\lambda} \mathbf{R} f'_* \mathbf{C}_{X'}[n] = \begin{cases} \psi_{t,\lambda} R^n f'_* \mathbf{C}_{X'} = \psi_{t,\lambda} R^n f_* \mathbf{C}_X & \text{if } \lambda \neq 1, \\ \text{Im can} \oplus \text{Ker var} & \text{if } \lambda = 1, \end{cases} \tag{1.8.2}$$

where $\text{can} : \psi_{t,1} \to \varphi_{t,1}$ and $\text{var} : \varphi_{t,1} \to \psi_{t,1}(-1)$ are as in [Sai88, §5], and we apply these to ${}^p \mathcal{H}^0 \mathbf{R} f'_*(\mathbf{C}_{X'}[\dim X'])$ (see [BBD82] for ${}^p \mathcal{H}^j$). The assertion for $\lambda = 1$ follows from the decomposition theorem in the work cited above. Moreover, we have

$$\text{Im can} = \text{Im } N \subset \psi_{t,1} R^n f'_* \mathbf{C}_{X'} = \psi_{t,1} R^n f_* \mathbf{C}_X, \tag{1.8.3}$$

and the action of N on Ker var is trivial. We thus get, for any λ,

$$\nu^n_{g_0,\lambda} = \nu^n_{f,\lambda}. \tag{1.8.4}$$

(Here, it is not necessary to assume that the restriction morphism induces a surjection from $H^n(X_t, \mathbf{C})$ to the Milnor cohomology.)

On the other hand, we have

$$\varphi_t {}^p \mathcal{H}^j \mathbf{R} f'_*(\mathbf{C}_{X'}[\dim X']) = 0 \quad \text{if } j \neq 0, \tag{1.8.5}$$

since f' has only isolated singularities and the vanishing cycle functor commutes with the direct image by proper morphisms. This implies that the local systems

$$R^j f'_* \mathbf{C}_{X'}|_{\Delta^*} = R^j f_* \mathbf{C}_X|_{\Delta^*}$$

are constant for $j \neq n$, and hence $\nu^j_{f,\lambda} = 0$ if $j \in [1, n-1]$ or $j = 0$ with $\lambda \neq 1$, where $\nu^0_{f,1} = 1$. So the assertion follows from Theorem 2.

2. Partial generalization to the singular case

In this section, we partially generalize Theorem 2 to the singular case in Theorem 2.2.

2.1 Singular case

Theorem 2 for $\nu^j_{f_U,\lambda}$ cannot be generalized to the singular case (see Example 2.3 below). However, we can generalize the assertion for $\nu^j_{c,f_U,\lambda}$ in Theorem 2 to the singular case as follows. Let $f : X \to \Delta$ be a projective morphism of a reduced analytic space X to Δ, and let D be a closed reduced analytic subspace of X such that any irreducible components of X and D are dominant over Δ. Set $U := X \backslash D$, with $f_U : U \to \Delta$ the morphism induced by f. Let $n := \dim X - 1$.

Let $\nu^j_{c,f_U,\lambda}$ and $\nu^{2n-j}_{c,f_U,\lambda}$ be, respectively, the number of Jordan blocks of size j and eigenvalue λ for the monodromy on $H^j_c(U_t)$ and $H^{2n-j}_c(U_t)$ with $j \leqslant n$. For the statement of Theorem 2.2 below for $\nu^{2n-j}_{c,f_U,\lambda}$ $(j \leqslant n)$, it is enough to take a resolution of singularities $\pi_{(0)} : X_{(0)} \to X$ with $\pi_{(0)}$ projective. For $\nu^j_{c,f_U,\lambda}$ $(j \leqslant n)$, however, the preparation for Theorem 2.2 is more complicated. We have to construct complex manifolds $X_{(0)}$, $X_{(1)}$, $D_{(0)}$ together with projective morphisms $\pi_{(k)} : X_{(k)} \to X (k = 1, 2)$, $\pi'_{(0)} : D_{(0)} \to D$ and an analytic cycle γ_X on $X_{(1)} \times X_{(0)}$ which is a $\mathbf{Z}$-linear combination of graphs of morphisms from connected components of $X_{(1)}$ to $X_{(0)}$ over X

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(where there may be many morphisms defined on one connected component). They have to satisfy the following conditions.

(i) The composition $f_{(k)} := f \circ \pi_{(k)} : X_{(k)} \to \Delta$ is flat, (i.e. any connected component is dominant over Δ), its restriction over Δ^* is smooth, and $f_{(k)}^{-1}(0)$ is a divisor with simple normal crossings on $X_{(k)}$ ($k = 1.2$).

(ii) We have $\Gamma_{\pi_{(0)}} \circ \gamma_X = 0$ as a cycle on $X_{(1)} \times X$ (without any equivalence relation), where $\Gamma_{\pi_{(0)}}$ is the graph of $\pi_{(0)}$, and the composition of correspondences $\Gamma_{\pi_{(0)}} \circ \gamma_X$ is defined in this case by using the composition of morphisms.

(iii) Setting $X_t := f^{-1}(t)$, $X_{(k),t} := f_{(k)}^{-1}(t)$, we have the following exact sequence for any $j \in \mathbf{Z}$ and $t \in \Delta^*$:

$$0 \to \operatorname{Gr}_j^W H^j(X_t, \mathbf{Q}) \xrightarrow{\pi_{(0)}^*} H^j(X_{(0),t}, \mathbf{Q}) \xrightarrow{\gamma_X^*} H^j(X_{(1),t}, \mathbf{Q}), \qquad (2.1.1)$$

where W is the weight filtration of the canonical mixed Hodge structure on $H^j(X_t, \mathbf{Q})$, and γ_X^* is defined by using the pull-backs by the morphisms in the definition of γ_X.

(iv) The above condition (i) for $k = 0$ with X replaced by D is satisfied, where we denote the restriction of f to D by h, and the morphism $D_{(0)} \to \Delta$ by $h_{(0)}$. Moreover, $\pi'_{(0)}$ is surjective and there is a morphism $\rho_{(0)} : D_{(0)} \to X_{(0)}$, giving a commutative diagram.

$$\begin{array}{ccc}
D_{(0)} & \xrightarrow{\pi'_{(0)}} & D \\
\downarrow{\scriptstyle \rho_{(0)}} & & \downarrow{\scriptstyle i} \\
X_{(1)} \xrightarrow{\gamma_X} X_{(0)} & \xrightarrow{\pi_{(0)}} & X
\end{array} \qquad (2.1.2)$$

(Here, $X_{(1)}$ is also noted, since this will be useful for Theorem 2.2 below.)

This can be done, for instance, by using an argument similar to [GNPP88] together with resolution of singularities. If X, D are defined algebraically (i.e. if they are base changes of algebraic varieties over a curve C by an open inclusion $\Delta \hookrightarrow C^{\mathrm{an}}$), then the above assumptions are satisfied by using simplicial resolutions [Del74] or cubic resolutions [GNPP88].

Let $B_{f_{(k)},\lambda}^\bullet$ be as in Theorem 2 applied to $f_{(k)} : X_{(k)} \to \Delta$ ($k = 1, 2$), and similarly for $B_{h_{(0)},\lambda}^\bullet$. For any morphism g of a connected component of $X_{(1)}$ to $X_{(0)}$, we have a morphism of complexes

$$g^* : B_{f_{(0)},\lambda}^\bullet \to B_{f_{(1)},\lambda}^\bullet,$$

by choosing an irreducible component of $f_{(0)}^{-1}(0)$ containing the image of each irreducible component of $f_{(1)}^{-1}(0)$ by g. This induces a morphism of complexes

$$\gamma_X^* : B_{f_{(0)},\lambda}^\bullet \to B_{f_{(1)},\lambda}^\bullet,$$

and similarly for

$$\rho_{(0)}^* : B_{f_{(0)},\lambda}^\bullet \to B_{h_{(0)},\lambda}^\bullet.$$

THEOREM 2.2. *With the above notation and assumptions we have, for $j \in [0, n]$,*

$$\nu_{c,f_U,\lambda}^j = \dim \operatorname{Ker}((\gamma_X^*, \rho_{(0)}^*) : H^j B_{f_{(0)},\lambda}^\bullet \to H^j B_{f_{(1)},\lambda}^\bullet \oplus H^j B_{h_{(0)},\lambda}^\bullet), \qquad (2.2.1)$$

$$\nu_{c,f_U,\lambda}^{2n-j} = \dim H^j B_{f_{(0)},\lambda}^\bullet. \qquad (2.2.2)$$

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Proof. We first consider $\nu_{c,f_U,\lambda}^j$ $(j \leqslant n)$ and prove (2.2.1). We have a long exact sequence of mixed Hodge structures for $t \in \Delta^*$

$$\to H^{j-1}(D_t, \mathbf{Q}) \to H_c^j(U_t, \mathbf{Q}) \to H^j(X_t, \mathbf{Q}) \xrightarrow{i^*} H^j(D_t, \mathbf{Q}) \to . \qquad (2.2.3)$$

Since $H^{j-1}(D_t)$ has weights at most $j-1$, this induces an isomorphism

$$\mathrm{Gr}_j^W H_c^j(U_t, \mathbf{Q}) = \mathrm{Ker}(i^* : \mathrm{Gr}_j^W H^j(X_t, \mathbf{Q}) \to \mathrm{Gr}_j^W H^j(D_t, \mathbf{Q})).$$

Combining this with (2.1.1) and using (2.1.2), we see that $\mathrm{Gr}_j^W H_c^j(U_t, \mathbf{Q})$ is isomorphic to the kernel of

$$(\gamma_X^*, \rho_{(0)}^*) : H^j(X_{(0),t}, \mathbf{Q}) \to H^j(X_{(1),t}, \mathbf{Q}) \oplus H^j(D_{(0),t}, \mathbf{Q}).$$

By [Del74] (and using (2.2.3)), we have

$$\mathrm{Gr}_F^p W_{j-1} H_c^j(U_t, \mathbf{Q}) = 0 \quad \text{for } p \notin [0, j-1].$$

So the assertion (2.2.1) follows from the same argument as in the proof of Proposition 1.5.

We now consider $\nu_{2n-c,f_U,\lambda}^j$ $(j \leqslant n)$. For the proof of (2.2.2), note first that we may replace $X_{(0)}$ by any resolution of singularities of X by Remark 1.6. Here, we can neglect any complex manifold Y of pure dimension $m < n$, since we use duality and the dual of $\mathbf{Q}_Y$ is

$$\mathbf{Q}_Y(m)[2m] = \big(\mathbf{Q}_Y(m)[2n]\big)[2r] \quad \text{with } r := m - n < 0.$$

(Without using duality, it is related to the fact that the level of $H^j(Y, \mathbf{Q})$ is strictly less than j if $j > \dim Y$.) We can also construct $X_{(1)}$ and $D_{(0)}$ as in the above case by using an argument as in [GNPP88] so that we may assume, moreover,

$$\dim X_{(1)} < n, \quad \dim D_{(0)} < n.$$

Using the dual argument of the proof of (2.2.1), we get only the Cech–Gysin morphisms. So (2.2.2) follows from the same argument as in the proof of Proposition 1.5. This finishes the proof of Theorem 2.2. $\qquad \square$

The following example shows that Theorem 2 for $\nu_{f_U,\lambda}$ cannot be generalized to the singular case.

Example 2.3. Let $Z' = \mathbf{P}^1$, with $\Sigma := \{0, \infty\}$. Let $\sigma : Z' \to Z$ be a morphism inducing an isomorphism outside Σ and such that $\sigma(\Sigma)$ is one point. Let $\iota' : \Delta \hookrightarrow \mathbf{P}^1$ be the natural inclusion of an open disk Δ of radius <1. This induces an inclusion $\iota : \Delta \hookrightarrow Z$, and $1 \in \mathbf{P}^1 \backslash \iota'(\Delta)$ is identified with a point of $Z \backslash \iota(\Delta)$ which is also denoted by 1. Set $X := Z \times \Delta$, with $f : X \to \Delta$ the second projection. Let $D \subset X$ be the union of the graph of ι and $\{1\} \times \Delta$. Set $U := X \backslash D$. Then, for $t \in \Delta^*$, we have isomorphisms

$$H^1(U_t) = H^1(Z \backslash \{1, t\}, \sigma(\Sigma)) = H^1(\mathbf{P}^1 \backslash \{1, t\}, \Sigma),$$

where the cohomology is with $\mathbf{Q}$-coefficients, and $\iota'(t)$, $\iota(t)$ are denoted by t to simplify the notation. We have a long exact sequence

$$H^0(\mathbf{P}^1 \backslash \{1, t\}) \to H^0(\Sigma) \to H^1(\mathbf{P}^1 \backslash \{1, t\}, \Sigma) \to H^1(\mathbf{P}^1 \backslash \{1, t\}) \to 0,$$

inducing a short exact sequence

$$0 \to \mathbf{Q} \to H^1(\mathbf{P}^1 \backslash \{1, t\}, \Sigma) \to \mathbf{Q}(-1) \to 0.$$

We see that the monodromy around the origin in Δ is nontrivial as follows. There is a relative cycle class γ in $H_1(\mathbf{P}^1 \backslash \{1, t\}, \Sigma)$, represented by a path between 0 and ∞ which is slightly

below the real positive half line. Let $t_0 \in \Delta^*$ be a sufficiently small real positive number. Take a loop $\alpha \in \pi_1(\Delta^*, t_0)$ going around the origin of Δ counterclockwise. Deform the relative cycle γ continuously when $t \in \Delta^*$ moves along α. Then the relative cycle γ becomes slightly above the real positive half line. Thus the action of the monodromy T on the relative cycle γ is given by

$$T\gamma = \gamma + \eta,$$

where η is a small circle around t_0. This implies the nonvanishing of

$$N : H^1(U_\infty) \to H^1(U_\infty)(-1).$$

However, we have

$$\mathrm{Gr}_k^L H^1(U_\infty) = \mathrm{Gr}_k^W H^1(U_\infty) = \begin{cases} \mathbf{Q} & \text{if } k = 0, \\ \mathbf{Q}(-1) & \text{if } k = 2, \\ 0 & \text{if } k \neq 0, 2, \end{cases}$$

where L is induced by the weight filtration W on $H^1(U_t)$ for $t \in \Delta^*$, as in the proof of Proposition 1.5. Thus Theorem 2 for $\nu_{f_U, \lambda}$ is false in the singular case.

3. Global triviality of certain nearby cycle local systems

In this section, we provide a method to show Theorem 1 in § 4, and to prove Theorem 4.

3.1 Global factorization of functions

Let f be a holomorphic function on a complex manifold. Assume that $Y := f^{-1}(0)$ is a divisor with simple normal crossings. Set $X^* := X \setminus Y$ with the inclusion $j : X^* \hookrightarrow X$. For a locally closed analytic subset Z of Y with the inclusion $i_Z : Z \hookrightarrow X$, set

$$\mathcal{M}_Z^* := i_Z^{-1} j_*^{\mathrm{mer}} \mathcal{O}_{X^*}^*,$$

where j_*^{mer} denotes the meromorphic extension over X. If Y is locally the union of $\{x_i = 0\}$ for $0 \leqslant i < r$, where $x_0, \dots, x_n$ are local coordinates of X around $x \in Y$, then

$$\mathcal{M}_{Z,x}^* = \left\{ u \prod_{0 \leqslant i < r} x_i^{a_i} \mid u \in \mathcal{O}_{X,x}^*, a_i \in \mathbf{Z} \right\}.$$

For an integer $m' \geqslant 2$, let $\mathcal{M}_Z^{*m'}$ be the image of the m'th power endomorphism of $\mathcal{M}_Z^*$. We have a short exact sequence of sheaves of multiplicative groups over Z

$$1 \to \mu_{Z,m'} \to \mathcal{M}_Z^* \xrightarrow{m'} \mathcal{M}_Z^{*m'} \to 1,$$

where $\mu_{Z,m'}$ is the constant sheaf on Y with stalks $\mu_{m'}$ (the multiplicative group consisting of the roots of unity of order m' in $\mathbf{C}^*$). We have the associated long exact sequence

$$1 \to \mu_{m'} \to \Gamma(Z, \mathcal{M}_Z^*) \xrightarrow{m'} \Gamma(Z, \mathcal{M}_Z^{*m'}) \xrightarrow{c_{m'}} H^1(Z, \mu_{Z,m'}),$$

where the last morphism $c_{m'}$ gives the cohomology class of $u \in \Gamma(Z, \mathcal{M}_Z^{*m'})$. This is the same as the cohomology class of the finite unramified covering of Z defined by $m'^{-1}(u)$, which is a principal $\mu_{m'}$-bundle. (Indeed, consider the Cech cocycle associated to local pull-backs of u by m' for an sufficiently fine open covering of Z.) Anyway, we have a primitive m'th root of u globally over Z if and only if $c_{m'}(u) = 0$.

Assume that the restriction of f to a sufficiently small neighborhood of Z defines an element u_f of $\Gamma(Z, \mathcal{M}^{*m'})$; that is, there is a solution of $\xi^{m'} = f$ with $\xi \in \mathcal{O}_{X,x}$, for any $x \in Z$. Then we have a global solution of $\xi^{m'} = f$ on a sufficiently small open neighborhood of Z if and only if $c_{m'}(u_f) = 0$.

3.2 The globally factorized case

With the notation of §3.1, assume that there is a global solution $\xi^{m'} = f$ on X, where $f : X \to \Delta$ is not necessarily proper. We have a factorization

$$f : X \xrightarrow{\xi} \widetilde{\Delta}' \xrightarrow{\pi_{m'}} \Delta,$$

where $\pi_{m'}$ is a totally ramified covering of degree m'. Let $\widetilde{X}'$ be the normalization of the base change of $f : X \to \Delta$ by $\pi_{m'} : \widetilde{\Delta}' \to \Delta$. Let $\pi' : \widetilde{X}' \to X$ be the canonical morphism. Set $\widetilde{Y}' := \pi'^{-1}(Y)$, with $\pi'_0 : \widetilde{Y}' \to Y$ the canonical morphism. Note that this is a trivial covering space; that is, $\widetilde{Y}'$ is a disjoint union of m' copies of Y.

Let $V_{m'}$ be a complex vector space endowed with a basis $(e_0, \ldots, e_{m'-1})$ and an action of T defined by $T e_i = e_{i+1}$ for $i = 0, \ldots, m' - 1 \bmod m'$. Let $V_{m',Y}$ denote the constant sheaf with stalks $V_{m'}$. Then, choosing a section of π'_0, we have canonical isomorphisms

$$\bigoplus_{\lambda^{m'}=1} \mathcal{H}^0 \psi_{f,\lambda} \mathbf{C}_X = (\pi'_0)_* \mathbf{C}_{\widetilde{Y}'} = V_{m',Y}, \tag{3.2.1}$$

in a compatible way with the action of T, where T on the middle term is given by the action of an appropriate generator of the covering transformation group of π'_0. Indeed, the first isomorphism is shown by using the Milnor fiber at each point. The second isomorphism follows from the triviality of the covering $\pi'_0 : \widetilde{Y}' \to Y$ by choosing a section of π'_0.

For $I \subset J(\lambda)$, set

$$L'_{\lambda,I} := L_{\lambda,I}|_{U'_{\lambda,I}} \quad \text{with } U'_{\lambda,I} := U_{\lambda,I} \cap Y^{(\lambda)}.$$

Then, using the projection from $V_{m'}$ to $\mathbf{C}e_0 \subset V_{m'}$, (3.2.1) induces canonical isomorphisms

$$L'_{\lambda,I} = \mathbf{C}_{U'_{\lambda,I}}, \tag{3.2.2}$$

in a compatible way with the restriction morphisms (1.2.8).

PROPOSITION 3.3. *For an integer $m' \geqslant 2$, let Z be a closed subvariety of $Y^{(\lambda')}$ with $\lambda' := \exp(2\pi i/m')$ in the notation of Theorem 4. Let $\pi_Z : \widetilde{Z} \to Z$ be the unramified covering of degree m' defined by local solutions of $\xi^{m'} = f$, as in §3.1. Then, with the notation of (3.2.2), we have a canonical isomorphism*

$$(\pi_Z)_* \mathbf{C}_{\widetilde{Z}} \xrightarrow{\sim} \bigoplus_{\lambda^{m'}=1} \left(\mathcal{H}^0 \psi_{f,\lambda} \mathbf{C}_X \right)|_Z \tag{3.3.1}$$

in a compatible way with the action of T, where T on the left-hand side is defined by the action of an appropriate generator of the covering transformation group of π_Z.

Proof. Let $\widetilde{X}'$ be the normalization of the base change of $f : X \to \Delta$ by the m'-fold ramified covering $\pi_{m'} : \widetilde{\Delta}' \to \Delta$. Restricting over a sufficiently small open neighborhood of each z of Z in X, this coincides with the construction in §3.2. Note that the restriction of $\widetilde{X}' \to X$ over Z is identified with $\pi_Z : \widetilde{Z} \to Z$. Let $\tilde{f}' : \widetilde{X}' \to \widetilde{\Delta}'$ be the natural morphism. We have natural isomorphism and inclusion

$$(\pi_Z)_* \mathbf{C}_{\widetilde{Z}} \xrightarrow{\sim} (\pi_Z)_* (\mathcal{H}^0 \psi_{\tilde{f}',1} \mathbf{C}_{\widetilde{X}'}|_{\widetilde{Z}}) \hookrightarrow \mathcal{H}^0 \psi_f \mathbf{C}_X|_Z$$

compatible with the action of T, where T on the first and second terms is induced by the action of an appropriate generator of the covering transformation group of π_Z. So the assertion follows from the local calculation in (3.2.1) (by counting the dimension). This finishes the proof of Proposition 3.3. □

COROLLARY 3.4. *With the above notation and assumption, $\pi_Z : \widetilde{Z} \to Z$ is trivial if and only if the $\mathcal{H}^0\psi_{f,\lambda}\mathbf{C}_X$ are trivial local systems for any λ with $\lambda^{m'} = 1$.*

Proof. This follows from Proposition 3.3 by applying the global section functor to (3.3.1). □

3.5 Proof of Theorem 4

We apply § 3.1 to the case $Z = Y^{(\lambda)}$ and $m' = m_\lambda$. If $H^1(Y^{(\lambda)}, \mu_{m_\lambda}) = 0$, then we have a global solution of $\xi_\lambda^{m_\lambda} = f$ on a sufficiently small open neighborhood $X^{(\lambda)}$ of $Y^{(\lambda)}$. So the assertion (ii) follows from § 3.2. The argument is similar for the remaining assertions. This finishes the proof of Theorem 4.

PROPOSITION 3.6. *With the notation of Theorem 4, assume that there is a subset $Z^{(\lambda)}$ of $Y^{(\lambda)}$ which is homotopy equivalent to a dense Zariski-open subset $U^{(\lambda)}$ of $Y^{(\lambda)}$ and, moreover, there is a holomorphic function g_λ on a sufficiently small open neighborhood of $Z^{(\lambda)}$ in X satisfying $g_\lambda^{m_\lambda} = f$ on this neighborhood. Then $B_{f,\lambda}^\bullet = C_{f,\lambda}^\bullet$. If the above condition holds by replacing X, $Y^{(\lambda)}$ and f respectively with D_k, $Y_k^{(\lambda)}$ and $f_k = f|_{D_k}$ for any k, then $B_{f_k,\lambda}^\bullet = C_{f_k,\lambda}^\bullet$.*

Proof. By the same argument as in the proof of Theorem 4, it is sufficient to show that we have a global solution of $\xi^{m_\lambda} = f$ on a sufficiently small open neighborhood $X^{(\lambda)}$ of $Y^{(\lambda)}$. Here, we may replace $Y^{(\lambda)}$ with the dense Zariski-open subset $U^{(\lambda)}$. Indeed, local solutions of $\xi^{m_\lambda} = f$ form a finite unramified covering as in § 3.1, and it is trivial over $Y^{(\lambda)}$ if its restriction over any dense Zariski-open subset is trivial. Moreover, the triviality of the covering is determined by its cohomology class in the first cohomology with coefficients in μ_{m_λ} (see § 3.1). This triviality can be seen by restricting to the subspace $Z^{(\lambda)}$, which is homotopy equivalent to $U^{(\lambda)}$ by the hypothesis of Theorem 4. So the assertion follows. □

PROPOSITION 3.7. *With the notation and the assumption of Theorem 3, assume that $n = 2$ and that the embedded resolution is obtained by iterating blowing-ups with point or $\mathbf{P}^1$-centers. Then $B_{f,\lambda}^j = C_{f,\lambda}^j$ for any j, and hence Theorem 3 holds with $B_{f,\lambda}^\bullet$ replaced by $C_{f,\lambda}^\bullet$.*

Proof. Since projective spaces $\mathbf{P}^k$ $(k = 1, 2)$ and $\mathbf{P}^1$-bundles over $\mathbf{P}^1$ are simply connected, and simple connectedness does not change by point-center blow-ups, the assertion follows from Theorem 4(i). □

The following is closely related with results in [Art94a, Mar12], where similar constructions are used.

PROPOSITION 3.8. *With the notation and the assumption of Theorem 3, assume that $n = 2$ and g_0 defines a super-isolated singularity [Lue87] or, more generally, a Yomdin singularity [Yom75]. Then $B_{f,\lambda}^j = C_{f,\lambda}^j$ for any j, and hence Theorem 3 holds with $B_{f,\lambda}^\bullet$ replaced by $C_{f,\lambda}^\bullet$.*

Proof. We have the expansion $g_0 = \sum_{j \geqslant d} g_{0,j}$, with $g_{0,j}$ a homogeneous polynomial of degree j, and $g_{0,d} \neq 0$. Set

$$Z := g_{0,d}^{-1}(0) \subset \mathbf{P}^2.$$

Then the condition that $g_0^{-1}(0)$ is a Yomdin singularity [Yom75] means that Z has only isolated singularities, $g_{0,j} = 0$ for $d < j < k$, and $g_{0,d+k}^{-1}(0) \cap \mathrm{Sing}\, Z = \emptyset$ (see [ALM06]). It is a super-isolated singularity [Lue87] if $k = 1$. We show that the embedded resolution can be obtained by repeating blowing-ups with point or $\mathbf{P}^1$-centers.

We first take the blow-up $\sigma_1 : X_1 \to X_0 = X'$ at $0 \in X'$. Its exceptional divisor E_0 is $\mathbf{P}^2$, and the intersection of E_0 with the proper transform of $g_0^{-1}(0)$ is identified with $Z \subset \mathbf{P}^1$. Moreover, the total transform of $g_0^{-1}(0)$ around a singular point of Z can be defined locally by an equation of the form

$$w^d(h(u,v) + w^k) = 0, \tag{3.8.1}$$

where (u, v, w) is a local coordinate system such that the exceptional divisor $E_0 = \mathbf{P}^2$ is locally defined by $w = 0$, and $Z \subset \mathbf{P}^2$ is defined by $h(u,v) = 0$. Here, the restrictions of x, y to $\mathbf{P}^2$ are identified with local coordinates of $\mathbf{P}^2$. Indeed, take a coordinate system (x, y, z) of $\mathbf{C}^3$. Set $h_j := g_{0,j}/z^j$. This is viewed as a function on the complement of $\{z = 0\} \subset E_0 = \mathbf{P}^2$. Then, the pull-back of g_0 to the complement of the proper transform of $\{z = 0\} \subset \mathbf{C}^3$ is expressed as

$$z^d(h_d + z^k(h_{d+k} + z h_{d+k+1} + \cdots)),$$

where z also denotes the pull-back of z, which locally defines the exceptional divisor E_0. So (3.8.1) follows by setting locally

$$h := h_d, \quad w := z(h_{d+k} + z h_{d+k+1} + \cdots)^{1/k}.$$

Repeating point-center blow-ups at singular points of the total transform of Z in the proper transform of $E_0 = \mathbf{P}^2$, we then get a morphism

$$\sigma_2 : X_2 \to X_1,$$

such that the intersection of the total transform of Z with the proper transform $\widetilde{E}_0$ of E_0 is a divisor with simple normal crossings on $\widetilde{E}_0$. (Here, we use the fact that the restriction of a point-center blow-up to the proper transform of a smooth divisor is a point-center blow-up.) Moreover, we may assume that any two irreducible components of the proper transform of Z do not intersect each other (taking a point-center blow-up at the intersection point if necessary).

Applying a point-center blow-up to (3.8.1), the local coordinate system (u, v, w) is substituted by (u, uv, uw) or (uv, v, vw) near the proper transform of E_0. Repeating this, the total transform of $g_0^{-1}(0)$ by $\sigma_1 \circ \sigma_2$ is locally defined by

$$u^i v^j w^l(u^a v^b + w^c) = 0 \quad \text{with } i, j \geqslant 0,\ l, a, b, c > 0, \tag{3.8.2}$$

using a local coordinate system (u, v, w), where $l = d$, $c = k$. We have $a = 1$ if $i = 0$, and $b = 1$ if $j = 0$. Note that the nonnormal crossing points of (3.8.2) are contained in the union of $\{u = w = 0\}$ and $\{v = w = 0\}$.

By the above construction, the nonnormal crossing points of the total transform of $g_0^{-1}(0)$ consist of a union of smooth rational curves. In order to apply Proposition 3.7, it is then sufficient to show that (3.8.2) is essentially *stable* by blowing-ups along the origin or along the coordinate axes. Indeed, it is known that Hironaka's resolution can be obtained by repeating blow-ups with smooth centers contained in the set of nonnormal crossing points, and the *new components* of the set of nonnormal crossing points which are obtained by a blow-up of the divisor defined by the equation of the form (3.8.2) are also *rational curves*. Here, 'essentially' means that we allow $a, b, c \geqslant 0$ together with a *permutation of variables*, and that we may get an equation which is not of the form (3.8.2) if the equation defines a divisor with normal crossings, as explained below. (It may be possible to give a more explicit algorithm by induction on the maximum

of a, b, c, although this seems more complicated than we might imagine. Indeed, a resolution of singularities is *global* on X_2 and a *local description* using the Euclidean algorithm at each point of X_2 is not enough. Here, permutations of variables make the argument rather complicated.)

In case of a point-center blow-up, (u, v, w) is substituted in (3.8.2) by

$$(u, uv, uw) \quad \text{or} \quad (uv, v, vw) \quad \text{or} \quad (uw, vw, w).$$

In case of the blow-up along $\{u = w = 0\}$, (u, v, w) is substituted in (3.8.2) by

$$(u, v, uw) \quad \text{or} \quad (uw, v, w).$$

and similarly for $\{u = w = 0\}$ with u replaced by v. By these substitutions, (3.8.2) is essentially stable except for the case when we get a local equation of the form

$$u^i v^j w^l (u^a v^b w^c + 1) = 0. \tag{3.8.3}$$

However, this defines a divisor with normal crossings, and we do not have to consider it. So the assertion follows from Proposition 3.7. More precisely, under a substitution by (u, uv, uw) or (u, v, uw) for instance, only a changes and b, c do not change in (3.8.2) if we allow a negative, and we need a permutation of variables if we want to get $a, b, c \geqslant 0$. We may have (3.8.3) under a substitution by (uw, vw, w) or (uw, v, w) or (u, vw, w). Then, repeating the blow-ups consisting of Hironaka's resolution (or using the Euclidean algorithm essentially), we will reach local equations of the form

$$u^i v^j w^l (v^c + w^c) = 0 \quad \text{or} \quad v^j w^l (uv^c + w^c) = 0 \quad \text{with } c \geqslant 1, \tag{3.8.4}$$

just before getting a divisor with normal crossings by blowing-up along $\{v = w = 0\}$. Here, the obtained equation depends on whether or not we started from (3.8.2) with $i, j \geqslant 1$, $ab \neq 0$. In the latter case, if we start from (3.8.2) with $i = 0$, $a = 1$, then we can apply the Euclidean algorithm to b, c in (3.8.1) since we get a divisor with normal crossings by an equation of the form

$$v^j w^l (u + v^b w^c) = 1.$$

This finishes the proof of Proposition 3.8. □

Remark 3.9. In the case of super-isolated singularities or, more generally, Yomdin singularities with $n \geqslant 2$, formulas are known for the Milnor number, the characteristic polynomial of the Milnor monodromy, and also for the spectrum (see [ALM06, LM95, Sie90, Ste89, Yom75]).

In fact, Steenbrink [Ste89, Theorem 6.1] proved a formula for the spectrum of a homogeneous polynomial f with one-dimensional singular locus, which can be expressed for instance (using the normalization as in [Sai91]) as follows:

$$\mathrm{Sp}(f, 0) = \left(\frac{t - t^{1/d}}{t^{1/d} - 1} \right)^{n+1} - \sum_{i,j} t^{\alpha'_{i,j}} \frac{t - 1}{t^{1/d} - 1}, \tag{3.9.1}$$

where $\alpha'_{i,j} := (\lfloor \alpha_{i,j} d \rfloor + 1)/d$ with $\lfloor \alpha \rfloor := \max\{p \in \mathbf{Z} \mid p \leqslant \alpha\}$, and the $\alpha_{i,j}$ are the exponents, that is, the spectral numbers counted with multiplicities at each singular point y_i of $f^{-1}(0) \subset \mathbf{P}^n$. Note that (3.9.1) is quite useful for calculations of the spectrum in this case; for instance, the formula in [BS10, Theorem 3] for the spectrum of reduced hyperplane arrangements in $\mathbf{C}^3$ follows from it.

We may view (3.9.1) as a special case (with $k = 0$) of Steenbrink's conjecture in [Ste89], which was proved there in the case when f is homogeneous and the isolated singularities are of

Brieskorn type (and in [Sai91] in general). In this case, the latter can be expressed as follows:

$$\mathrm{Sp}(f + h^{d+k}, 0) - \mathrm{Sp}(f, 0) = \sum_{i,j} t^{\alpha''_{i,j}(k)} \frac{t-1}{t^{1/d+k} - 1} \quad (k \geqslant 0), \qquad (3.9.2)$$

where $\alpha''_{i,j}(k) := (k\alpha_{i,j} + \lfloor \alpha_{i,j}d \rfloor + 1)/(d+k)$, and h is a sufficiently general linear function (see also [ALM06, Theorem 1.4]). Indeed, in the homogeneous polynomial case, there is a well-known relation between the Milnor monodromy and the local system monodromy along $\mathbf{C}^* \subset \mathrm{Sing}\, f^{-1}(0)$ so that $\beta_{i,j}$ in [Sai91, (0.1)] satisfies the relation

$$\alpha_{i,j}d + \beta_{i,j} \in \mathbf{Z}. \qquad (3.9.3)$$

Combining this with the condition $\beta_{i,j} \in (0, 1]$, we get

$$\alpha_{i,j}d + \beta_{i,j} = \lfloor \alpha_{i,j}d \rfloor + 1 \quad \text{and} \quad \alpha''_{i,j}(k) = ((d+k)\alpha_{i,j} + \beta_{i,j})/(d+k). \qquad (3.9.4)$$

The lower bound of k in (3.9.2) is 0, since the number R in [Sai91, Theorem 2.5] is d in this case. (This can be shown by using the natural $\mathbf{C}^*$-action.)

Note that ((3.9.1)–(3.9.2)) imply a formula for the spectrum of Yomdin singularities as in [ALM06, Theorem 1.4] (using the constancy of the spectrum by μ-constant deformations). We can verify that the normalization of the formulas ((3.9.1)–(3.9.2)) is correct, for instance, in a simple case when $f := xyz$ (i.e. of type $T_{\infty,\infty,\infty}$: see [Ste89]) with $n = 2$, $d = 3$, and $f' := f + x^p + y^p + z^p$ (i.e. of type $T_{p,p,p}$) for $p = k + 3 > 3$. In this case, we have $\alpha_{i,1} = \beta_{i,1} = 1$ for $i = 1, 2, 3$, and

$$\mathrm{Sp}(f, 0) = t - 2t^2, \quad \mathrm{Sp}(f', 0) = \mathrm{Sp}(f, 0) + 3 \sum_{l=1}^{p} t^{1+l/p}.$$

(There is a shift by one between the normalizations of the spectrum in [Ste89] and in [Sai91].)

Since Steenbrink's conjecture is generalized to the case of spectral pairs [NS94], it would imply a certain formula for the number of Jordan blocks of the Milnor monodromy of Yomdin singularities by using the monodromical property of the weight filtration [Ste77].

3.10 A criterion

In the case of Theorem 3, we can determine whether the equality $B^j_{f,\lambda} = C^j_{f,\lambda}$ for $\lambda \neq 1$ holds in certain cases as follows. Here, we consider a slightly more general situation where $f : X \to \Delta$ is obtained by an embedded resolution of the singular fiber $f'^{-1}(0)$ of a morphism of complex manifolds $f' : X' \to \Delta$, where the singularities of $f'^{-1}(0)$ are not necessarily isolated. We assume that the resolution is given by the composition of blow-ups with connected smooth centers

$$\sigma_i : X_i \to X_{i-1} \quad (i = 1, \ldots, r),$$

where $X_0 = X'$ and $X_r = X$. Let $E_i \subset X_i$ be the exceptional divisor of σ_i with D_i its proper transform in X. Let m_i be the multiplicity of Y along D_i. Let g_i be the pull-back of f' to X_i.

Fix some $i \in [1, r]$ with $m_i/m_\lambda \in \mathbf{Z}$. Let Z be a closed subvariety of $D_i \cap Y^{(\lambda)}$ such that the canonical morphism $\pi_i : X \to X_i$ induces a morphism of Z to its image Z' in X_i with *connected fibers*. Assume that there is a meromorphic function h_i on a neighborhood $U_{Z'}$ of $Z' \subset X_i$ (in classical topology) satisfying the following three conditions.

(i) The zeros of the pull-back of h_i in a sufficiently small open neighborhood U_Z of Z in $\pi_i^{-1}(U_{Z'})$ are contained in Y.

(ii) The order of zero of h_i along E_i is m_i/m_λ.

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(iii) The restriction of $g_i' := g_i / h_i^{m_\lambda}$ to $U_{Z'} \cap E_i$ is a meromorphic function having finite values on dense Zariski-open subsets of any intersections of irreducible components of Z'.

We then have the following (which will be used in § 4.3 below).

PROPOSITION 3.11. *With the above notation and assumption, there is a global solution of the equation $\xi^{m_\lambda} = f$ on a sufficiently small neighborhood of Z if and only if there is a global solution of $\xi'^{m_\lambda} = g_i'|_{E_i}$ on Z'.*

Proof. Let g' and h, respectively, denote the pull-backs of g_i' and h_i to $U_Z \subset X$. Then $g' = f / h^{m_\lambda}$, and it is enough to consider the global solvability of $\xi^{m_\lambda} = g'$. By hypothesis, the zeros and poles of g' are contained in Y, and it has finite values generically on $U_Z \cap D_i$. Hence we can take the pull-back of g_i' after restricting it to $U_{Z'} \cap E_i$. Then the assertion follows from the hypothesis on the connectivity of the fibers of the morphism $Z \to Z'$. This finishes the proof of Proposition 3.11. $\square$

Remarks 3.12. (i) In Proposition 3.11, it is essential to consider the restriction of g_i' to the intersection with E_i, since $h_i^{-1}(0)$ is not necessarily contained in $g_i^{-1}(0)$ on a neighborhood of Z' in X_i, even though we have the inclusion on a neighborhood of Z in X after taking the pull-back because of a blow-up with center contained in the proper transform of $h_i^{-1}(0) \cap E_i$. This will be used in § 4.3.

(ii) By Proposition 3.6 for $Z^{(\lambda)} = Y^{(\lambda)}$, the global solvability of the equation $\xi^{m_\lambda} = f$ on a sufficiently small open neighborhood of $Y^{(\lambda)}$ implies the equality $B_{f,\lambda}^\bullet = C_{f,\lambda}^\bullet$.

4. Examples

In this section, we give some interesting examples and prove Theorem 1 in 4.3.

Example 4.1. Let E be an elliptic curve with the origin O. Let P be a torsion point of E with order $m > 1$. Let X be the blow-up of $E \times E$ along the two points (O, P), (P, O). Let

$$D_0 = E \times \{O\}, \quad D_0' = \{O\} \times E, \quad D_\infty = E \times \{P\}, \quad D_\infty' = \{P\} \times E,$$

and $\widetilde{D}_0$, $\widetilde{D}_0'$, $\widetilde{D}_\infty$, $\widetilde{D}_\infty'$ be their proper transforms. Then we have a rational function f on X defining a morphism of algebraic varieties $f : X \to \mathbf{P}^1$, and satisfying

$$\mathrm{div}\, f = m\widetilde{D}_0 + m\widetilde{D}_0' - m\widetilde{D}_\infty - m\widetilde{D}_\infty'.$$

Indeed, there is a rational function g on E with $\mathrm{div}\, g = mO - mP$ by Abel's theorem for elliptic curves, and f is the pull-back of $pr_1^* g \cdot pr_2^* g$, where pr_1, pr_2 are the first and second projections.

However, there is no univalued holomorphic function g with $g^a = f$ for $a > 1$, even on a sufficiently small analytic neighborhood of $f^{-1}(0)$ in X, since the general fibers of f are connected. Indeed, we have finite morphisms $\mathbf{P}^1 \to S \xrightarrow{\rho} \mathbf{P}^1$, where the first $\mathbf{P}^1$ is an exceptional divisor of the blow-up, and ρ is the Stein factorization of f. The composition is given by the restriction of f, and is a ramified covering of degree m which is ramified only at 0 and ∞. Then ρ is an isomorphism (i.e. the general fibers of f are connected), since otherwise there is a rational function g on X with $g^a = f$ for $a > 1$, contradicting the fact that there is no rational function g' on E with $\mathrm{div}\, g' = m'O - m'P$ for $0 < m' < m$ (by restricting to $E \times \{Q\}$ for a general point $Q \in E$).

A similar assertion holds by restricting to a neighborhood of $\widetilde{D}_0$ or $\widetilde{D}_0'$. Here, we use the first cohomology $H^1(f^{-1}(0), \mu_m)$ as in § 3.1. This gives an example with $\chi(B_{f,\lambda}^\bullet) \neq \chi(C_{f,\lambda}^\bullet)$

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for $\lambda \in \mu_m \backslash \{1\}$. More precisely, we have for $\lambda \in \mu_m \backslash \{1\}$

$$B_{f,\lambda}^0 = 0, \quad C_{f,\lambda}^0 = \mathbf{C} \oplus \mathbf{C}, \quad B_{f,\lambda}^1 = C_{f,\lambda}^1 = \mathbf{C}.$$

In this case, a general fiber X_t is a connected curve of genus $m + 1$ (using, for instance, the Riemann–Roch theorem on X). Let $H^j(X_\infty, \mathbf{Q})$ be the limit mixed Hodge structure, and let $H^j(X_\infty, \mathbf{C})_\lambda$ be the λ-eigenspace of the monodromy. Calculating the E_1-complex of the weight spectral sequence, we get

$$\mathrm{Gr}_k^W H^j(X_\infty, \mathbf{Q})_1 = \begin{cases} \mathbf{Q} & \text{if } (j,k) = (0,0), \\ H^1(E, \mathbf{Q}) \oplus H^1(E, \mathbf{Q}) & \text{if } (j,k) = (1,1), \\ \mathbf{Q}(-1) & \text{if } (j,k) = (2,2), \\ 0 & \text{otherwise,} \end{cases}$$

and for $\lambda \in \mu_m \backslash \{1\}$

$$\mathrm{Gr}_k^W H^j(X_\infty, \mathbf{C})_\lambda = \begin{cases} \mathbf{C} & \text{if } (j,k) = (1,0), \\ \mathbf{C}(-1) & \text{if } (j,k) = (1,2), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\nu_{f,1}^1 = 0$ and $\nu_{f,\lambda}^1 = 1$ for any $\lambda \in \mu_m \backslash \{1\}$. (This is the first example with $B_{f,\lambda}^j \neq C_{f,\lambda}^j$, and it was rather surprising.)

Example 4.2. Let C be an elliptic curve embedded in $\mathbf{P}^2$, and let L_i be three lines in $\mathbf{P}^2$ intersecting C only at one point P_i with intersection multiplicity 3 for $i = 1, 2, 3$ (i.e. the P_i are inflection points), and such that $\bigcap_{i=1}^3 L_i = \emptyset$. Let h, h' be homogeneous polynomials h, h' of degree 3 defining C and $\bigcup_{i=1}^3 L_i$, respectively. Using coordinates, we have

$$h = x^3 + 3\alpha^2 x^2 y + 3\alpha xy^2 + y^3 + 3(x^2 + y^2)z + 3(x + y)z^2 + z^3 + cxyz,$$
$$h' = xyz, \quad \text{where } \alpha^3 = 1, \text{ and } c \in \mathbf{C} \text{ is generic.}$$

We assume that $\alpha \neq 1$. This is equivalent to the following condition:

$$\text{The three points } P_1, P_2, P_3 \text{ are } not \text{ on the same line in } \mathbf{P}^2. \tag{A}$$

Here, we may assume that P_3 is the origin O of the elliptic curve. Then P_1, P_2 are torsion points of order 3, and condition (A) is equivalent to the condition $P_1 + P_2 \neq O$.

Set

$$g' := h^3 h' : \mathbf{C}^3 \to \mathbf{C}.$$

We have an embedded resolution $U' \to \mathbf{C}^3$ of $g'^{-1}(0)$ by first blowing up the origin, and then repeating the blowing-ups along the proper transforms of the affine cone of $C \cap L_i$ in $\mathbf{C}^3$ three times for each i. Then, the composition $U' \to \mathbf{C}$ of the resolution and g' can be extended to a projective morphism $f : X \to \mathbf{C}$ such that X is smooth and $(X \backslash U') \cup f^{-1}(0)$ is a divisor with normal crossings. However, U' may be different from U in the introduction, since $X \backslash U'$ may contain some vertical divisors.

Let $D_0' \subset U'$ be the proper transform of the exceptional divisor $\mathbf{P}^2$ of the first blow-up. Let $D_i' \subset U'$ be the exceptional divisor of the last blow-up of the successive three blow-ups along the proper transforms of the affine cone of $C \cap L_i$ for $i = 1, 2, 3$. Let $D_4' \subset U'$ be the proper transform of the affine cone of C. Let D_i be the closure of D_i' in X for $i = 0, \dots, 4$, where $D_0 = D_0'$.

Let m_i be the multiplicity of D_i. Then $m_i = 12$ for $i = 0, \dots, 3$, and $m_4 = 3$. The multiplicities of the exceptional divisors of the first and second blow-ups along the proper transforms

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of the affine cone of $C \cap L_i$ are 4 and 8, respectively, and are not divisible by 3. So the D_i for $i = 0, \ldots, 4$ are irreducible components of $f^{-1}(0)$ with multiplicities divisible by 3, and $D_{\{1,4\}} := D_0 \cap D_4 \subset U'$ does not intersect the irreducible components of $f^{-1}(0)$ other than D_i ($i = 0, \ldots, 4$). We thus get a unramified covering of degree 3

$$\widetilde{D}_{\{1,4\}} \to D_{\{1,4\}},$$

which is nontrivial by condition (A). (Indeed, using the coordinates $u = x/z$, $v = y/z$, $w = z$ of the blow-up at the origin, the pull-back of g' is written as $\big(h(u, v, 1)w^4\big)^3 uv$. So it is enough to show the nonexistence of a rational function ξ on C satisfying $\xi^3 = uv|_C$. Since $\mathrm{div}(uv|_C) = 3P_1 + 3P_2 - 6P_3$, the assertion follows from the remark after condition (A).) We thus get

$$B^1_{f,\omega} \neq C^1_{f,\omega} \quad \text{with } \omega = \exp(\pm 2\pi i/3).$$

In this case, the local monodromy is semisimple, since f is homogeneous. In particular, $\nu^j_{f_U, \lambda} = 0$ for $j = 1, 2$. This example is needed for the proof of Theorem 1 below. Note that some related results are obtained in [Art94b, AC98].

4.3 Proof of Theorem 1

With the notation of Example 4.2, set

$$g_0 := h^3 h' + h'',$$

where h'' is a homogeneous polynomial of degree 16 such that $h''^{-1}(0) \subset \mathbf{P}^2$ is smooth and transversely intersects $\bigcup_i L_i \cup C$ at smooth points. Let f be a desingularization of a good projective compactification g of g_0 as in Theorem 3. Here, the desingularization is given by the embedded resolution of $g_0^{-1}(0) \subset (\mathbf{C}^3, 0)$ constructed below.

Blow up the origin of $\mathbf{C}^3$ with E_0 the exceptional divisor. This contains $\bigcup_i L_i \cup C$ as its intersection with the proper transform of $g_0^{-1}(0)$. At each singular point P_i of $L_i \cup C \subset E_0 = \mathbf{P}^2$, the pull-back of g_0 can be written locally as

$$(v^3(v - u^3) - w^4)w^{12},$$

using appropriate analytic local coordinates u, v, w. Here, E_0 is locally defined by $w = 0$, and u, v induce local coordinates of E_0 such that C and L_i are, respectively, defined by $v = 0$ and $v = u^3$ locally on E_0. (Note that we have w^4 in the above function, since $\deg h'' = 16$. The following argument about the point-center blow-ups does not work well unless $\deg h'' = 16$.) We repeat point-center blow-ups three times at the singular point P_i. Here, u, v, w are, respectively, substituted by u, uv, uw each time. After these three blow-ups, we get

$$(v^3(v - 1) - w^4)u^{48}w^{12}.$$

Here, the proper transform E_0' of E_0 is locally defined by $w = 0$, and the proper transforms of C, L_i, which will be denoted by C', L_i' respectively, are defined by $v = 0$ and $v = 1$ locally on E_0'. So C' and L_i' do not intersect each other. Let E_i denote the exceptional divisor of the last blow-up for each $i = 1, 2, 3$. This is locally defined by $u = 0$ using the above coordinates after taking the three blow-ups, and transversally intersects C' and L_i', as is seen by the above description.

The total transform of $g_0^{-1}(0)$ still has singularities along C'. These can be resolved by repeating the blow-ups with center isomorphic to C' four times. Indeed, the pull-back of g_0 is generically given by the function

$$(v^3 - w^4)w^{12},$$

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after restricting to a hyperplane transversal to C'. Here, v, w are, respectively, replaced with vw, w by the first blow-ups, and with v, vw by the remaining three blow-ups. We do not have a problem at the intersection point of C' and E_i, since the intersection is transversal, as is seen by the above equation. However, the calculation at the intersection of C' with the proper transform of $h''^{-1}(0)$ is rather nontrivial. (The latter does not intersect E_i for $i = 1, 2, 3$ by the assumption on h''.) Using appropriate analytic local coordinates u, v, w, the pull-back of g_0 can be written as

$$(v^3 - uw^4)w^{12},$$

where E'_0, C', and the intersection of E'_0 with the proper transform of $h''^{-1}(0)$ are, respectively, defined by $w = 0$, $v = w = 0$, and $u = w = 0$. By the successive blow-ups, u, v, w are substituted by u, vw, w or u, v, vw, depending on the two affine charts each time. By the first blow-up, we get

$$(v^3 - uw)w^{15} \quad \text{and} \quad (1 - uvw^4)v^{15}w^{12},$$

on the two affine charts. Here, we do not have to consider the second, since $1 - uvw^4 \neq 0$ if $w = 0$. By the second blow-up, we then get

$$(v^3 w^2 - u)w^{16} \quad \text{and} \quad (v^2 - uw)v^{16}w^{15}.$$

Here, we do not have to consider the first, since $(v^3 w^2 - u)w^{16}$ defines a divisor with normal crossings. The argument is similar for the third and fourth blow-ups.

Let E_4 and E_5, respectively, denote the exceptional divisor of the first and the last blow-up of the successive four blow-ups. Let D_i be the proper transform of E_i in X for $i = 0, \ldots, 5$. These are the irreducible components with multiplicity divisible by 3, and D_4 does not intersect the irreducible components with multiplicity nondivisible by 3. Moreover, $D_{\{i,j\}} := D_i \cap D_j$ does not intersect the irreducible components with multiplicity nondivisible by 3 if and only if $4 \in \{i, j\}$ (i.e. $\{i, j\} = \{0, 4\}$, $\{4, 5\}$, $\{i, 4\}$ with $i = 1, 2, 3$). Here, $D_{\{0,4\}}$ and $D_{\{4,5\}}$ are isomorphic to the original elliptic curve C. We have the unramified coverings of degree 3

$$\widetilde{D}_4 \to D_4, \quad \widetilde{D}_{\{0,4\}} \to D_{\{0,4\}}, \quad \widetilde{D}_{\{4,5\}} \to D_{\{4,5\}},$$

which are compatible with the base changes by the inclusions

$$D_{\{0,4\}} \hookrightarrow D_4, \quad D_{\{4,5\}} \hookrightarrow D_4,$$

and also by the canonical projections

$$D_4 \to D_{\{0,4\}}, \quad D_4 \to D_{\{4,5\}}.$$

These coverings are *nontrivial* by condition (A). (Indeed, we apply Proposition 3.11 to the case when Z, E_i, and h_i in Proposition 3.11 are, respectively, $D_{\{0,4\}}$, D_0, and $h(u, v, 1)w^4$ using the coordinates u, v, w as in Example 4.2. Then the nontriviality follows from the remark after condition (A).) On the other hand, $D_{\{i,4\}}$ is $\mathbf{P}^1$ for $i = 1, 2, 3$, and we have the triviality of the unramified covering

$$\widetilde{D}_{\{i,4\}} \to D_{\{i,4\}} \quad (i = 1, 2, 3).$$

Setting $b^j_{f,\lambda} := \dim B^j_{f,\lambda}$, $c^j_{f,\lambda} := \dim C^j_{f,\lambda}$, we then get, for $\omega = \exp(\pm 2\pi i/3)$,

$$b^0_{f,\omega} = 0, \quad b^1_{f,\omega} = 3, \quad b^2_{f,\omega} = 6,$$
$$c^0_{f,\omega} = 1, \quad c^1_{f,\omega} = 5, \quad c^2_{f,\omega} = 6.$$

Hence $\chi(B^\bullet_{f,\omega}) \neq \chi(C^\bullet_{f,\omega})$, and we have $\nu^2_{g_0,\omega} = \chi(B^\bullet_{f,\omega}) = 3$ by Theorem 3.

A similar argument shows that $b_{f,\omega}^j = c_{f,\omega}^j$ and hence $\nu_{g_0,\omega}^2 = 2$ in the case when condition (A) is *not* satisfied; that is, if $\alpha = 1$. This shows that there is no simple formula for $\nu_{g_0,\lambda}^j$ using only the combinatorial data of the desingularization of g_0 in general. So Theorem 1 follows.

Example 4.4. Assume that f is obtained by taking the minimal resolution of a good projective compactification $f' : X' \to \Delta$ of a germ of a holomorphic function at $0 \in \mathbf{C}^2$ defined by

$$g_0 := (x^{2a} + y^2)(x^2 + y^{2a}) \quad \text{for } a \geqslant 2.$$

In this case, f is obtained by repeating point-center blow-ups $2a - 1$ times, where all the exceptional divisors have even multiplicities, but the proper transforms of the irreducible components of $g^{-1}(0)$ have multiplicity 1. (This coincides with the resolution obtained by taking a smooth subdivision of the dual fan of the Newton polygon.)

We have $B_{f,-1}^\bullet = C_{f,-1}^\bullet$ for $\lambda = -1$ by Theorem 4 and, moreover,

$$\dim C_{f,-1}^0 = 2a - 3, \quad \dim C_{f,-1}^1 = 2a - 2.$$

So we get $\nu_{g_0,-1}^1 = 1$ by Theorem 3. This assertion also follows from a theorem in [Ste77] for the mixed Hodge numbers of the Milnor cohomology in the nondegenerate Newton boundary case with $\dim X = 2$. (This example shows that the estimate in [MT12], which is given by $\dim C_{f,\lambda}^j$, is not very good in general.) Note that some related argument using a $\mathbf{Q}$-resolution is given in [Mar12].

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