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A MULTIPLIER INCLUSION THEOREM ON PRODUCT DOMAINS

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Abstract In this note it is shown that the class of all multipliers from the d-parameter Hardy space $H^1_{\mathrm{prod}}(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$ is properly contained in the class of all multipliers from $L\log^{d/2} L(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$.

Keywords: multiplier inclusion; product Hardy spaces; higher-dimensional Zygmund inequality; Littlewood–Paley square function

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1. Introduction

Let d be a positive integer. If X is a subspace of $L^1(\mathbb{T}^d)$, then we denote by $\mathcal{M}_{X\to L^2(\mathbb{T}^d)}$ the class of all multipliers from X to $L^2(\mathbb{T}^d)$, that is, the class $\mathcal{M}_{X\to L^2(\mathbb{T}^d)}$ consists of all functions $m : \mathbb{Z}^d \to \mathbb{C}$ such that for every $f \in X$ one has

$$\sum_{(k_1,\dots,k_d)\in\mathbb{Z}^d} |m(k_1,\dots,k_d)\widehat{f}(k_1,\dots,k_d)|^2 < \infty.$$

In [**1**], it is shown that the class of all multipliers from the (real) Hardy space $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$ is properly contained in the class of all multipliers from $L\log^{1/2} L(\mathbb{T})$ to $L^2(\mathbb{T})$. Our aim in this note is to extend this result to the multi-parameter setting. First of all, note that if $H^1_{\mathrm{prod}}(\mathbb{T}^d)$ denotes the d-parameter (real) Hardy space over the d-torus, then $L\log^d L(\mathbb{T}^d) \subset H^1_{\mathrm{prod}}(\mathbb{T}^d)$ (see § 2.2), and hence one automatically has $\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^d)\to L^2(\mathbb{T}^d)} \subset \mathcal{M}_{L\log^d L(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}$. On the other hand, by adapting the argument given in [**1**] to the multi-parameter case, one deduces that the best we can expect is that $\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}$ is contained in $\mathcal{M}_{L\log^{d/2} L(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}$. In this note we prove that this is indeed the case, that is, we strengthen the trivial exponent $r = d$ in $L\log^r L(\mathbb{T}^d)$ to the optimal one, $r = d/2$. In particular, our main result in this note is the following theorem.

Theorem 1.1. *One has the inclusion*

$$\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^d)\to L^2(\mathbb{T}^d)} \subset \mathcal{M}_{L\log^{d/2} L(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}. \tag{1.1}$$

Moreover, this inclusion is proper and it is sharp, in the sense that the exponent $r = d/2$ in $L\log^{d/2} L(\mathbb{T}^d)$ cannot be improved.

The multiplier inclusion (1.1) is obtained by a series of reductions. First, arguing as in [1] and using Oberlin's characterization of the class $\mathcal{M}_{H^1_{\text{prod}}(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}$ given in [14], it follows that the proof of (1.1) is reduced to showing the following higher-dimensional version of an inequality due to Zygmund (see [20, Chapter XII, Theorem 7.6]), a result of independent interest. To state this version of Zygmund's inequality on $\mathbb{T}^d$, let $\mathcal{J}$ denote the set of all 'intervals' of integers of the form $\pm\{2^n-1,\dots,2^{n+1}-2\}$, $n\in\mathbb{N}_0$; in other words, $\mathcal{J}$ consists of all the sets in $\mathbb{Z}$ of the form $\{2^k-1,\dots,2^{k+1}-2\}$, $k\in\mathbb{N}_0$ and $\{-2^{l+1}+2,\dots,-2^l+1\}$, $l\in\mathbb{N}_0$.

Proposition 1.2. *Let $\mathcal{J}$ be as above. If $E\subset\mathbb{Z}^d$ is a non-empty set satisfying the condition*

$$D_E = \sup_{I_1,\dots,I_d\in\mathcal{J}} \#\{E\cap(I_1\times\cdots\times I_d)\} < \infty, \tag{1.2}$$

then there exists a positive constant A_{D_E}, depending only on D_E, such that

$$\Bigg(\sum_{(k_1,\dots,k_d)\in E} |\widehat{f}(k_1,\dots,k_d)|^2\Bigg)^{1/2} \le A_{D_E}\Bigg[1+\int_{\mathbb{T}^d}|f|\log^{d/2}(1+|f|)\Bigg]. \tag{1.3}$$

In turn, (1.3) will be a corollary of a higher-dimensional extension of a result due to Seeger and Trebels [19] concerning sharp bounds of sums involving 'smooth' Littlewood–Paley projections on $\mathbb{T}^d$. To state this result, fix a Schwartz function η supported in $(-2,2)$ such that $\eta|_{[-1,1]}\equiv 1$, and consider $\phi(\xi)=\eta(\xi)-\eta(2\xi)$. For $k\in\mathbb{N}$, set $\phi_k(\xi)=\phi(2^{-k}\xi)$, and for $k=0$, set $\phi_0=\eta$. One can easily see that $\sum_{k\in\mathbb{N}_0}\phi_k(\xi)=1$ for every $\xi\in\mathbb{R}$. Then, for $k\in\mathbb{N}_0$, the corresponding 'smooth' Littlewood–Paley projection in the periodic setting is defined by

$$\widetilde{\Delta}_k(f)(x) = \sum_{r\in\mathbb{Z}} \phi_k(r)\widehat{f}(r)\mathrm{e}^{\mathrm{i}2\pi rx}$$

for any, say, trigonometric polynomial f on $\mathbb{T}$. On the d-torus we put

$$\begin{aligned}\widetilde{\Delta}_{k_1,\dots,k_d}(f)(x_1,\dots,x_d) &= \widetilde{\Delta}_{k_1}\otimes\cdots\otimes\widetilde{\Delta}_{k_d}(f)(x_1,\dots,x_d)\\ &= \sum_{r_1,\dots,r_d\in\mathbb{Z}} \phi_{k_1}(r_1)\cdots\phi_{k_d}(r_d)\widehat{f}(r_1,\dots,r_d)\mathrm{e}^{\mathrm{i}2\pi(r_1x_1+\cdots+r_dx_d)}\end{aligned}$$

initially defined over trigonometric polynomials f on $\mathbb{T}^d$. Then Proposition 1.2 is a consequence of the following result.

Proposition 1.3. *There exists a constant $C_d>0$, depending only on the dimension d and our choice of ϕ, such that the inequality*

$$\|f\|_{L^p(\mathbb{T}^d)} \le C_d p^{d/2}\Bigg(\sum_{k_1,\dots,k_d\in\mathbb{N}_0} \|\widetilde{\Delta}_{k_1,\dots,k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)}\Bigg)^{1/2} \tag{1.4}$$

holds for every trigonometric polynomial f on $\mathbb{T}^d$ and for each $p>2$.

The proof of Proposition 1.3 is an adaptation of the work of Seeger and Trebels [19] to the higher-dimensional setting combined with a well-known inequality on multiple martingales; see § 2.3.

At this point, it should be mentioned that, as also remarked by Bourgain, Brezis, and Mironescu in [5] for $d = 1$, one expects that the constant $C_p(d)$ in the Littlewood–Paley inequality

$$\|f\|_{L^p(\mathbb{T}^d)} \leq C_p(d) \left\| \left(\sum_{k_1,\dots,k_d \in \mathbb{N}_0} |\widetilde{\Delta}_{k_1,\dots,k_d}(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T}^d)} \tag{1.5}$$

behaves like $C_p(d) \sim p^{d/2}$ as $p \to \infty$, which of course, if true, would imply (1.4). Note that it is well known that $C_p(d) \lesssim p^d$ as $p \to \infty$; see, for example, [10, (6.1.31), p. 430]. Since (1.4) is sharp (see Remark 4.1), we deduce that $p^{d/2} \lesssim C_p(d) \lesssim p^d$ as $p \to \infty$. In this direction, see also Remark 5.3 where a stronger version of (1.4) is obtained. However, as our primary aim is to establish Theorem 1.1 and since (1.4) is enough for that purpose, we shall not pursue the problem of studying the sharp behaviour of $C_p(d)$ in (1.5) as $p \to \infty$ in the present note.

The paper is organized as follows. In § 2 we give some notation and background and in § 3 we show how the proof of our multiplier inclusion theorem follows from Proposition 1.2. In § 4 we prove that Proposition 1.3 implies Proposition 1.2, and then in § 5 we give a proof of Proposition 1.3. In the final section we briefly present some further applications of our work.

2. Notation and background

2.1. Notation

We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of positive integers, and by $\mathbb{N}_0$ the set of non-negative integers.

The cardinality of a finite set A is denoted by $\#\{A\}$.

If X and Y are positive quantities such that $X \leq CY$, where $C > 0$ is a constant, then we write $X \lesssim Y$. To specify the dependence of this constant on some additional parameters $\alpha_1, \dots, \alpha_n$ we write $X \lesssim_{\alpha_1,\dots,\alpha_n} Y$. If $X \lesssim Y$ and $Y \lesssim X$, we write $X \sim Y$.

In this note, we identify $\mathbb{T}$ with $[0, 1)$ in the usual way.

2.2. Product Hardy spaces and the class $\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^d) \to L^2(\mathbb{T}^d)}$

For $0 < r < 1$, let P_r denote the Poisson kernel on $\mathbb{T}$ given by

$$P_r(x) = (1 - r^2)/(1 - 2r\cos(2\pi x) + r^2),$$

$x \in \mathbb{T}$. For $x \in \mathbb{T}$, let $\Gamma(x) = \{z \in \mathbb{D} : |z - \mathrm{e}^{\mathrm{i}2\pi x}| \leq 2(1 - |z|)\}$, where $\mathbb{D}$ denotes the unit disc in the complex plane. Following [8], the d-parameter (real) Hardy space $H^1_{\mathrm{prod}}(\mathbb{T}^d)$ is defined as the space of all integrable functions f on the d-torus such that $f^* \in L^1(\mathbb{T}^d)$, where for $(x_1, \dots, x_d) \in \mathbb{T}^d$ one has

$$f^*(x_1, \dots, x_d) = \sup_{r_1 \mathrm{e}^{\mathrm{i}2\pi y_1} \in \Gamma(x_1), \dots, r_d \mathrm{e}^{\mathrm{i}2\pi y_d} \in \Gamma(x_d)} |f * (P_{r_1} \otimes \cdots \otimes P_{r_d})(y_1, \dots, y_d)|.$$

If $f \in H^1_{\mathrm{prod}}(\mathbb{T}^d)$, we set $\|f\|_{H^1_{\mathrm{prod}}(\mathbb{T}^d)} := \|f^*\|_{L^1(\mathbb{T}^d)}$.

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For $r \geq 0$, $L\log^r L(\mathbb{T}^d)$ denotes the class of all measurable functions f on $\mathbb{T}^d$ such that $\int_{\mathbb{T}^d} |f| \log^r(1+|f|) < \infty$. As mentioned in the introduction, one has the inclusion

$$L\log^d L(\mathbb{T}^d) \subset H^1_{\mathrm{prod}}(\mathbb{T}^d). \tag{2.1}$$

Indeed, to see this, note that if M denotes the centred Hardy–Littlewood maximal operator on $\mathbb{T}$, then there exists an absolute constant $C_0 > 0$ such that for each $g \in L^1(\mathbb{T})$ one has

$$\sup_{r\mathrm{e}^{\mathrm{i}2\pi y} \in \Gamma(x)} |g * P_r(y)| \leq C_0 M(g)(x) \tag{2.2}$$

for every $x \in \mathbb{T}$; see, for example, [**8**, p. 91]. Therefore, if M_i denotes the centred Hardy–Littlewood maximal operator acting on the ith variable ($i = 1, \ldots, d$), then it follows from (2.2) that

$$f^*(x_1, \ldots, x_d) \leq C_0^d M_1(M_2(\cdots(M_d(f))\cdots))(x_1, \ldots, x_d) \tag{2.3}$$

for every $f \in L^1(\mathbb{T}^d)$. Since M_i is bounded from $L\log^k L$ to $L\log^{k-1} L$ for $k \geq 1$ (see, for example, [**12**, Lemma E]), (2.3) implies that

$$\|f\|_{H^1_{\mathrm{prod}}(\mathbb{T}^d)} \lesssim_d 1 + \int_{\mathbb{T}^d} |f| \log^d(1+|f|)$$

and we thus deduce that (2.1) holds.

It follows from the work of Oberlin [**14**] that $m : \mathbb{Z}^d \to \mathbb{C}$ belongs to the class $\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^d) \to L^2(\mathbb{T}^d)}$ if and only if

$$\sup_{N_1, \ldots, N_d \in \mathbb{N}_0} \sum_{N_1 \leq |k_1| \leq 2N_1} \cdots \sum_{N_d \leq |k_d| \leq 2N_d} |m(k_1, \ldots, k_d)|^2 < \infty. \tag{2.4}$$

2.3. Dyadic square functions

If $f \in L^1(\mathbb{T})$ and $m \in \mathbb{N}_0$, then the mth conditional expectation of f is given by

$$\mathbb{E}_m(f)(x) = 2^m \int_I f(x')\,\mathrm{d}x',$$

where I is the unique dyadic interval in $\mathbb{T}$ of the form $I = [s2^{-m}, (s+1)2^{-m})$, $s = 0, 1, \ldots, 2^m - 1$, such that $x \in I$.

For $m \in \mathbb{N}$, let $\mathbb{D}_m = \mathbb{E}_m - \mathbb{E}_{m-1}$ denote the martingale differences acting on functions defined on $\mathbb{T}$. For $m = 0$, we set $\mathbb{D}_0 = \mathbb{E}_0$.

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For a given d-tuple $(m_1, \ldots, m_d)$ of non-negative integers, we define the corresponding operators acting on functions on the d-torus by

$$\mathbb{E}_{m_1,\ldots,m_d} = \mathbb{E}_{m_1} \otimes \cdots \otimes \mathbb{E}_{m_d}$$

and

$$\mathbb{D}_{m_1,\ldots,m_d} = \mathbb{D}_{m_1} \otimes \cdots \otimes \mathbb{D}_{m_d}.$$

More precisely, if $d > 1$ then, given $\mathbb{D}_{m_1,\ldots,m_{d-1}}$, we define

$$\mathbb{D}_{m_1,\ldots,m_d} = \mathbb{D}_{m_1,\ldots,m_{d-1}} \otimes \mathbb{D}_{m_d},$$

and so if $m_d = 0$ then we set $\mathbb{D}_{m_1,\ldots,m_d} = \mathbb{D}_{m_1,\ldots,m_{d-1}} \otimes \mathbb{E}_0$, and if $m_d \geq 1$ then $\mathbb{D}_{m_1,\ldots,m_d} = \mathbb{D}_{m_1,\ldots,m_{d-1}} \otimes (\mathbb{E}_{m_d} - \mathbb{E}_{m_{d-1}})$.

It follows from the work of Chang *et al.* [**6**], in particular from [**6**, Corollary 3.1], that

$$\|f\|_{L^p(\mathbb{T})} \leq Cp^{1/2} \left\| \left(\sum_{m\in\mathbb{N}_0} |\mathbb{D}_m(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})} \tag{2.5}$$

for all $p > 2$, where $C > 0$ is an absolute constant; see also, for example, [**19**, p. 152]. Moreover, Chang *et al.* obtained in [**6**] a result analogous to (2.5) involving Lusin area integrals. See also [**2**] and the references therein. In [**15**], Pipher extended (2.5) and its analogous version on Lusin area integrals to the two-parameter setting, and in [**9**], Fefferman and Pipher extended the aforementioned inequality of Chang *et al.* involving Lusin area integrals to ℓ^2-valued functions. The argument of Fefferman and Pipher [**9**] can easily be adapted to obtain an ℓ^2-valued extension of (2.5); see [**7**]. By using this ℓ^2-valued extension of (2.5) together with induction on d, one deduces that there exists a constant $C_d > 0$, depending only on the dimension $d \in \mathbb{N}$, such that

$$\|f\|_{L^p(\mathbb{T}^d)} \leq C_d p^{d/2} \left\| \left(\sum_{m_1,\ldots,m_d\in\mathbb{N}_0} |\mathbb{D}_{m_1,\ldots,m_d}(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T}^d)} \tag{2.6}$$

for every $p > 2$; see also [**3**] and [**7**, Proposition 4.5].

2.4. Thin sets in Harmonic analysis

Let G be a compact abelian group and let Λ be a non-empty set in its dual $\widehat{G}$. In this note, we shall only consider the case $G = \mathbb{T}^d$, $d \in \mathbb{N}$. A trigonometric polynomial f on G whose spectrum lies in Λ is said to be a Λ-polynomial.

Let $p > 2$. We say that $\Lambda \subset \widehat{G}$ is a $\Lambda(p)$ set if there exists a constant $A(p, \Lambda) > 0$ such that

$$\|f\|_{L^p(G)} \leq A(p, \Lambda)\|f\|_{L^2(G)}$$

for every Λ-polynomial f. The smallest constant $A(p, \Lambda)$ such that the above inequality holds is called the $\Lambda(p)$ constant of Λ.

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A set $\Lambda \subset \widehat{G}$ is called Sidon if there is a constant $S_\Lambda > 0$ such that

$$\sum_{\gamma \in \Lambda} |\widehat{f}(\gamma)| \leq S_\Lambda \|f\|_{L^\infty(G)} \tag{2.7}$$

for every Λ-polynomial. It follows from the work of Rudin [18] and Pisier [16] that a spectral set Λ is Sidon if, and only if, it is a $\Lambda(p)$ set for any $p > 2$ and its $\Lambda(p)$ constant grows like $p^{1/2}$ as $p \to \infty$.

Let $q \geq 1$. A set $\Lambda \subset \widehat{G}$ is said to be q-Rider if there is a constant $R_{\Lambda,q} > 0$ such that

$$\left(\sum_{\gamma \in \Lambda} |\widehat{f}(\gamma)|^q \right)^{1/q} \leq R_{\Lambda,q}[|f|] \tag{2.8}$$

for every Λ-polynomial. Here, we use the notation

$$[|f|] = \mathbb{E}\bigg[\Big\| \sum_{\gamma \in \widehat{G}} r_\gamma \widehat{f}(\gamma)\gamma \Big\|_{L^\infty(G)} \bigg],$$

where $(r_\gamma)_\gamma$ denotes the set of Rademacher functions.

It is well known that if Λ is a $\Lambda(p)$ set for all $p > 2$ with $\Lambda(p)$ constant growing as $p^{k/2}$, $k \in \mathbb{N}$, then Λ is a q-Rider set with $q = 2k/(k+1)$; see [17, Théorème 6.3].

3. Proposition 1.2 implies Theorem 1.1

To prove that Proposition 1.2 implies Theorem 1.1, we adapt the argument given in [1] to the multi-parameter setting by using the characterization of $\mathcal{M}_{H^1_{\text{prod}}(\mathbb{T}^d) \to L^2(\mathbb{T}^d)}$. To be more specific, assume that Proposition 1.2 holds and take an arbitrary m in the class $\mathcal{M}_{H^1_{\text{prod}}(\mathbb{T}^d) \to L^2(\mathbb{T}^d)}$. Then, by definition, we need to show that for every $f \in L \log^{d/2} L(\mathbb{T}^d)$ one has

$$\sum_{(k_1,\ldots,k_d) \in \mathbb{Z}^d} |m(k_1,\ldots,k_d)\widehat{f}(k_1,\ldots,k_d)|^2 < \infty.$$

Towards this aim, fix an $f \in L \log^{d/2} L(\mathbb{T}^d)$ and note that the sum

$$\sum_{(k_1,\ldots,k_d) \in \mathbb{Z}^d} |m(k_1,\ldots,k_d)\widehat{f}(k_1,\ldots,k_d)|^2$$

is bounded by

$$\sum_{I_1,\ldots,I_d \in \mathcal{J}} \max_{(k_1,\ldots,k_d) \in I_1 \times \cdots \times I_d} |\widehat{f}(k_1,\ldots,k_d)|^2 \left(\sum_{k_1 \in I_1} \cdots \sum_{k_d \in I_d} |m(k_1,\ldots,k_d)|^2 \right),$$

where $\mathcal{J}$ is as in the introduction and the statement of Proposition 1.2. Hence, by (2.4), it follows that

$$\sum_{(k_1,\ldots,k_d) \in \mathbb{Z}^d} |m(k_1,\ldots,k_d)\widehat{f}(k_1,\ldots,k_d)|^2 \lesssim_m \sum_{(\widetilde{k}_1,\ldots,\widetilde{k}_d) \in E_f} |\widehat{f}(\widetilde{k}_1,\ldots,\widetilde{k}_d)|^2,$$

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where E_f is a set in $\mathbb{Z}^d$ defined as follows. Given $I_1, \dots, I_d \in \mathcal{J}$, choose $(\widetilde{k}_1, \dots, \widetilde{k}_d)$ in $I_1 \times \cdots \times I_d$ so that

$$|\widehat{f}(\widetilde{k}_1, \dots, \widetilde{k}_d)| = \max_{(k_1,\dots,k_d)\in I_1\times\cdots\times I_d} |\widehat{f}(k_1, \dots, k_d)|.$$

Then, having chosen a set of d-tuples $(\widetilde{k}_1, \dots, \widetilde{k}_d)$ as above, we define

$$E_f = \{(\widetilde{k}_1, \dots, \widetilde{k}_d) \in \mathbb{Z}^d : \text{ for } I_1, \dots, I_d \in \mathcal{J},\ (\widetilde{k}_1, \dots, \widetilde{k}_d) \in I_1 \times \cdots \times I_d \text{ being as above}\}.$$

Notice that as the choice of d-tuples $(\widetilde{k}_1, \dots, \widetilde{k}_d)$ is not necessarily unique, there might be several choices of sets E_f. We just choose one of them to write

$$\sum_{I_1,\dots,I_d\in\mathcal{J}} \max_{(k_1,\dots,k_d)\in I_1\times\cdots\times I_d} |\widehat{f}(k_1, \dots, k_d)|^2 = \sum_{(\widetilde{k}_1,\dots,\widetilde{k}_d)\in E_f} |\widehat{f}(\widetilde{k}_1, \dots, \widetilde{k}_d)|^2.$$

Note that any such set E_f satisfies condition (1.2) in Proposition 1.2 with $D_{E_f} = 1$. Therefore, as $f \in L\log^{d/2} L(\mathbb{T}^d)$, it follows from (1.3) that

$$\sum_{(k_1,\dots,k_d)\in\mathbb{Z}^d} |m(k_1, \dots, k_d)\widehat{f}(k_1, \dots, k_d)|^2 < \infty,$$

as desired.

3.1. Sharpness of (1.1)

We remark that, in fact, the above argument shows that if $m \in \mathcal{M}_{H^1_{\text{prod}}(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}$, then there is a constant $C_m > 0$, depending only on m, such that

$$\left(\sum_{(k_1,\dots,k_d)\in\mathbb{Z}^d} |m(k_1, \dots, k_d)\widehat{f}(k_1, \dots, k_d)|^2\right)^{1/2} \le C_m\left[1 + \int_{\mathbb{T}^d} |f|\log^{d/2}(1 + |f|)\right].$$

To see that the exponent $r = d/2$ in $L\log^{d/2} L(\mathbb{T}^d)$ in (1.1) cannot be improved, we argue as in [**1**]. More specifically, assume that for some $r > 0$ every multiplier from $H^1_{\text{prod}}(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$ is a multiplier from $L\log^r L(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$. We shall prove that $r \ge d/2$. To this end, for a large positive integer N, take f to be a trigonometric polynomial on $\mathbb{T}^d$ given by $f = V_{2^N} \otimes \cdots \otimes V_{2^N}$, where $V_{2^N} = 2K_{2^{N+1}+1} - K_{2^N}$ denotes the de la Vallée Poussin kernel of order 2^N and K_n is the Fejér kernel on $\mathbb{T}$ of order $n \in \mathbb{N}$. Since $\|K_n\|_{L^1(\mathbb{T})} = 1$ and $\|K_n\|_{L^\infty(\mathbb{T})} \lesssim n$, we deduce that

$$\int_{\mathbb{T}^d} |f(x_1, \dots, x_d)|\log^r(1 + |f(x_1, \dots, x_d)|)\,\mathrm{d}x_1 \dots \mathrm{d}x_d \lesssim_{r,d} N^r.$$

So, if we take $M = (m(k_1, \dots, k_d))_{k_1,\dots,k_d\in\mathbb{Z}}$ with $m(k_1, \dots, k_d) = 1/\sqrt{k_1 \dots k_d}$ for $k_1 > 0, \dots, k_d > 0$ and $m(k_1, \dots, k_d) = 0$ otherwise, namely when at least one of the coordinates is less or equal than 0, then $M \in \mathcal{M}_{H^1_{\text{prod}}(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}$, and hence

$$\left(\sum_{(k_1,\dots,k_d)\in\mathbb{Z}^d} |m(k_1, \dots, k_d)\widehat{f}(k_1, \dots, k_d)|^2\right)^{1/2} \lesssim_{r,d} N^r.$$

Since

$$\Big(\sum_{(k_1,\dots,k_d)\in\mathbb{Z}^d} |m(k_1,\dots,k_d)\widehat{f}(k_1,\dots,k_d)|^2\Big)^{1/2} \geq \Big(\sum_{1\leq k_1,\dots,k_d\leq 2^N} \frac{1}{k_1\dots k_d}\Big)^{1/2}$$
$$= \prod_{i=1}^{d}\Big(\sum_{1\leq k_i\leq 2^N} \frac{1}{k_i}\Big)^{1/2}$$
$$\sim N^{d/2},$$

we see that, by choosing N to be large enough, we must have $r \geq d/2$.

Remark 3.1. A similar argument shows that the Orlicz space $L\log^{d/2} L(\mathbb{T}^d)$ in (1.3) cannot be improved. Indeed, if E is a set satisfying (1.2), then by making use of the argument presented above, we see that the exponent $r = d/2$ in $L\log^{d/2} L(\mathbb{T}^d)$ on the right-hand side of higher-dimensional Zygmund inequality (1.3) is sharp.

To show that the inclusion (1.1) is proper, take Λ to be a Sidon set in $\mathbb{Z}$ that cannot be written as a finite union of lacunary sequences; see [**18**, Remark 2.5(3)]. Then $M = \chi_{\Lambda\times\cdots\times\Lambda}$ belongs to the class $\mathcal{M}_{L\log^{d/2} L(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}$; see, for example, [**1**, Proposition 4]. However, it can easily be checked that $M = \chi_{\Lambda\times\cdots\times\Lambda}$ does not satisfy (2.4) and we thus deduce that $\chi_{\Lambda\times\cdots\times\Lambda} \in \mathcal{M}_{L\log^{d/2} L(\mathbb{T}^d)\to L^2(\mathbb{T}^d)} \setminus \mathcal{M}_{H^1_{\text{prod}}(\mathbb{T}^d)\to L^2(\mathbb{T}^d)}$.

4. Proposition 1.3 implies Proposition 1.2

Our aim in this section is to prove that Proposition 1.3 implies Proposition 1.2. Towards this aim, take $E \subset \mathbb{Z}^d$ to be a set satisfying the assumption of Proposition 1.2, that is, condition (1.2). By duality (see, for example, [**4**, Remarque, pp. 350–351]), (1.3) is equivalent to the fact that E is a $\Lambda(p)$ set in $\mathbb{Z}^d$ for every $p > 2$ with $\Lambda(p)$ constant growing like $A(p,E) \leq C_{D_E} p^{d/2}$ as $p \to \infty$. In other words, to prove (1.3), it is enough to show that for every E-polynomial f one has for every $p > 2$,

$$\|f\|_{L^p(\mathbb{T}^d)} \leq C_{D_E} p^{d/2} \|f\|_{L^2(\mathbb{T}^d)}, \tag{4.1}$$

where C_{D_E} is an absolute constant, independent of p and f. As we will soon see, if $D_E = 1$, then, in fact, C_{D_E} depends only on d and, in particular, can be taken to be independent of E.

Assume first that E satisfies (1.2) with $D_E = 1$. To prove (4.1), fix an E-polynomial f and note that for every $(k_1,\dots,k_d) \in \mathbb{N}_0^d$ one has, by the triangle inequality,

$$\|\widetilde{\Delta}_{k_1,\dots,k_d}(f)\|_{L^\infty(\mathbb{T}^d)} \leq \sum_{(r_1,\dots,r_d)\in E\cap(I_{k_1}\times\cdots\times I_{k_d})} |\phi_{k_1}(r_1)\cdots\phi_{k_d}(r_d)\widehat{f}(r_1,\dots,r_d)|$$
$$\lesssim_{d,\phi} \sum_{(r_1,\dots,r_d)\in E\cap(I_{k_1}\times\cdots\times I_{k_d})} |\widehat{f}(r_1,\dots,r_d)|,$$

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where I_{k_l} denotes the set $\mathbb{Z} \cap \{(-2^{k_l+1}, -2^{k_l-1}] \cup [2^{k_l-1}, 2^{k_l+1})\}$, $l = 1, \ldots, d$. Observe that, thanks to condition (1.2) for $D_E = 1$, the sum

$$\sum_{(r_1,\ldots,r_d)\in E\cap(I_{k_1}\times\cdots\times I_{k_d})} |\widehat{f}(r_1, \ldots, r_d)|$$

consists of at most 6^d terms. Hence,

$$\|\widetilde{\Delta}_{k_1,\ldots,k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \lesssim_{d,\phi} \sum_{(r_1,\ldots,r_d)\in E\cap(I_{k_1}\times\cdots\times I_{k_d})} |\widehat{f}(r_1, \ldots, r_d)|^2$$

and we thus deduce that

$$\left(\sum_{k_1,\ldots,k_d\in\mathbb{N}_0} \|\widetilde{\Delta}_{k_1,\ldots,k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \right)^{1/2} \lesssim_{d,\phi} \left(\sum_{(r_1,\ldots,r_d)\in E} |\widehat{f}(r_1, \ldots, r_d)|^2 \right)^{1/2}. \tag{4.2}$$

Observe that the quantity on the right-hand side of the last inequality equals $\|f\|_{L^2(\mathbb{T}^d)}$, as $\mathrm{supp}(\widehat{f}) \subset E$. Hence, (4.1) follows from (1.4) and (4.2) in the case where $D_E = 1$. Moreover, note that, in the case where $D_E = 1$, the implied constant in (4.2) depends only on the dimension d and on our choice of ϕ and, in particular, it is independent of E.

In the case where $D_E > 1$, write $f = \sum_{i=1}^{D_E} f_i$, with f_i being trigonometric polynomials on $\mathbb{T}^d$ such that $\mathrm{supp}(\widehat{f_i}) \subset E_i$, where $E = \bigcup_{i=1}^{D_E} E_i$ and $D_{E_i} = 1$. Then, by using the triangle inequality and the previous step, we have

$$\|f\|_{L^p(\mathbb{T}^d)} \leq \sum_{i=1}^{D_E} \|f_i\|_{L^p(\mathbb{T}^d)} \leq Cp^{d/2} \sum_{i=1}^{D_E} \|f_i\|_{L^2(\mathbb{T}^d)} \leq CD_E p^{d/2} \|f\|_{L^2(\mathbb{T}^d)},$$

since, by our construction and the L^2-theory of Fourier series, $\|f_i\|_{L^2(\mathbb{T}^d)} \leq \|f\|_{L^2(\mathbb{T}^d)}$ for all $i = 1, \ldots, D_E$.

Remark 4.1. As mentioned in Remark 3.1, the exponent $r = d/2$ in $L\log^{d/2} L(\mathbb{T}^d)$ in (1.3) is sharp. Using this fact, we deduce that the behaviour of the constant on the right-hand side of (1.4) with respect to p as $p \to \infty$ is best possible. That is, the exponent $r = d/2$ in $p^{d/2}$ in (1.4) cannot be improved.

5. Proof of Proposition 1.3

To prove Proposition 1.3, note that, as $p > 2$, it follows from Minkowski's inequality that

$$\left\| \left(\sum_{m_1,\cdots,m_d\in\mathbb{N}_0} |\mathbb{D}_{m_1,\cdots,m_d}(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T}^d)} \leq \left(\sum_{m_1,\ldots,m_d\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\ldots,m_d}(f)\|^2_{L^p(\mathbb{T}^d)} \right)^{1/2}.$$

Moreover, since one trivially has

$$\left(\sum_{m_1,\ldots,m_d\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\ldots,m_d}(f)\|^2_{L^p(\mathbb{T}^d)} \right)^{1/2} \leq \left(\sum_{m_1,\ldots,m_d\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\ldots,m_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \right)^{1/2},$$

we deduce from (2.6) that

$$\|f\|_{L^p(\mathbb{T}^d)} \leq C_d p^{d/2} \bigg(\sum_{m_1,\ldots,m_d \in \mathbb{N}_0} \|\mathbb{D}_{m_1,\ldots,m_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \bigg)^{1/2} \tag{5.1}$$

for all $p > 2$. Hence, to prove that (1.4) holds, it suffices, in view of (5.1), to show that

$$\bigg(\sum_{m_1,\ldots,m_d \in \mathbb{N}_0} \|\mathbb{D}_{m_1,\ldots,m_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \bigg)^{1/2} \lesssim_d \bigg(\sum_{k_1,\ldots,k_d \in \mathbb{N}_0} \|\widetilde{\Delta}_{k_1,\ldots,k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \bigg)^{1/2}.$$

This last inequality follows from the next lemma which is a d-dimensional analogue of [**19**, Lemma 2.3].

Lemma 5.1. *Let δ be a Schwartz function that is even, supported in $(-4, 4)$ and such that $\delta|_{[-2,2]} \equiv 1$.*

Define $\psi(\xi) = \delta(\xi) - \delta(8\xi)$. For $k \in \mathbb{N}$, put $\psi_k(\xi) = \psi(2^{-k}\xi)$, and for $k = 0$, put $\psi_0 = \delta$. Consider the operator

$$\Psi_k(f)(x) = \sum_{r\in\mathbb{Z}} \psi_k(r)\widehat{f}(r)\mathrm{e}^{\mathrm{i}2\pi rx}$$

acting on functions defined over the torus. For $k_1, \ldots, k_d \in \mathbb{N}_0$ we use the notation $\Psi_{k_1,\ldots,k_d} = \Psi_{k_1} \otimes \cdots \otimes \Psi_{k_d}$.

There exists a constant $C_d > 0$, depending only on the dimension d and on ψ, such that for all d-tuples of non-negative integers $(m_1, \ldots, m_d)$ and $(k_1, \ldots, k_d)$ one has

$$\|\mathbb{D}_{m_1,\ldots,m_d} \Psi_{k_1,\ldots,k_d}\|_{L^\infty(\mathbb{T}^d)\to L^\infty(\mathbb{T}^d)} \leq C_d \prod_{j=1}^{d} 2^{-|k_j - m_j|}. \tag{5.2}$$

The proof of Lemma 5.1 will be given in the next subsection. By using the above lemma and in particular estimate (5.2), one can easily complete the proof of Proposition 1.3. Towards this aim, we argue as in the proof of [**19**, Proposition 2.2]. More precisely, we consider a trigonometric polynomial f on $\mathbb{T}^d$ and write $f = \sum_{k_1,\ldots,k_d\in\mathbb{N}_0} \widetilde{\Delta}_{k_1,\ldots,k_d}(f)$. For fixed η (and ϕ), if ψ is as in the statement of Lemma 5.1, then $\psi\phi = \phi$, and hence $\Psi_{k_1,\ldots,k_d}\widetilde{\Delta}_{k_1,\ldots,k_d} = \widetilde{\Delta}_{k_1,\ldots,k_d}$. So, by using (5.2), we obtain

$$\begin{aligned}
&\|\mathbb{D}_{m_1,\ldots,m_d}(f)\|_{L^\infty(\mathbb{T}^d)} \\
&\quad\leq \sum_{k_1,\ldots,k_d\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\ldots,m_d}[\widetilde{\Delta}_{k_1,\ldots,k_d}(f)]\|_{L^\infty(\mathbb{T}^d)} \\
&\quad\leq \sum_{k_1,\ldots,k_d\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\ldots,m_d}\Psi_{k_1,\ldots,k_d}\|_{L^\infty(\mathbb{T}^d)\to L^\infty(\mathbb{T}^d)} \|\widetilde{\Delta}_{k_1,\ldots,k_d}(f)\|_{L^\infty(\mathbb{T}^d)} \\
&\quad\lesssim_d \sum_{k_1,\ldots,k_d\in\mathbb{N}_0} \bigg(\prod_{j=1}^{d} 2^{-|m_j-k_j|} \bigg) \|\widetilde{\Delta}_{k_1,\ldots,k_d}(f)\|_{L^\infty(\mathbb{T}^d)},
\end{aligned}$$

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and it thus follows that

$$\left(\sum_{m_1,\ldots,m_d \in \mathbb{N}_0} \|\mathbb{D}_{m_1,\ldots,m_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \right)^{1/2}$$
$$\lesssim_d \left[\sum_{m_1,\ldots,m_d \in \mathbb{N}_0} \left(\sum_{k_1,\ldots,k_d \in \mathbb{N}_0} \left(\prod_{j=1}^{d} 2^{-|m_j-k_j|} \right) \|\widetilde{\Delta}_{k_1,\ldots,k_d}(f)\|_{L^\infty(\mathbb{T}^d)} \right)^2 \right]^{1/2},$$

where the implied constant depends only on the dimension d (and on our choice of ψ). Hence, by Minkowski's integral inequality,

$$\left[\sum_{m_1,\ldots,m_d \in \mathbb{N}_0} \left(\sum_{k_1,\ldots,k_d \in \mathbb{N}_0} \left(\prod_{j=1}^{d} 2^{-|m_j-k_j|} \right) \|\widetilde{\Delta}_{k_1,\ldots,k_d}(f)\|_{L^\infty(\mathbb{T}^d)} \right)^2 \right]^{1/2}$$
$$\leq \sum_{m_1,\ldots,m_d \in \mathbb{Z}} \left(\prod_{j=1}^{d} 2^{-|m_j|} \right) \left(\sum_{k_1 \geq -m_1} \cdots \sum_{k_d \geq -m_d} \|\widetilde{\Delta}_{k_1+m_1,\ldots,m_d+k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \right)^{1/2}.$$

Since we have

$$\sum_{m_1,\ldots,m_d \in \mathbb{Z}} \left(\prod_{j=1}^{d} 2^{-|m_j|} \right) \left(\sum_{k_1 \geq -m_1} \cdots \sum_{k_d \geq -m_d} \|\widetilde{\Delta}_{k_1+m_1,\ldots,m_d+k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \right)^{1/2}$$
$$\lesssim \left(\sum_{k_1,\ldots,k_d \in \mathbb{N}_0} \|\widetilde{\Delta}_{k_1,\ldots,k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \right)^{1/2},$$

the proof of Proposition 1.3 will be complete once we prove Lemma 5.1. This will be done in the following subsection.

5.1. Proof of Lemma 5.1

The proof of Lemma 5.1 can easily be obtained by iterating the corresponding one-dimensional result of Seeger and Trebels [**19**, Lemma 2.3], which, in particular, asserts that for all $m, k \in \mathbb{N}_0$ one has

$$\|\mathbb{D}_m \Psi_k\|_{L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})} \leq C 2^{-|m-k|}, \tag{5.3}$$

where $C > 0$ is an absolute constant. More precisely, to prove Lemma 5.1 we shall induct on the dimension $d \in \mathbb{N}$. Note that the one-dimensional case is (5.3). Assume now that, for some $d > 1$, estimate (5.2) holds for the $(d-1)$-dimensional case, namely

$$\|\mathbb{D}_{m_1,\ldots,m_{d-1}} \Psi_{k_1,\ldots,k_{d-1}}\|_{L^\infty(\mathbb{T}^{d-1}) \to L^\infty(\mathbb{T}^{d-1})} \leq C_{d-1} \prod_{j=1}^{d-1} 2^{-|k_j-m_j|}. \tag{5.4}$$

To establish the d-dimensional case, fix a trigonometric polynomial f on $\mathbb{T}^d$ and observe that one has

$$\mathbb{D}_{m_1,\ldots,m_d}[\Psi_{k_1,\ldots,k_d}(f)] = \mathbb{D}_{m_d} \Psi_{k_d} [\mathbb{D}_{m_1,\ldots,m_{d-1}} \Psi_{k_1,\ldots,k_{d-1}}(f)]. \tag{5.5}$$

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Indeed, to show (5.5), note that for $m_d \geq 1$ one has

$$\mathbb{D}_{m_1,\ldots,m_d} = \mathbb{D}_{m_1,\ldots,m_{d-1}} \otimes \mathbb{E}_{m_d} - \mathbb{D}_{m_1,\ldots,m_{d-1}} \otimes \mathbb{E}_{m_{d-1}},$$

and hence, for $(x_1,\ldots,x_d) \in \mathbb{T}^d$, we may write

$$\begin{aligned}
&\mathbb{D}_{m_1,\ldots,m_d}[\Psi_{k_1,\ldots,k_d}(f)](x_1,\ldots,x_d) \\
&\quad = 2^{m_d} \int_{I_d} \sum_{r_d \in \mathbb{Z}} \psi(2^{-k_d} r_d) \mathbb{D}_{m_1,\ldots,m_{d-1}}[\Psi_{k_1,\ldots,k_{d-1}}(f_{r_d})](x_1,\ldots,x_{d-1}) \mathrm{e}^{\mathrm{i}2\pi r_d x_d'} \,\mathrm{d}x_d' \\
&\qquad - 2^{m_d-1} \int_{\widetilde{I}_d} \sum_{r_d \in \mathbb{Z}} \psi(2^{-k_d} r_d) \mathbb{D}_{m_1,\ldots,m_{d-1}}[\Psi_{k_1,\ldots,k_{d-1}}(f_{r_d})](x_1,\ldots,x_{d-1}) \mathrm{e}^{\mathrm{i}2\pi r_d x_d'} \,\mathrm{d}x_d',
\end{aligned}$$

where I_d is the unique interval in $\mathbb{T}$ of length 2^{-m_d} containing x_d, $\widetilde{I}_d$ is the unique interval in $\mathbb{T}$ of length $2^{-(m_d-1)}$ containing x_d, and for $r_d \in \mathbb{Z}$ we use the notation

$$f_{r_d}(x_1,\ldots,x_{d-1}) = \sum_{r_1,\ldots,r_{d-1} \in \mathbb{Z}} \widehat{f}(r_1,\ldots,r_d) \mathrm{e}^{\mathrm{i}2\pi(r_1 x_1 + \cdots + r_{d-1} x_{d-1})}.$$

Note that we may write

$$\begin{aligned}
&\sum_{r_d \in \mathbb{Z}} \mathbb{D}_{m_1,\ldots,m_{d-1}}[\Psi_{k_1,\ldots,k_{d-1}}(f_{r_d})](x_1,\ldots,x_{d-1}) \mathrm{e}^{\mathrm{i}2\pi r_d x_d} \\
&\qquad = \mathbb{D}_{m_1,\ldots,m_{d-1}}[\Psi_{k_1,\ldots,k_{d-1}}(f_{x_d})](x_1,\ldots,x_{d-1}),
\end{aligned}$$

where for fixed $x_d \in \mathbb{T}$ we use the notation $f_{x_d}(x_1,\ldots,x_{d-1}) = f(x_1,\ldots,x_d)$. We thus obtain that

$$\begin{aligned}
&\mathbb{D}_{m_1,\ldots,m_d}[\Psi_{k_1,\ldots,k_d}(f)] \\
&\quad = \mathbb{E}_{m_d} \Psi_{k_d}[\mathbb{D}_{m_1,\ldots,m_{d-1}} \Psi_{k_1,\ldots,k_{d-1}}(f)] - \mathbb{E}_{m_{d-1}} \Psi_{k_d}[\mathbb{D}_{m_1,\ldots,m_{d-1}} \Psi_{k_1,\ldots,k_{d-1}}(f)],
\end{aligned}$$

and this completes the proof of (5.5) in the case where $m_d \geq 1$. If $m_d = 0$, one shows (5.5) similarly.

Hence, using (5.5) and applying (5.3) to the dth variable, for fixed $(x_1,\ldots,x_{d-1}) \in \mathbb{T}^{d-1}$, we get

$$\begin{aligned}
&|\mathbb{D}_{m_1,\ldots,m_d}[\Psi_{k_1,\ldots,k_d}(f)](x_1,\ldots,x_d)| \\
&\qquad \leq C 2^{-|m_d-k_d|} \sup_{x_d' \in \mathbb{T}} |\mathbb{D}_{m_1,\ldots,m_{d-1}} \Psi_{k_1,\ldots,k_{d-1}}(f_{x_d'})(x_1,\ldots,x_{d-1})|
\end{aligned}$$

for all $x_d \in \mathbb{T}$. By using the inductive hypothesis (5.4), we obtain

$$\begin{aligned}
&|\mathbb{D}_{m_1,\ldots,m_{d-1}} \Psi_{k_1,\ldots,k_{d-1}}(f_{x_d})(x_1,\ldots,x_{d-1})| \\
&\qquad \leq C_{d-1} \prod_{j=1}^{d-1} 2^{-|m_j-k_j|} \sup_{x_d' \in \mathbb{T}} \left(\sup_{(x_1',\ldots,x_{d-1}') \in \mathbb{T}^{d-1}} |f_{x_d'}(x_1',\ldots,x_{d-1}')| \right)
\end{aligned}$$

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for all $(x_1, \dots, x_{d-1}) \in \mathbb{T}^{d-1}$. We thus deduce that

$$|\mathbb{D}_{m_1,\dots,m_d}[\Psi_{k_1,\dots,k_d}(f)](x_1,\dots,x_d)| \le CC_{d-1} \prod_{j=1}^{d} 2^{-|m_j-k_j|} \|f\|_{L^\infty(\mathbb{T}^d)}$$

for all $(x_1, \dots, x_d) \in \mathbb{T}^d$, and this implies the desired result. Hence, the proof of the lemma is complete.

Note that the argument above gives $C_d = C^d$, where $C > 0$ is the constant in (5.3).

Remark 5.2. We remark that one can give an alternative proof to Lemma 5.1 by adapting the argument in the proof of [**19**, Lemma 2.3] to higher dimensions.

Remark 5.3. At this point, it is worth noting that, by using (5.3) together with a result due to Grafakos and Kalton [**11**, Proposition 4.4] (see also [**10**, Theorem 6.4.8]), one obtains a stronger version of (1.4), where the $L^\infty(\mathbb{T}^d)$-norms of the Littlewood–Paley projections on the right-hand side of (1.4) are replaced by $L^p(\mathbb{T}^d)$-norms, $p > 2$.

To be more specific, by adapting the argument of Grafakos and Kalton [**11**] to the torus, it follows that there exist absolute constants $C, c_0 > 0$ such that

$$\|\mathbb{D}_m \Psi_k\|_{L^2(\mathbb{T}) \to L^2(\mathbb{T})} \le C2^{-c_0|m-k|}, \tag{5.6}$$

where in the one-dimensional periodic case one can take $c_0 = 1/2$. Hence, by interpolating between (5.3) and (5.6), we deduce that for every $p > 2$ one has

$$\|\mathbb{D}_m \Psi_k\|_{L^p(\mathbb{T}) \to L^p(\mathbb{T})} \le A2^{-|m-k|/2}, \tag{5.7}$$

where $A > 0$ is an absolute constant. Therefore, by using (5.7) and arguing as in the proof of Lemma 5.1, it follows that

$$\|\mathbb{D}_{m_1,\dots,m_d} \Psi_{k_1,\dots,k_d}\|_{L^p(\mathbb{T}^d) \to L^p(\mathbb{T}^d)} \lesssim_d \prod_{j=1}^{d} 2^{-|k_j-m_j|/2}. \tag{5.8}$$

Hence, by using (5.8) and arguing as in the proof of Proposition 1.3, we deduce that, for every trigonometric polynomial f on $\mathbb{T}^d$ and for each $p > 2$, a stronger version of (1.4),

$$\|f\|_{L^p(\mathbb{T}^d)} \lesssim_d p^{d/2} \left(\sum_{k_1,\dots,k_d \in \mathbb{N}_0} \|\widetilde{\Delta}_{k_1,\dots,k_d}(f)\|^2_{L^p(\mathbb{T}^d)} \right)^{1/2}, \tag{5.9}$$

holds true.

6. Some further remarks and applications

6.1. Applications in thin sets

Proposition 1.2 gives examples of $\Lambda(p)$ sets in $\mathbb{Z}^d$ whose corresponding $\Lambda(p)$ constant grows like $p^{d/2}$ as $p \to \infty$ and they cannot be written as products of Sidon sets. Moreover, those sets, namely the class of the sets $E \subset \mathbb{Z}^d$ that cannot be written as d-fold products of

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sets in $\mathbb{Z}$ and satisfy the condition $\sup_{I_1,\dots,I_d\in\mathcal{J}} \#\{E\cap(I_1\times\cdots\times I_d)\}<\infty$, are examples of $2d/(d+1)$-Rider sets in $\mathbb{Z}^d$ that cannot be written as products of Sidon sets in $\mathbb{Z}$.

Note that if $\Lambda_1,\dots,\Lambda_d$ are lacunary sequences in $\mathbb{Z}$, then $\Lambda_1\times\cdots\times\Lambda_d$ satisfies (1.2) and we thus recover the well-known fact that $\Lambda_1\times\cdots\times\Lambda_d$ is a $\Lambda(p)$ set in $\mathbb{Z}^d$ whose constant grows like $p^{d/2}$ as $p\to\infty$. However, Proposition 1.2 cannot handle spectral sets of the form $\Lambda_1\times\cdots\times\Lambda_d$, where Λ_j is a Sidon set that is not a finite union of lacunary sequences $(j=1,\dots,d)$.

6.2. A version of (1.4) for 'rough' projections

For $k\in\mathbb{N}$, consider the classical Littlewood–Paley projection Δ_k given by

$$\Delta_k(f)(x)=\sum_{n=2^{k-1}}^{2^k-1}\widehat{f}(n)\mathrm{e}^{\mathrm{i}2\pi nx}+\sum_{n=-2^k+1}^{-2^{k-1}}\widehat{f}(n)\mathrm{e}^{\mathrm{i}2\pi nx}.$$

For $k=0$, set $\Delta_0(f)(x)=\widehat{f}(0)$. For $k_1,\dots,k_d\in\mathbb{N}_0$, we write

$$\Delta_{k_1,\dots,k_d}=\Delta_{k_1}\otimes\cdots\otimes\Delta_{k_d}.$$

Since for every trigonometric polynomial f on the d-torus we may write

$$f=\sum_{m_1,\dots,m_d\in\mathbb{N}_0}\Delta_{m_1,\dots,m_d}(f),$$

we have

$$\widetilde{\Delta}_{k_1,\dots,k_d}(f)=\sum_{m_1,\dots,m_d\in\mathbb{N}_0}\widetilde{\Delta}_{k_1,\dots,k_d}\Delta_{m_1,\dots,m_d}(f).$$

Observe that $\widetilde{\Delta}_{k_1,\dots,k_d}\Delta_{m_1,\dots,m_d}=0$ whenever there exists an index $j_0\in\{1,\dots,d\}$ such that $|k_{j_0}-m_{j_0}|>1$. Hence, for every $p>2$ one has

$$\begin{aligned}\|\widetilde{\Delta}_{k_1,\dots,k_d}(f)\|_{L^p(\mathbb{T}^d)}&\le\sum_{\substack{(m_1,\dots,m_d)\in\mathbb{N}_0^d:\\|k_j-m_j|\le1\text{ for all }j\in\{1,\dots,d\}}}\|\widetilde{\Delta}_{k_1,\dots,k_d}\Delta_{m_1,\dots,m_d}(f)\|_{L^p(\mathbb{T}^d)}\\&\lesssim_d\sum_{\substack{(m_1,\dots,m_d)\in\mathbb{N}_0^d:\\|k_j-m_j|\le1\text{ for all }j\in\{1,\dots,d\}}}\|\Delta_{m_1,\dots,m_d}(f)\|_{L^p(\mathbb{T}^d)}.\end{aligned}$$

Therefore,

$$\Bigg(\sum_{k_1,\dots,k_d\in\mathbb{N}_0}\|\widetilde{\Delta}_{k_1,\dots,k_d}(f)\|_{L^p(\mathbb{T}^d)}^2\Bigg)^{1/2}\lesssim_d\Bigg(\sum_{k_1,\dots,k_d\in\mathbb{N}_0}\|\Delta_{k_1,\dots,k_d}(f)\|_{L^p(\mathbb{T}^d)}^2\Bigg)^{1/2},$$

and hence it follows from (5.9) that

$$\|f\|_{L^p(\mathbb{T}^d)}\lesssim_d p^{d/2}\Bigg(\sum_{k_1,\dots,k_d\in\mathbb{N}_0}\|\Delta_{k_1,\dots,k_d}(f)\|_{L^p(\mathbb{T}^d)}^2\Bigg)^{1/2}.$$

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We thus deduce that for every trigonometric polynomial f on $\mathbb{T}^d$ one has

$$\|f\|_{L^p(\mathbb{T}^d)} \lesssim_d p^{d/2} \Bigg(\sum_{k_1,\dots,k_d \in \mathbb{N}_0} \|\Delta_{k_1,\dots,k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} \Bigg)^{1/2} \tag{6.1}$$

for each $p > 2$. Note that (6.1) also follows directly from (1.4). Estimate (6.1) is a multi-parameter version of an inequality due to Moore [**13**]. In particular, we obtain the following multi-parameter extension of [**13**, Theorem, p. 30].

Proposition 6.1. *There exist positive constants $c_1(d)$ and $c_2(d)$, depending only on the dimension d, such that whenever*

$$\sum_{k_1,\dots,k_d \in \mathbb{N}_0} \|\Delta_{k_1,\dots,k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)} < \infty$$

one has

$$\int_{\mathbb{T}^d} \exp \left\{ c_1(d) \left[\frac{|f(x_1,\dots,x_d)|}{\big(\sum_{k_1,\dots,k_d \in \mathbb{N}_0} \|\Delta_{k_1,\dots,k_d}(f)\|^2_{L^\infty(\mathbb{T}^d)}\big)^{1/2}} \right]^{2/d} \right\} \mathrm{d}x_1 \dots \mathrm{d}x_d < c_2(d).$$

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