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STABILITY OF GORENSTEIN FLAT MODULES

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Abstract. Sather-Wagstaff et al. proved in [**8**] (S. Sather-Wagsta, T. Sharif and D. White, Stability of Gorenstein categories, *J. Lond. Math. Soc.*(2), **77**(2) (2008), 481–502) that iterating the process used to define Gorenstein projective modules exactly leads to the Gorenstein projective modules. Also, they established in [**9**] (S. Sather-Wagsta, T. Sharif and D. White, AB-contexts and stability for Goren-stein at modules with respect to semi-dualizing modules, *Algebra Represent. Theory* **14**(3) (2011), 403–428) a stability of the subcategory of Gorenstein flat modules under a procedure to build R-modules from complete resolutions. In this paper we are concerned with another kind of stability of the class of Gorenstein flat modules via-à-vis the very Gorenstein process used to define Gorenstein flat modules. We settle in affirmative the following natural question in the setting of a left GF-closed ring R: Given an exact sequence of Gorenstein flat R-modules $\mathbf{G} = \cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$ such that the complex $H \otimes_R \mathbf{G}$ is exact for each Gorenstein injective right R-module H, is the module $M := \mathrm{Im}(G_0 \to G_{-1})$ a Gorenstein flat module?

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1. Introduction. Throughout this paper, R denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left R-modules. Let $\mathcal{M}(R)$ denote the category of (left) R-modules and let $\mathcal{F}(R)$ stand for the subcategory of flat R-modules.

The development of the Gorenstein homological algebra has reached an advanced level since the pioneering works of Auslander and Bridger (cf. [**1, 2**]). One of the key points of this theory is its ability to identify Gorenstein rings. In the Gorenstein homological algebra one replaces projective and injective modules, the elementary entities on which the classical homological algebra is based, with the Gorenstein projective and the Gorenstein injective modules. Recall that an R-module M is said to be Gorenstein flat (G-flat for short) if there exists an exact sequence $\mathbf{F}$ of flat modules, called a complete flat resolution, with

$$\mathbf{F} = \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$$

such that the complex $I \otimes_R \mathbf{F}$ is exact for each injective right R-module I (see [**5, 6**]), and $M = \mathrm{Im}(F_0 \to F_{-1})$. Let $\mathcal{G}F(R)$ denote the class of Gorenstein flat modules over R.

In [**8**], Sather-Wagstaff et al. investigated the modules that arise from an iteration of the very procedure that leads to the Gorenstein projective modules. Indeed, let $\mathcal{P}(R)$ and $\mathcal{G}P(R)$ denote the subcategories of projective modules and Gorenstein projective modules, respectively. For each subcategory $\mathcal{X}$ of the category of R-modules, they denoted by $\mathcal{G}^1(\mathcal{X})$ the category of all R-modules isomorphic to $\mathrm{Coker}(\delta_1^X)$ for some exact complex X in $\mathcal{X}$ such that the complexes $\mathrm{Hom}_R(X', X)$ and $\mathrm{Hom}_R(X, X')$ are exact for each module X' in $\mathcal{X}$. Inductively, they set $\mathcal{G}^{n+1}\mathcal{G}(X) = \mathcal{G}(\mathcal{G}^n(\mathcal{X}))$ for each integer $n \geq 1$. They answered a question from the folklore of the subject by proving, when R is commutative, the next equality

$$\mathcal{G}^n(\mathcal{G}P(R)) = \mathcal{G}P(R)$$

for each integer $n \geq 1$ (Theorem A in **[8]**) as well as its dual version for Gorenstein injective R-modules. Moreover, in **[9]**, they studied the properties of the subcategory of G_C-flat R-modules, where C is a semi-dualising module over a commutative Noetherian ring R. In this context, they proved that if C is a semi-dualising module over a commutative Noetherian ring R and $n \geq 1$ is an integer, then

$$\mathcal{G}^n(\mathcal{G}F_C(R) \cap \mathcal{B}_C(R)) = \mathcal{G}F_C(R) \cap \mathcal{B}_C(R) \text{ (Theorem II(a) in } \mathbf{[9]}\text{)},$$

where $\mathcal{B}_C(R)$ is the Bass class associated to C and $\mathcal{G}F_C(R)$ denotes the category of G_C-flat modules. Thus, in the particular case where $C = R$, this yields

$$\mathcal{G}^n(\mathcal{G}F(R)) = \mathcal{G}F(R)$$

for each integer $n \geq 1$, which explicits a certain stability of $\mathcal{G}F(R)$ under the above-introduced Gorenstein procedure $\mathcal{G}$.

On the other hand, in **[3]**, Bennis defined and studied the notion of left GF-closed rings. These are rings for which $\mathcal{G}F(R)$ is closed under extension, that is for any exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules if $A, C \in \mathcal{G}F(R)$, then $B \in \mathcal{G}F(R)$. It is worthwhile to note in this respect that the class of left GF-closed rings includes (strictly) right coherent rings together with rings of finite weak global dimension. Bennis showed that over a left GF-closed ring R, $\mathcal{G}F(R)$ is actually projectively resolving Theorem 2.3 in **[3]**. Also, he transfers in **[3]** all properties and characteristics of the Gorenstein flat modules over right coherent rings established in **[6]** to the case of left GF-closed rings.

The main purpose of this paper is to establish, in the setting of a left GF-closed ring R, the stability of the Gorenstein flat modules under the very process used to define these entities. Denote by $\mathcal{G}^{(2)}\mathcal{F}(R)$ (resp., $\mathcal{G}_i^{(2)}\mathcal{F}(R)$) the subcategory of $\mathcal{M}(R)$ for which there exists an exact sequence of Gorenstein flat R-modules $\mathbf{G} = \cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$ such that the complex $H \otimes_R \mathbf{G}$ is exact for each Gorenstein injective right R-module H (resp., $I \otimes_R \mathbf{G}$ is exact for each injective right R-module I) and $M = \mathrm{Im}(G_0 \to G_{-1})$. It is routine to check that

$$\mathcal{G}\mathcal{F}(R) \subseteq \mathcal{G}^{(2)}\mathcal{F}(R) \subseteq \mathcal{G}_i^{(2)}\mathcal{F}(R).$$

Our main theorem proves that these inequalities turn out to be equalities when R is a left GF-closed ring as is stated next.

1.2. Main theorem. Let R be a left GF-closed ring. Then

$$\mathcal{G}\mathcal{F}(R) = \mathcal{G}^{(2)}\mathcal{F}(R) = \mathcal{G}_i^{(2)}\mathcal{F}(R).$$

2. Proof of the main theorem. First, let us call the Gorenstein G-flat module any element of $\mathcal{G}_i^{(2)}\mathcal{F}(R)$, which is defined above.

For their simple characteristics the strongly Gorenstein flat modules were introduced and studied in [**4**]. Recall that an R-module M is called a strongly Gorenstein flat (SG-flat for short) module if there exists an exact sequence of R-modules,

$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$

such that F is a flat R-module and $I \otimes_R -$ leaves this sequence exact whenever I is an injective right module over R (Definition 3.1 in [**4**]). It is proved in this respect that each Gorenstein flat module is a direct summand of a strongly Gorenstein flat module (Theorem 3.5 in [**4**]). The proof of our main theorem relies heavily on the use of these new entities and cast light on the crucial role they play in the Gorenstein homological algebra.

Next, we introduce strongly Gorenstein G-flat modules.

Definition 2.1. An R-module M is called a strongly Gorenstein G-flat module if there exists an exact sequence of R-modules $0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$ such that F is Gorenstein flat over R and $I \otimes_R -$ leaves this sequence exact for each injective right R-module I.

Proposition 2.2. (1) *Any strongly Gorenstein G-flat module is Gorenstein G-flat.*
(2) *The family of Gorenstein G-flat modules is stable under arbitrary direct sums.*

Proof. (1) It is straightforward.

(2) It is straightforward, since any direct sum of Gorenstein flat modules is Gorenstein flat (Proposition 3.2 in [**6**]) and since, for each positive integer n, $\mathrm{Tor}_n^R(B, \bigoplus_i A_i) \cong \bigoplus_i \mathrm{Tor}_n^R(B, A_i)$ for any family of modules A_i and any right module B (Theorem 8.10 in [**7**]). $\square$

Proposition 2.3. *Let M be an R-module. Then the following statements hold.*

(1) *Given an exact sequence of R-modules*

$$0 \longrightarrow K \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots \longrightarrow G_n \longrightarrow M \longrightarrow 0$$

such that $G_1, G_2, \ldots, G_n$ are Gorenstein flat modules, then

$$Tor_{n+i}^R(Q, M) \cong Tor_i^R(Q, K)$$

for each right R-module Q with finite injective dimension and each integer $i \geq 1$.

(2) *If M is a Gorenstein G-flat R-module, then $Tor_i^R(Q, M) = 0$ for each right R-module Q with finite injective dimension and each integer $i \geq 1$.*

Proof. (1) It suffices to handle the case $n = 1$. So, let $0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$ be an exact sequence such that G is Gorenstein flat. Let Q be a right R-module with finite injective dimension. Applying the functor $Q \otimes_R -$ to this sequence yields the following exact sequence:

$$\mathrm{Tor}_{i+1}^R(Q, G) = 0 \longrightarrow \mathrm{Tor}_{i+1}^R(Q, M) \longrightarrow \mathrm{Tor}_i^R(Q, K) \longrightarrow \mathrm{Tor}_i^R(Q, G) = 0$$

for each integer $i \geq 1$. This ensures that $\mathrm{Tor}_{i+1}^R(Q, M) \cong \mathrm{Tor}_i^R(Q, K)$ for each integer $i \geq 1$, as desired.

(2) We proceed by induction on $s := \mathrm{id}_R(Q_R) < +\infty$. Let M be a Gorenstein G-flat module. Then there exists a short exact sequence $0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$ such that G is a Gorenstein flat module, K is a Gorenstein G-flat module and

$$0 \longrightarrow Q \otimes_R K \longrightarrow Q \otimes_R G \longrightarrow Q \otimes_R M \longrightarrow 0$$

is exact whenever Q is an injective right R-module. Hence, $\mathrm{Tor}_1^R(Q, M) = 0$ for each injective right module Q. Reiterating this process and using (1), we get $\mathrm{Tor}_i^R(Q, M) = 0$ for each injective right module Q and each integer $i \geq 1$. Then the case $s = 0$ holds. So let $s \geq 1$ and let Q be a right R-module of injective dimension s. Let $0 \longrightarrow Q \longrightarrow I \longrightarrow Q' \longrightarrow 0$ be an exact sequence of right R-modules such that I is injective. Tensoring with M yields the following exact sequence, for each integer $i \geq 1$:

$$\cdots \to \mathrm{Tor}_{i+1}^R(I, M) = 0 \to \mathrm{Tor}_{i+1}^R(Q', M) \to \mathrm{Tor}_i^R(Q, M) \to \mathrm{Tor}_i^R(I, M) = 0 \to \cdots,$$

by the first step so that, by inductive assumptions, $\mathrm{Tor}_i^R(Q, M) \cong \mathrm{Tor}_{i+1}^R(Q', M) = 0$ for each integer $i \geq 1$, as claimed. □

The next result establishes an analog version of Proposition 3.6 in [**4**] for the Gorenstein G-flat notion.

PROPOSITION 2.4. *Let M be an R-module. Then the following statements are equivalent:*

(1) *M is a strongly Gorenstein G-flat module.*
(2) *There exists an exact sequence $0 \longrightarrow M \longrightarrow G \longrightarrow M \longrightarrow 0$ such that G is a Gorenstein flat module, and $\mathrm{Tor}_1^R(Q, M) = 0$ for any injective right R-module Q.*
(3) *There exists an exact sequence $0 \longrightarrow M \longrightarrow G \longrightarrow M \longrightarrow 0$ such that G is a Gorenstein flat module, and $\mathrm{Tor}_1^R(Q, M) = 0$ for any right R-module Q with finite injective dimension.*
(4) *There exists an exact sequence $0 \longrightarrow M \longrightarrow G \longrightarrow M \longrightarrow 0$ such that G is a Gorenstein flat module and, for any right R-module Q with finite injective dimension, the following sequence is exact*

$$0 \longrightarrow Q \otimes_R M \longrightarrow Q \otimes_R G \longrightarrow Q \otimes_R M \longrightarrow 0.$$

Proof. (1) $\Rightarrow$ (2) holds by Proposition 2.3(2).
(2) $\Rightarrow$ (3), it is similar to the proof of Proposition 2.3(2).
(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are straightforward, completing the proof. □

The following result is analog to Theorem 3.5 in [**4**].

PROPOSITION 2.5. *Let M be a Gorenstein G-flat R-module. Then M is a direct summand of a strongly Gorenstein G-flat module.*

Proof. Let M be a Gorenstein G-flat module and $\mathbf{G} = \cdots \longrightarrow G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \xrightarrow{d_0} G_{-1} \xrightarrow{d_{-1}} G_{-2} \xrightarrow{d_{-2}} \cdots$ be a complete Gorenstein flat resolution such that $M = \mathrm{Im}(d_0)$. Let $M_i := \mathrm{Im}(d_i)$ for each integer i. As $\mathcal{GF}(R)$ is stable under direct sums, it is easily seen that the following sequence is a complete Gorenstein flat resolution:

$$\mathbf{G}' = \cdots \longrightarrow \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \cdots$$

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such that $\mathrm{Im}(\oplus_i d_i) = \bigoplus_i M_i$. Then $\bigoplus_i M_i$ is a strongly Gorenstein G-flat module so that M is a direct summand of a strongly Gorenstein G-flat module, as contended. □

For brevity, we adopt the following definition.

DEFINITION 2.6. Let M be a strongly Gorenstein G-flat module. An R-module N is called an M-type module if there exists an exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow H \longrightarrow 0$ such that H is a Gorenstein flat module.

Proposition 2.7 and Corollary 2.8 start the proof of the Main Theorem.

PROPOSITION 2.7. *Let M be a strongly Gorenstein G-flat module and N an M-type module. Then,*

(1) *$Tor_i^R(Q, N) = 0$ for each injective right R-module Q and for each integer $i \geq 1$.*

(2) *If R is a left GF-closed ring, then there exists an exact sequence $0 \longrightarrow N \longrightarrow F \longrightarrow L \longrightarrow 0$ such that F is a flat module and L is an M-type module.*

Proof. (1) If $0 \longrightarrow M \longrightarrow N \longrightarrow H \longrightarrow 0$ is an exact sequence such that H is a Gorenstein flat R-module, then, by considering the corresponding long exact sequence and by Proposition 2.3, we have $\mathrm{Tor}_i^R(Q, N) \cong \mathrm{Tor}_i^R(Q, M) = 0$ for each injective right module Q and each integer $i \geq 1$.

(2) Assume that R is a left GF-closed ring. Let $0 \longrightarrow M \longrightarrow G \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow M \longrightarrow N \longrightarrow H \longrightarrow 0$ be exact sequences such that G and H are Gorenstein flat R-modules. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & T \arrow[r, dashed] \arrow[d] & M \arrow[r] & 0 \\
 & H \arrow[r, equal] \arrow[d] & H \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since G and H are Gorenstein flat modules, we get, as R is left GF-closed, T is Gorenstein flat over R by Theorem 2.3 in [**3**]. Then there exists a short exact sequence $0 \longrightarrow T \longrightarrow F \longrightarrow K \longrightarrow 0$ such that F is a flat R-module and K is a Gorenstein flat R-module. Hence, we get the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, dashed] & F \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

as desired. □

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COROLLARY 2.8. *Let R be a left GF-closed ring. Let M be a strongly Gorenstein G-flat module and N an M-type module. Then N is a Gorenstein flat R-module.*

Proof. First, observe that by Proposition 2.7 there exist a flat module F_0 and an M-type module L such that the following sequence $0 \longrightarrow N \longrightarrow F_0 \longrightarrow L \longrightarrow 0$ is exact and stays exact after applying the functor $Q \otimes_R -$ for each injective right module Q. Then, it suffices to iterate Proposition 2.7(2) to get a resolution

$$0 \longrightarrow N \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

of flat modules, which remains exact after applying the functor $Q \otimes_R -$ for each injective right R-module Q. Now, Proposition 2.7(1) completes the proof. $\square$

Proof of the main theorem. In view of the inclusions $\mathcal{GF}(R) \subseteq \mathcal{G}^{(2)}\mathcal{F}(R) \subseteq \mathcal{G}_i^{(2)}\mathcal{F}(R)$, it suffices to prove that $\mathcal{G}_i^{(2)}\mathcal{F}(R) \subseteq \mathcal{GF}(R)$. Since R is left GF-closed, by Corollary 2.6 in [**3**], $\mathcal{G}F(R)$ is stable under direct summands. Thus, it suffices, by Proposition 2.5, to prove that any strongly Gorenstein G-flat module is Gorenstein flat. Then, let M be a strongly Gorenstein G-flat module. There exists an exact sequence $0 \longrightarrow M \longrightarrow G \longrightarrow M \longrightarrow 0$ such that G is a Gorenstein flat module and $\mathrm{Tor}_i^R(Q, M) = 0$ for each injective right module Q and each integer $i \geq 1$ by Proposition 2.4. As G is Gorenstein flat, there exists an exact sequence of R-modules $0 \longrightarrow G \longrightarrow F \longrightarrow G_1 \longrightarrow 0$ such that F is a flat module and G_1 is a Gorenstein flat module. Then we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M & \longrightarrow & G & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & M & \dashrightarrow & F & \to & M_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_1 & = & G_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence, M_1 is an M-type R-module. It follows from Corollary 2.8 that M_1 is a Gorenstein flat module. As R is left GF-closed and G_1 is Gorenstein flat, we get M is a Gorenstein flat R-module, as desired. $\square$

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