VOLUME INEQUALITIES FOR L_p-JOHN ELLIPSOIDS AND THEIR DUALS*

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Abstract. In this paper, we establish some inequalities among the L_p-centroid body, the L_p-polar projection body, the L_p-John ellipsoid and its dual, which are the strengthened version of known results. We also prove inequalities among the polar of the L_p-centroid body, the L_p-polar projection body, the L_p-John ellipsoid and its dual.

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1. Introduction. In [**6**, **7**] Lutwak showed that the Fiery L_p-combination (see [**2**]) led to L_p-Brunn-Minkowski theory for $p \geq 1$, which is also called Brunn-Minkowski-Firey theory. For many notions, there are natural extensions of projection bodies, centroid bodies, John ellipsoids etc. in the L_p-Brunn-Minkowski theory (see [**7**, **9**, **11**]).

Further, Lutwak, Yang, and Zhang proved that there are extensions of all of the known inequalities involving projection bodies, centroid bodies and John ellipsoids to the new L_p-version of them (see [**7**, **8**, **9**, **11**]). More results regarding the L_p-analogues are to be found in [**1**, **4**, **5**, **12**, **13**, **15**].

In this paper, we continue the study of the volume inequalities among the L_p-centroid body the L_p-polar projection body, the L_p-John ellipsoid and its dual.

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors). Let $\mathcal{K}_o^n$ denote the subset of $\mathcal{K}^n$ that contains the origin in their interiors in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and ω_n denote the n-dimensional volume of the unit ball B in $\mathbb{R}^n$, namely

$$\omega_n = \pi^{\frac{n}{2}}/\Gamma\left(1 + \frac{n}{2}\right).$$

For real $p \geq 1$, define

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

If K is a star body about the origin in $\mathbb{R}^n$, and $p \geq 1$, the L_p-centroid body $\Gamma_p K$ of K is the origin-symmetric convex body whose support function is given by [**4**, **8**, **9**]

$$h(\Gamma_p K, u)^p = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx, \qquad (1.1)$$

where the integration is with respect to Lebesgue measure.

If $K \in \mathcal{K}_o^n$ and $p \geq 1$, then the L_p-polar projection body $\Gamma_{-p}K$ is an origin-symmetric body whose radial function is given by [11]

$$\rho(\Gamma_{-p}K, u)^{-p} = \frac{1}{nc_{n-2,p}V(K)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v), \qquad (1.2)$$

where $S_p(K, \cdot)$ denotes the L_p-surface area measure.

Note that our definition of $\Gamma_{-p}K$ is different from the definition given by Lutwak, Yang and Zhang in [11]. That is for $K = B$, we have $\Gamma_{-p}B = B$.

Lutwak, Yang, and Zhang (see [11]) have generalized the classical John ellipsoid JK to the L_p-John ellipsoid E_pK which can be associated with a fixed convex body K: If $K \in \mathcal{K}_o^n$ and $p > 0$, amongst all origin-centered ellipsoids E, the unique ellipsoid E_pK that solves the constrained maximization problem

$$V(E_pK) = \max_E V(E) \quad \text{subject to} \quad V_p(K, E) \leq V(K), \qquad (1.3)$$

will be called the L_p-John ellipsoid of K, denoted by E_pK.

Corresponding to Lutwak, Yang and Zhang's research, Yu, Leng and Wu in [15] proposed the notion that a family of dual L_p-John ellipsoids $\tilde{E}_pK$ which can be associated with a fixed convex body K: If $K \in \mathcal{K}_o^n$ and $p > 0$, amongst all origin-centered ellipsoids E, the unique ellipsoid $\tilde{E}_pK$ that solves the constrained maximization problem

$$V(\tilde{E}_pK) = \max_E \left(\frac{1}{V(E)} \right) \quad \text{subject to} \quad \tilde{V}_{-p}(K, E) \leq V(K), \qquad (1.4)$$

will be called the dual L_p-John ellipsoid of K, denoted by $\tilde{E}_pK$.

Lutwak, Yang, and Zhang [11] showed that when $p \geq 1$, the volume of E_pK is always dominated by the volume of K.

THEOREM A. *Let $K \subset \mathbb{R}^n$ be a convex body that contains the origin in their interiors and $p \geq 1$, then*

$$V(E_pK) \leq V(K),$$

with equality if and only if K is an ellipsoid centered at the origin.

Yu, Leng and Wu [15] also showed that when $p \geq 1$, the volume of K is always dominated by the volume of $\tilde{E}_pK$.

THEOREM B. *Let $K \subset \mathbb{R}^n$ be a convex body that contains the origin in their interiors and $p \geq 1$, then*

$$V(K) \leq V(\tilde{E}_pK),$$

with equality if and only if K is an ellipsoid centered at the origin.

One of the aims of this paper is to establish the following strengthened versions of Theorem A and B.

THEOREM 1.1. *Let $K \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$V(E_pK) \leq V(\Gamma_{-p}K) \leq V(K) \leq V(\Gamma_pK) \leq V(\tilde{E}_pK), \qquad (1.5)$$

with equality if and only if K is an ellipsoid centered at the origin.

Another aim of this paper is to establish several polar versions for inequalities (1.5). In fact, we prove the following volume inequalities among the polar of L_p-centroid body, L_p-John ellipsoid and its dual, between the polar of L_p-polar projection body and the polar of L_p-John ellipsoid, respectively.

THEOREM 1.2. *Let $K \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$V(E_p^* K) \geq V(\Gamma_p^* K) \geq V(\tilde{E}_p^* K), \tag{1.6}$$

with equality if and only if K is an ellipsoid centered at the origin.

THEOREM 1.3. *Let $K \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$V(E_p^* K) \geq V(\Gamma_{-p}^* K), \tag{1.7}$$

with equality if and only if K is an ellipsoid centered at the origin.

Section 2 contains some notation and background material. We shall prove these theorems in the final section.

2. Notation and preliminaries.   If $K \in \mathcal{K}^n$, then the support function of K, $h(K, \cdot)$: $\mathbb{R}^n \longrightarrow (0, \infty)$, is defined by [**3**, **14**]

$$h(K, u) = \max\{u \cdot x : x \in K\}, \quad u \in S^{n-1}$$

where $u \cdot x$ denotes the standard inner product of u and x.

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L, \cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$ [**3**, **14**]

$$\rho(L, u) = \max\{\lambda > 0 : \lambda u \in L\}.$$

If $\rho(L, \cdot)$ is continuous and positive, L will be called a star body. Let $\mathcal{S}_o^n$ denote the set of star bodies in $\mathbb{R}^n$. Two star bodies K and L are said to be dilates if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is a convex body that contains the origin in its interior, the polar body K^* of K, with respect to the origin, is defined by [**3**, **14**]

$$K^* = \{x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K\}.$$

For $K \in \mathcal{K}_o^n$, it follows from the definitions of support and radial functions, and the definition of polar body, that [**3**, **14**]

$$h_{K^*} = 1/\rho_K \quad \text{and} \quad \rho_{K^*} = 1/h_K.$$

For $p \geq 1$, $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L$ is defined as the convex body whose support function is given by [**6**]

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p. \tag{2.1}$$

Firey combinations of convex bodies were defined and studied by Firey [**2**] (who called them p-means of convex bodies).

For $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of the convex bodies $K, L \in \mathcal{K}_o^n$, can be defined by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

That this limit exists was demonstrated in [6].

It was shown in [6], that corresponding to each convex body K contained the origin in its interior in $\mathbb{R}^n$, there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p dS_p(K, u), \tag{2.2}$$

for each convex body $Q \in \mathcal{K}_o^n$. The measure $S_1(K, \cdot)$ is just the classical surface area measure of K and usually denoted by $S(K, \cdot)$ or S_K.

From the definition of the L_p-mixed volume, it follows immediately that for each $K \in \mathcal{K}_o^n$,

$$V_p(K, K) = V(K). \tag{2.3}$$

We shall require a basic inequalities for the L_p-mixed volume. The L_p-Minkowski inequality states that for $K, L \in \mathcal{K}_o^n$ and $p \geq 1$ (see [6, 7])

$$V_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{2.4}$$

with equality if and only if K and L are dilates.

For star bodies K, L and $p \geq 1$, $\varepsilon > 0$, the L_p-harmonic radial combination $K +_{-p} \varepsilon \cdot L$ is defined as the star body whose radial function is given by [7]

$$\rho(K +_{-p} \varepsilon \cdot L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}. \tag{2.5}$$

The L_p-dual mixed volume $\tilde{V}_{-p}(K, L)$ of the star bodies K, L can be defined by

$$\frac{n}{-p} \tilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the dual mixed volume $\tilde{V}_{-p}(K, L)$ of the star bodies K, L

$$\tilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_L^{-p}(v) \, dS(v), \tag{2.6}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the definition of the L_p-dual mixed volume, it follows immediately that for each $K \in \mathcal{S}_o^n$,

$$\tilde{V}_{-p}(K, K) = V(K). \tag{2.7}$$

We shall also require a basic inequality for the L_p-dual mixed volumes. The L_p-Minkowski inequality for the L_p-dual mixed volumes states that for star bodies K, L and $p \geq 1$ (see [7])

$$\tilde{V}_{-p}(K, L) \geq V(K)^{n+p/n} V(L)^{-p/n}, \tag{2.8}$$

with equality if and only if K and L are dilates.

3. Proof of the results. The following lemmas will be used later.

LEMMA 3.1. **[9]** *If K is a star body about the origin in $\mathbb{R}^n$ and $p \geq 1$, then*

$$V(\Gamma_p K) \geq V(K), \tag{3.1}$$

with equality if and only if K is an ellipsoid centered at the origin.

LEMMA 3.2. **[8]** *If K is a star body about the origin in $\mathbb{R}^n$ and $p \geq 1$, then*

$$V(K)V(\Gamma_p^* K) \leq \omega_n^2, \tag{3.2}$$

with equality if and only if K is an ellipsoid centered at the origin.

LEMMA 3.3. *Let $K \in \mathcal{S}_o^n$, $L \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$\frac{V_p(L, \Gamma_p K)}{V(L)} = \frac{\tilde{V}_{-p}(K, \Gamma_{-p} L)}{V(K)}. \tag{3.3}$$

Proof. From the integral representation (2.6), definition (1.2), Fubini's theorem, definition (1.1), and the integral representation (2.2), it follows that

$$
\begin{aligned}
\tilde{V}_{-p}(L, \Gamma_{-p} K) &= \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_{\Gamma_{-p}L}^{-p}(v) \, dS(v) \\
&= \frac{1}{nnc_{n-2,p}V(L)} \int_{S^{n-1}} \rho_K^{n+p}(v) \int_{S^{n-1}} |u \cdot v|^p dS_p(L, v) \, dS(v) \\
&= \frac{1}{nnc_{n-2,p}V(L)} \int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) dS(v) \, dS_p(L, v) \\
&= \frac{V(K)}{nV(L)} \int_{S^{n-1}} h_{\Gamma_p K}^p(v) \, dS_p(L, v) \\
&= \frac{V(K)}{V(L)} V_p(L, \Gamma_p K).
\end{aligned}
$$

REMARK 1. Identity (3.3) for $p = 2$ can be found in **[10]**.

Proof of Theorem 1.1. Taking $E_p K = K$ in inequality (3.3) and noticing that $\Gamma_p E_p K = E_p K$, we obtain

$$V_p(L, E_p K) = V_p(L, \Gamma_p E_p K) = \frac{V(L)}{V(E_p K)} \tilde{V}_{-p}(E_p K, \Gamma_{-p} L).$$

Letting $K = L$ in the above equality and combining with inequality (2.8), we have

$$
\begin{aligned}
V_p(K, E_p K) &= \frac{V(K)}{V(E_p K)} \tilde{V}_{-p}(E_p K, \Gamma_{-p} K) \\
&\geq \frac{V(K)}{V(E_p K)} V(E_p K)^{(n+p)/n} V(\Gamma_{-p} K)^{-p/n} \\
&= V(K)V(E_p K)^{p/n} V(\Gamma_{-p} K)^{-p/n}. \tag{3.4}
\end{aligned}
$$

According to the equality conditions of inequality (2.8), we know that equality in (3.4) holds if and only if E_pK and $\Gamma_{-p}K$ are dilates, that is, equality in (3.4) holds if and only if $\Gamma_{-p}K$ is an ellipsoid centered at the origin.

From the definition (1.3) of L_p-John ellipsoid E_pK, it follows that

$$V(K) \geq V_p(K, E_pK), \tag{3.5}$$

with equality if and only if K is an ellipsoid centered at the origin.

Combining inequalities (3.4) and (3.5) we have

$$V(K) \geq V(K)V(E_pK)^{p/n}V(\Gamma_{-p}K)^{-p/n},$$

and therefore we get

$$V(E_pK) \leq V(\Gamma_{-p}K), \tag{3.6}$$

which is the first inequality of (1.5).

Concerning the equality conditions of inequality (3.4) and (3.5), we know that equality in (3.6) holds if and only if $\Gamma_{-p}K$ is an ellipsoid centered at the origin and K is an ellipsoid centered at the origin, that is, K must be an ellipsoid centered at the origin.

In Lemma 3.3, let $K = \Gamma_{-p}L$, and noting (2.3), we can get

$$V(L) = V_p(L, \Gamma_p\Gamma_{-p}L).$$

By inequality (2.4), taking $K = L$ and using Lemma 3.1, we get

$$V(K) = V_p(K, \Gamma_p\Gamma_{-p}K) \geq V(K)^{(n-p)/n}V(\Gamma_p\Gamma_{-p}K)^{p/n}$$
$$\geq V(K)^{(n-p)/n}V(\Gamma_{-p}K)^{p/n},$$

that is

$$V(\Gamma_{-p}K) \leq V(K), \tag{3.7}$$

which is just the second inequality of inequalities (1.5).

According to the equality conditions of inequality (2.4) and (3.1), we know that equality in (3.7) holds if and only if K and $\Gamma_p\Gamma_{-p}K$ are dilates and $\Gamma_{-p}K$ is an ellipsoid centered at the origin, that is, K must be an ellipsoid centered at the origin.

Taking $\tilde{E}_pK = L$ in inequality (3.3), noticing that $\Gamma_{-p}\tilde{E}_pK = \tilde{E}_pK$, and combining with inequality (2.4), we have

$$\tilde{V}_{-p}(K, \tilde{E}_pK) = \tilde{V}_{-p}(K, \Gamma_{-p}\tilde{E}_pK)$$
$$= \frac{V(K)}{V(\tilde{E}_pK)}V_p(\tilde{E}_pK, \Gamma_pK)$$
$$\geq \frac{V(K)}{V(\tilde{E}_pK)}V(\tilde{E}_pK)^{(n-p)/n}V(\Gamma_pK)^{p/n}$$
$$= V(K)V(\tilde{E}_pK)^{-p/n}V(\Gamma_pK)^{p/n}. \tag{3.8}$$

According to the equality conditions of inequality (2.4), we know that equality in (3.8) holds if and only if $\tilde{E}_pK$ and Γ_pK are dilates, that is, equality in inequality (3.8) holds if and only if Γ_pK is an ellipsoid centered at the origin.

From the definition (1.4) of dual L_p-John ellipsoid $\tilde{E}_p K$, it follows that

$$V(K) \geq \tilde{V}_{-p}(K, \tilde{E}_p K), \tag{3.9}$$

with equality if and only if K is an ellipsoid centered at the origin.

Combining inequalities (3.8) and (3.9), we have

$$V(K) \geq V(K)V(\tilde{E}_p K)^{-p/n} V(\Gamma_p K)^{p/n},$$

and therefore we get

$$V(\Gamma_p K) \leq V(\tilde{E}_p K), \tag{3.10}$$

which is the fourth inequality of inequalities (1.5).

Concerning the equality conditions of inequality (3.8) and (3.9), we know that equality in (3.10) holds if and only if $\Gamma_p K$ is an ellipsoid centered at the origin and K is an ellipsoid centered at the origin, that is, K must be an ellipsoid centered at the origin.

Combining with inequalities (3.6), (3.7), (3.1) and (3.10), we immediately obtain inequalities (1.5), and we also know that equality holds if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.2. Because $V(E_p K)V(E_p^* K) = \omega_n^2$, from Theorem A and Lemma 3.2, we have

$$V(\Gamma_p^* K)V(E_p K) \leq V(\Gamma_p^* K)V(K) \leq \omega_n^2 = V(E_p K)V(E_p^* K),$$

that is

$$V(E_p^* K) \geq V(\Gamma_p^* K), \tag{3.11}$$

which is the first inequality of (1.6).

According to the equality conditions of Theorem A and Lemma 3.2, we know that equality in inequality (3.11) holds if and only if K is an ellipsoid centered at the origin.

According to the integral representation (2.6), definition (1.1) and Fubini's theorem, we immediately get

$$\frac{\tilde{V}_{-p}(K, \Gamma_p^* L)}{V(K)} = \frac{\tilde{V}_{-p}(L, \Gamma_p^* K)}{V(L)}. \tag{3.12}$$

In equation (3.12), if we let $L = \tilde{E}_p^* K$, notice that $\Gamma_p \tilde{E}_p^* K = \tilde{E}_p^* K$, and combine with inequality (3.9), we have

$$\tilde{V}_{-p}(\tilde{E}_p^* K, \Gamma_p^* K) / V(\tilde{E}_p^* K) = \tilde{V}_{-p}(K, \Gamma_p^* \tilde{E}_p^* K) / V(K$$
$$= \tilde{V}_{-p}(K, \tilde{E}_p K) / V(K)$$
$$\leq V(K) / V(K) = 1.$$

By inequality (2.8), we have

$$\tilde{V}_{-p}(\tilde{E}_p^* K, \Gamma_p^* K) \geq V(\tilde{E}_p^* K)^{(n+p)/n} V(\Gamma_p^* K)^{-p/n},$$

therefore

$$V(\Gamma_p^* K) \geq V(\tilde{E}_p^* K), \tag{3.13}$$

which is the second inequality of inequalities (1.6).

According to the equality conditions of inequality (2.8) and inequality (3.9), we know that equality in inequality (3.13) holds if and only if K is an ellipsoid centered at the origin.

Combining inequalities (3.11) with (3.13), we immediately obtain inequalities (1.6), and we also know that equality holds if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.3. From the integral representation (2.2), definition (1.2), and Fubini's theorem, it follows that

$$\frac{V_p(K, \Gamma_{-p}^* L)}{V(K)} = \frac{V_p(L, \Gamma_{-p}^* K)}{V(L)}. \tag{3.14}$$

In equation (3.14), if we let $L = E_p^* K$, notice $\Gamma_{-p} E_p^* K = E_p^* K$, and combine with inequality (3.5), we have

$$\begin{aligned}
V_p(E_p^* K, \Gamma_{-p}^* K) / V(E_p^* K) &= V_p(K, \Gamma_{-p}^* E_p^* K) / V(K) \\
&= V_p(K, E_p K) / V(K) \\
&\leq V(K) / V(K) = 1.
\end{aligned}$$

By inequality (2.4), we get

$$V_p(E_p^* K, \Gamma_{-p}^* K) \geq V(E_p^* K)^{(n-p)/n} V(\Gamma_{-p}^* K)^{p/n},$$

and therefore

$$V(E_p^* K) \geq V(\Gamma_{-p}^* K).$$

According to the equality conditions of inequalities (2.4) and (3.5), we know that equality in (1.7) holds if and only if K is an ellipsoid centered at the origin.

REFERENCES

1. S. Campi and P. Gronchi, The L_p-Busemann-Petty centroid inequality, *Adv. Math.* **167** (2002), 128–141.

2. W. J. Firey, p-means of convex bodies, *Math. Scand.* **10** (1962), 17–24.

3. R. J. Gardner, *Geometric tomography* (Cambridge University Press, 1995).

4. E. Grinberg and G. Y. Zhang, Convolutions, transforms, and convex bodies, *Proc. London Math. Soc.* **78** (1999), 77–115.

5. C. Haberl and M. Ludwig, A characterization of L_p intersection bodies, *Int. Math. Res. Notices*, to appear.

6. E. Lutwak, The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem, *J. Differential Geom.* **38** (1993), 131–150.

7. E. Lutwak, The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas, *Adv. Math.* **118** (1996), 244–294.

8. E. Lutwak and G. Zhang, Blaschke-Santaló inequalities, *J. Differential Geom.* **47** (1997), 1–16.

9. E. Lutwak, D. Yang and G. Y. Zhang, L_p affine isoperimetric inequalities, *J. Differential Geom.* **56** (2000), 111–132.

10. E. Lutwak, D. Yang and G. Y. Zhang, A new ellipsoid associated with conves bodies, *Duke Math. J.* **104** (2000), 375–390.

11. E. Lutwak, D. Yang and G. Y. Zhang, L_p John ellipsoids, *Proc. London Math. Soc. (3)* **90** (2005), 497–520.

12. M. Meyer and E. Werner, On the p–affine surface area, *Adv. Math.* **152** (2000), 288–313.

13. D. Ryabogin and A. Zvavitch, The Fourier transform and Firey projections of convex bodies, *Indiana. Univ. Math. J.* **53** (2004), 667–682.

14. R. Schneider, *Convex Bodies: The Brunn-Minkowski theory* (Cambridge University Press, 1993).

15. W. Yu, G. Leng and D. Wu, Dual L_p-John ellipsoids, *Proc. Edinburgh Math. Soc. (2)*, to appear.