CORRESPONDENCE.

ON THE FORMULA FOR THE MARKET VALUE OF A COMPLETE ANNUITY.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—The error involved in the formula $\frac{1-p}{i+p}$, given by D. Jones, vol. i, page 189, of his work on Annuities, for the value of a complete annuity as an investment, has been so well pointed out and corrected in the letter of Mr. A. Baden, in vol. xvii, page 447, of the *Journal of the Institute*, that I do not hope to add to the completeness of the demonstration there given. The following considerations, however, on the subject have occurred to me, with which I beg to trouble you, in case you should think them worthy of insertion.

I will first show how the result arrived at by Mr. Baden can be deduced in another way. The annuity being 1, payable annually, the last payment, or that in respect of which such annuity is rendered complete, is on an average equal, or nearly so, to $\frac{1}{2}$ the annual payment, or simply to $\frac{1}{2}$; and this is assumed to be due, not at the instant of death, but at the end of a year from the preceding payment of the annuity. So that, in fact, if we suppose the annuity of 1 to be divided into two annuities of $\frac{1}{2}$ each, both payable at the same periods, one of these annuities will be payable during the life, and its value is given by the well known formula $\frac{1}{2}\left(\frac{1}{d+\varpi_x}-1\right)$, (using ϖ_x instead of p, the symbol adopted by Jones); but in the case of the other, one further payment is made, viz.: of $\frac{1}{2}$ at the end of the year in which the life drops.

Now a little consideration will show that if each payment of an annuity of $\frac{1}{2}$ be made one year later than is assumed in the formula $\frac{1}{2}\left(\frac{1}{d+\varpi_x}-1\right)$, that is, at the end of each year, provided the life was in existence at the *beginning* of the year, we have an annuity of $\frac{1}{2}$, payment of which is to commence at the end of the second year, to continue during life, and also be made at the end of the year of death. But since the present value of 1 payable a year hence is v, the value of the latter annuity is $\frac{v}{2}\left(\frac{1}{d+\varpi_x}-1\right)$; and as the payment of $\frac{1}{2}$ in respect of the first year is to be made whether the life continue or not, its value is $\frac{v}{2}$. Adding,

$$\frac{v}{2}\left(\frac{1}{d+\varpi_x}-1\right)+\frac{v}{2}=\frac{v}{2}\left(\frac{1}{d+\varpi_x}\right) \quad . \quad . \quad . \quad (1)$$

and this is therefore the value of an annuity of $\frac{1}{2}$ payable annually during life and one year longer. Adding to (1) the value of the other annuity of $\frac{1}{2}$, viz., $\frac{1}{2}\left(\frac{1}{d+\varpi_x}-1\right)$ we have

$$\frac{v-\frac{\varpi_x}{2}}{d+\varpi_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for the value of the complete annuity of 1; and this is the same result as that arrived at by Mr. Baden in the letter referred to. The amount for which the assurance is to be effected is $\frac{1-\frac{d}{2}}{d+\varpi_x}$.

The formula (1) may also be deduced as follows. Let 1 be the amount invested by the purchaser. The interest on 1 is i; and if f_x denote the annual premium on the assurance, then, after the payment of the first premium, there remains $1-f_x$, and the annuity which this should purchase is $i+f_x$. If a payment of such annuity is to be made in respect of the year in which the life fails, the assurance should be for $1-f_x$. Hence, $f_x=\varpi_x(1-f_x)=\frac{\varpi_x}{1+\varpi_x}$ $\therefore 1-f_x$ $=1-\frac{\varpi_x}{1+\varpi_x}=\frac{1}{1+\varpi_x}$, and $i+f_x=i+\frac{\varpi_x}{1+\varpi_x}=\frac{i+(1+i)\varpi_x}{1+\varpi_x}$. The value of an annuity of $\frac{1}{2}$ is $\frac{1}{2}\frac{1-f_x}{i+f_x}=\frac{1}{2}\frac{1}{i+(1+i)\varpi_x}=\frac{1}{2}\frac{v}{d+\varpi_x}$, which agrees with (1).

It can be readily seen that, as is pointed out by Mr. Baden, Jones's formula, $\frac{1-\varpi_x}{i+\varpi_x}$, gives a correct result when $\varpi_x=i$. It may be worth while to ascertain the difference between it and the more correct formula $\frac{v-\frac{\varpi_x}{2}}{d+\varpi_x}$.

$$\frac{v-\frac{\varpi_x}{2}}{d+\varpi_x}-\frac{1-\varpi_x}{i+\varpi_x}=\frac{vi+\varpi_x v-\frac{\varpi_x}{2}i-\frac{\varpi_x^2}{2}-d-\varpi_x+d\varpi_x+\varpi_x^2}{(d+\varpi_x)(i+\varpi_x)}$$

$$=\frac{d+\varpi_x v-\frac{\varpi_x}{2}i-\frac{\varpi_x^2}{2}-d-\varpi_x v+\varpi_x^2}{(d+\varpi_x)(i+\varpi_x)}$$

$$=\frac{\frac{\varpi_x}{2}(\varpi_x-i)}{(d+\varpi_x)(i+\varpi_x)}=\left(\frac{\varpi_x-i}{i+\varpi_x}\right)\left(\frac{\frac{\varpi_x}{2}}{d+\varpi_x}\right). \quad . \quad . \quad (3)$$

Hence, according as ϖ_x is $>$ or $<$ i, Jones's formula produces too small or too great a value. $\frac{\frac{\varpi_x}{2}}{d+\varpi_x}$ represents the value of an annuity-due of $\frac{\varpi_x}{2}$, and also the increase in the value of an annuity of 1 on account of its being complete, since $\frac{v-\frac{\varpi_x}{2}}{d+\varpi_x}=\frac{1}{d+\varpi_x}-1+\frac{\varpi_x}{2}\frac{1}{d+\varpi_x}$.

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Therefore the amount of error in computing such additional value by Jones's formula may be found from (3) for given values of i and ϖ_x.

Without wishing to devote more time to this subject than its importance warrants, I will consider the following other points connected with Jones's formula $\frac{1-\varpi_x}{i+\varpi_x}$.

1st. What is really the interpretation to be put on it? This may be easily seen on reference to vol. i, Articles 244 and 245, of his work on Annuities, but will perhaps be more obvious if the formula be converted into one more familiar to the reader. This can be done by multiplying its numerator and denominator by v. Thus we have

$$\frac{1-\varpi_x}{i+\varpi_x}=\frac{v(1-\varpi_x)}{v(i+\varpi_x)}=\frac{1-d-v\varpi_x}{d+v\varpi_x}=\frac{1}{d+v\varpi_x}-1=\frac{1}{d+\varpi'_x}-1,$$

if $v\varpi_x=\varpi'_x$, or ϖ'_x represent ϖ_x reduced in the ratio $1:1+i$.

The latter formula coincides with the well known one, $\frac{1}{d+\varpi_x}-1$, except that ϖ'_x is substituted for ϖ_x; or, in other words, Jones, in designing to alter the formula for a curtate annuity so as to obtain one for a complete annuity, really retained the same formula, only making a modification in one of its terms, viz., ϖ_x.

2nd. The increase in the value of the annuity on account of its being complete, if Jones's formula be used, is

$$\frac{1}{d+v\varpi_x}-1-\left(\frac{1}{d+\varpi_x}-1\right)=\frac{d\varpi_x}{(d+\varpi_x)(d+v\varpi_x)}.$$

Differentiating the Napierian logarithm of the latter quantity, viz., $\log_e d+\log_e \varpi_x-\log_e (d+\varpi_x)-\log_e (d+v\varpi_x)$, to x or ϖ_x, and equating the result to 0, according to the theory of maxima and minima, we have

$$\frac{1}{\varpi_x}-\frac{1}{d+\varpi_x}-\frac{v}{d+v\varpi_x}=0$$

$$\frac{d}{\varpi_x(d+v\varpi_x)}-\frac{1}{d+\varpi_x}=0$$

$$d^2+d\varpi_x-d\varpi_x-v\varpi_x^2=0$$

$$\varpi_x=\frac{d}{\sqrt{v}}=\sqrt{id}.$$

Differentiating again: d being the symbol of differentiation,

$$\frac{\mathrm{d}\left(\frac{1}{\varpi_x}-\frac{1}{d+\varpi_x}-\frac{v}{d+v\varpi_x}\right)}{\mathrm{d}x}=-\left\{\frac{1}{\varpi_x^2}-\frac{1}{(d+\varpi_x)^2}-\frac{v^2}{(d+v\varpi_x)^2}\right\}\frac{\mathrm{d}\varpi_x}{\mathrm{d}x}$$

$$=-\left\{\frac{(d+v\varpi_x)^2-v^2\varpi_x^2}{\varpi_x^2(d+v\varpi_x)^2}-\frac{1}{(d+\varpi_x)^2}\right\}\frac{\mathrm{d}\varpi_x}{\mathrm{d}x}$$

$\left(\text{or since from above } \frac{d}{\varpi_x(d+v\varpi_x)} = \frac{1}{d+\varpi_x}\right)$

$$= -\left\{\frac{(d+v\varpi_x)^2 - v^2\varpi_x^2}{\varpi_x^2(d+v\varpi_x)^2} - \frac{d^2}{\varpi_x^2(d+v\varpi_x)^2}\right\}\frac{\mathrm{d}\varpi_x}{\mathrm{d}x}$$

$$= -\frac{2dv\varpi_x}{\varpi_x^2(d+v\varpi_x)^2}\frac{\mathrm{d}\varpi_x}{\mathrm{d}x}.$$

The last result is negative. Hence $\log_e \frac{d\varpi_x}{(d+\varpi_x)(d+v\varpi_x)}$ is a maximum, and therefore also $\frac{d\varpi_x}{(d+\varpi_x)(d+v\varpi_x)}$, when $\varpi_x = \sqrt{id}$; that is, the difference between the values of a complete and curtate annuity, if Jones's formula be used for the former, attains a maximum value at the age at which the annual premium $= \sqrt{id}$, and thence decreases as the premium increases; whereas it will be seen that the difference, viz., $\frac{1}{2}\frac{\varpi_x}{d+\varpi_x}$, or $\frac{1}{2}\left(1-\frac{d}{d+\varpi_x}\right)$, between the values of the two annuities, when the formula (2), $\frac{v-\frac{\varpi_x}{2}}{d+\varpi_x}$, is used for the value of the complete annuity, increases as ϖ_x increases.

Since the value of a complete annuity as above deduced is equal to that of a curtate annuity increased by $\frac{1}{2}\frac{\varpi_x}{d+\varpi_x}$, it may appear at first sight that there is no difference between the two annuities, or between the methods of carrying out the transactions in respect of them, except in such additional payment of $\frac{1}{2}\frac{\varpi_x}{d+\varpi_x}$ being made for the complete annuity. In reference to this point, the following details as to the values, &c., of the two annuities are furnished.

	Value.	Total original cost of Annuity	Amount of Assurance	Premium.	Interest	Premium + Interest.
Complete annuity	$\frac{v-\frac{\varpi_x}{2}}{d+\varpi_x}$	$\frac{v\left(1+\frac{\varpi_x}{2}\right)}{d+\varpi_x}$	$\frac{1-\frac{d}{2}}{d+\varpi_x}$	$\varpi_x\left(\frac{1-\frac{d}{2}}{d+\varpi_x}\right)$	$d\left(\frac{1+\frac{\varpi_x}{2}}{d+\varpi_x}\right)$	1
Curtate annuity	$\frac{1}{d+\varpi_x}-1$	$\frac{v}{d+\varpi_x}$	$\frac{1}{d+\varpi_x}$	$\frac{\varpi_x}{d+\varpi_x}$	$\frac{d}{d+\varpi_x}$	1

It will thus be seen that not only the values and costs of the two annuities differ, but also the sums assured, premiums, and interest.

In the case of a complete annuity of 1, $\frac{1-\frac{d}{2}}{d+\varpi_x}$, representing the sum to be assured and $\frac{1}{2}$ the proportion of the annuity in respect of the year of death, the purchaser will receive at that time

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$\dfrac{1-\frac{d}{2}}{d+\varpi_x}+\dfrac{1}{2}=\dfrac{1+\frac{\varpi_x}{2}}{d+\varpi_x}=\left\{\dfrac{v\left(1+\frac{\varpi_x}{2}\right)}{d+\varpi_x}\right\}(1+i)$. But $\dfrac{v\left(1+\frac{\varpi_x}{2}\right)}{d+\varpi_x}$ is the original cost of the annuity: therefore he will be repaid the latter with a year's interest on it, as should be the case. Supposing, however, the assurance to be effected for $\dfrac{1}{d+\varpi_x}$, as in the case of a curtate annuity, since $\frac{1}{2}$ will be received as before for the proportion of the annuity in respect of the year of death, the purchaser will receive altogether $\dfrac{1}{d+\varpi_x}+\dfrac{1}{2}$, or $\dfrac{1+\frac{\varpi_x}{2}+\frac{d}{2}}{d+\varpi_x}$, that is, a larger sum than in the preceding case. The assurance for $\dfrac{1}{d+\varpi_x}$ is therefore greater than is necessary.

I am, Sir,

Your obedient servant,

THOMAS CARR.

7 Royal Exchange,
4 March 1874.

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