Ergod. Th. & Dynam. Sys., (2024), **44**, 184–203 © The Author(s), 2023. Published by Cambridge University Press.
doi:10.1017/etds.2023.5

Sequence entropy tuples and mean sensitive tuples

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(*Received* 6 *December* 2022 *and accepted in revised form* 16 *January* 2023)

Abstract. Using the idea of local entropy theory, we characterize the sequence entropy tuple via mean forms of the sensitive tuple in both topological and measure-theoretical senses. For the measure-theoretical sense, we show that for an ergodic measure-preserving system, the μ-sequence entropy tuple, the μ-mean sensitive tuple, and the μ-sensitive in the mean tuple coincide, and give an example to show that the ergodicity condition is necessary. For the topological sense, we show that for a certain class of minimal systems, the mean sensitive tuple is the sequence entropy tuple.

Key words: sequence entropy tuples, mean sensitive tuples, sensitive in the mean tuples
2020 Mathematics Subject Classification: 37A35, 37B05 (Primary)

1. *Introduction*

By a *topological dynamical system* (*t.d.s.* for short), we mean a pair (X, T), where X is a compact metric space with a metric d and T is a homeomorphism from X to itself. A point $x \in X$ is called a *transitive point* if $\mathrm{Orb}(x, T) = \{x, Tx, \ldots\}$ is dense in X. A t.d.s. (X, T) is called *minimal* if all points in X are transitive points. Denote by $\mathcal{B}_X$ all Borel measurable subsets of X. A Borel (probability) measure μ on X is called T-*invariant* if $\mu(T^{-1}A) = \mu(A)$ for any $A \in \mathcal{B}_X$. A T-invariant measure μ on X is called *ergodic* if $B \in \mathcal{B}_X$ with $T^{-1}B = B$ implies $\mu(B) = 0$ or $\mu(B) = 1$. Denote by $M(X, T)$ (respectively $M^e(X, T)$) the collection of all T-invariant measures (respectively all ergodic measures) on X. For $\mu \in M(X, T)$, the *support* of μ is defined by $\mathrm{supp}(\mu) = \{x \in X : \mu(U) > 0$ for any neighborhood U of $x\}$. Each measure $\mu \in M(X, T)$ induces a *measure-preserving system* (*m.p.s.* for short) $(X, \mathcal{B}_X, \mu, T)$.

It is well known that the entropy can be used to measure the local complexity of the structure of orbits in a given system. One may naturally ask how to characterize the entropy in a local way. The related research started from the series of pioneering papers of Blanchard *et al* [1–4], in which the notions of entropy pairs and entropy pairs for a measure were introduced. From then on, entropy pairs have been intensively studied by many researchers. Huang and Ye [16] extended the notions from pairs to finite tuples, and showed that if the entropy of a given system is positive, then there are entropy n-tuples for any $n \in \mathbb{N}$ in both topological and measurable settings.

The sequence entropy was introduced by Kušhnirenko [22] to establish the relation between spectrum theory and entropy theory. As in classical local entropy theory, the sequence entropy can also be localized. In [12, 15], the authors investigated the sequence entropy pairs, sequence entropy tuples, and sequence entropy tuples for a measure. Using tools from combinatorics, Kerr and Li [18, 19] studied (sequence) entropy tuples, (sequence) entropy tuples for a measure, and IT-tuples via independence sets. Huang and Ye [17] showed that a system has a sequence entropy n-tuple if and only if its maximal pattern entropy is no less than $\log n$ in both topological and measurable settings. More introductions and applications of the local entropy theory can refer to a survey [10].

In addition to the entropy, the sensitivity is another candidate to describe the complexity of a system, which was first used by Ruelle [30]. In [31], Xiong introduced a multi-variant version of the sensitivity, called the n-sensitivity. Motivated by the local entropy theory, Ye and Zhang [32] introduced the notion of sensitive tuples. Particularly, they showed that a transitive t.d.s. is n-sensitive if and only if it has a sensitive n-tuple; and a sequence entropy n-tuple of a minimal t.d.s. is a sensitive n-tuple. For the converse, Maass and Shao [29] showed that in a minimal t.d.s., if a sensitive n-tuple is a minimal point of the n-fold product t.d.s., then it is a sequence entropy n-tuple.

Recently, Li, Tu, and Ye [25] studied the sensitivity in the mean form. Li, Ye, and Yu [27, 28] further studied the multi-version of mean sensitivity and its local representation, namely, the mean n-sensitivity and the mean n-sensitive tuple. One naturally wonders if there is still a characterization of sequence entropy tuples via mean sensitive tuples. By the results of [6, 8, 18, 27], one can see that a sequence entropy tuple is not always a mean sensitive tuple even in a minimal t.d.s. Nonetheless, the works of [5, 11, 25] yield that every minimal mean sensitive t.d.s. (that is, has a mean sensitive pair by [27]) is not tame (that is, exists an IT pair by [18]). So generally, we conjecture that for any minimal t.d.s., a mean sensitive n-tuple is an IT n-tuple and so a sequence entropy n-tuple by [18, Theorem 5.9]. Now we can answer this question under an additional condition. Namely, the following theorem.

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THEOREM 1.1. *Let (X, T) be a minimal t.d.s. and $\pi : (X, T) \to (X_{eq}, T_{eq})$ be the factor map to its maximal equicontinuous factor which is almost one to one. Then for $2 \leq n \in \mathbb{N}$,*

$$MS_n(X, T) \subset IT_n(X, T),$$

where $MS_n(X, T)$ denotes all the mean sensitive n-tuples and $IT_n(X, T)$ denotes all the IT n-tuples.

In the parallel measure-theoretical setting, Huang, Lu, and Ye [14] studied measurable sensitivity and its local representation. The notion of μ-mean sensitivity for an invariant measure μ on a t.d.s. was studied by García-Ramos [7]. Li [23] introduced the notion of the μ-mean n-sensitivity, and showed that an ergodic m.p.s. is μ-mean n-sensitive if and only if its maximal pattern entropy is no less than $\log n$. The authors in [27] introduced the notion of the μ-n-sensitivity in the mean, which was proved to be equivalent to the μ-mean n-sensitivity in the ergodic case.

Using the idea of localization, the authors [28] introduced the notion of the μ-mean sensitive tuple and showed that every μ-entropy tuple of an ergodic m.p.s. is a μ-mean sensitive tuple. A natural question is left open in [28].

Question 1.2. Is there a characterization of μ-sequence entropy tuples via μ-mean sensitive tuples?

The authors in [24] introduced a weaker notion named the density-sensitive tuple and showed that every μ-sequence entropy tuple of an ergodic m.p.s. is a μ-density-sensitive tuple. In this paper, we give a positive answer to this question. Namely, the following theorem.

THEOREM 1.3. *Let (X, T) be a t.d.s., $\mu \in M^e(X, T)$ and $2 \leq n \in \mathbb{N}$. Then the μ-sequence entropy n-tuple, the μ-mean sensitive n-tuple and the μ-n-sensitive in the mean tuple coincide.*

By the definitions, it is easy to see that a μ-mean sensitive n-tuple must be a μ-n-sensitive in the mean tuple. Thus, Theorem 1.3 is a direct corollary of the following two theorems.

THEOREM 1.4. *Let (X, T) be a t.d.s., $\mu \in M(X, T)$, and $2 \leq n \in \mathbb{N}$. Then each μ-n-sensitive in the mean tuple is a μ-sequence entropy n-tuple.*

THEOREM 1.5. *Let (X, T) be a t.d.s., $\mu \in M^e(X, T)$, and $2 \leq n \in \mathbb{N}$. Then each μ-sequence entropy n-tuple is a μ-mean sensitive n-tuple.*

In fact, Theorem 1.4 shows a bit more than Theorem 1.3, as for a T-invariant measure μ which is not ergodic, every μ-n-sensitive in the mean tuple is still a μ-sequence entropy n-tuple. However, the following result shows that ergodicity of μ in Theorem 1.5 is necessary.

THEOREM 1.6. *For every $2 \leq n \in \mathbb{N}$, there exist a t.d.s. (X, T) and $\mu \in M(X, T)$ such that there is a μ-sequence entropy n-tuple but it is not a μ-n-sensitive in the mean tuple.*

It is fair to note that García-Ramos informed us that at the same time, he with Muñoz-López also reported a completely independent proof of the equivalence of the sequence entropy pair and the mean sensitive pair in the ergodic case [**9**]. Their proof relies on the deep equivalent characterization of measurable sequence entropy pairs developed by Kerr and Li [**19**] using the combinatorial notion of independence. Our results provide more information in the general case, and the proofs work on the classical definition of sequence entropy pairs introduced in [**15**]. It is worth noting that the proofs depend on a new interesting ergodic measure decomposition result (Lemma 4.3), which was applied to prove the profound Erdös's conjectures in the number theory by Kra *et al* [**20**, **21**]. This decomposition may have more applications because it has the hybrid topological and Borel structures.

The outline of the paper is the following. In §2, we recall some basic notions that we will use in the paper. In §3, we prove Theorem 1.4. In §4, we show Theorems 1.5 and 1.6. In §5, we study the mean sensitive tuple and the sequence entropy in the topological sense and show Theorem 1.1.

2. *Preliminaries*

Throughout the paper, denote by $\mathbb{N}$ and $\mathbb{Z}_+$ the collections of natural numbers $\{1, 2, \ldots\}$ and non-negative integers $\{0, 1, 2, \ldots\}$, respectively.

For $F \subset \mathbb{Z}_+$, denote by $\#\{F\}$ (or simply write $\#F$ when it is clear from the context) the cardinality of F. The *upper density* $\overline{D}(F)$ of F is defined by

$$\overline{D}(F) = \limsup_{n\to\infty} \frac{\#\{F \cap [0, n-1]\}}{n}.$$

Similarly, the *lower density* $\underline{D}(F)$ of F can be given by

$$\underline{D}(F) = \liminf_{n\to\infty} \frac{\#\{F \cap [0, n-1]\}}{n}.$$

If $\overline{D}(F) = \underline{D}(F)$, we say that the *density* of F exists and is equal to the common value, which is written as $D(F)$.

Given a t.d.s. (X, T) and $n \in \mathbb{N}$, denote by $X^{(n)}$ the n-fold product of X. Let $\Delta_n(X) = \{(x, x, \ldots, x) \in X^{(n)} : x \in X\}$ be the diagonal of $X^{(n)}$ and $\Delta'_n(X) = \{(x_1, x_2, \ldots, x_n) \in X^{(n)} : x_i = x_j \text{ for some } 1 \le i \ne j \le n\}$.

If a closed subset $Y \subset X$ is T-invariant in the sense of $TY = Y$, then the restriction $(Y, T|_Y)$ (or simply write (Y, T) when it is clear from the context) is also a t.d.s., which is called a *subsystem* of (X, T).

Let (X, T) be a t.d.s., $x \in X$, and $U, V \subset X$. Denote by

$$N(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\} \quad \text{and} \quad N(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}V \ne \emptyset\}.$$

A t.d.s. (X, T) is called *transitive* if $N(U, V) \ne \emptyset$ for all non-empty open subsets U, V of X. It is well known that the set of all transitive points in a transitive t.d.s. forms a dense G_δ subset of X .

Given two t.d.s. (X, T) and (Y, S), a map $\pi : X \to Y$ is called a *factor map* if π is surjective and continuous such that $\pi \circ T = S \circ \pi$, and in which case (Y, S) is referred

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to be a *factor* of (X, T). Furthermore, if π is a homeomorphism, we say that (X, T) is *conjugate* to (Y, S).

A t.d.s. (X, T) is called *equicontinuous* (respectively *mean equicontinuous*) if for any $\epsilon > 0$, there is $\delta > 0$ such that if $x, y \in X$ with $d(x, y) < \delta$, then $\max_{k\in\mathbb{Z}_+} d(T^k x, T^k y) < \epsilon$ (respectively $\limsup_{n\to\infty}(1/n)\sum_{k=0}^{n-1} d(T^k x, T^k y) < \epsilon$). Every t.d.s. (X, T) is known to have a maximal equicontinuous factor (or a maximal mean equicontinuous factor [25]). More studies on mean equicontinuous systems can be seen in the recent survey [26].

In the remainder of this section, we fix a t.d.s. (X, T) with a measure $\mu \in M(X, T)$. The *entropy of a finite measurable partition* $\alpha = \{A_1, A_2, \ldots, A_k\}$ *of* X is defined by $H_\mu(\alpha) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i)$, where $0 \log 0$ is defined to be 0. Moreover, we define the *sequence entropy of* T *with respect to* α *along an increasing sequence* $S = \{s_i\}_{i=1}^{\infty}$ *of* $\mathbb{Z}_+$ by

$$h_\mu^S(T, \alpha) = \limsup_{n\to\infty} \frac{1}{n} H_\mu\bigg(\bigvee_{i=1}^{n} T^{-s_i}\alpha\bigg).$$

The *sequence entropy of* T *along the sequence* S is

$$h_\mu^S(T) = \sup_{\alpha} h_\mu^S(T, \alpha),$$

where the supremum takes over all finite measurable partitions. Correspondingly, the *topological sequence entropy of* T *with respect to* S *and a finite open cover* $\mathcal{U}$ is

$$h^S(T, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log N\bigg(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}\bigg),$$

where $N(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U})$ is the minimum among the cardinalities of all sub-families of $\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}$ covering X. The *topological sequence entropy of* T *with respect to* S is defined by

$$h^S(T) = \sup_{\mathcal{U}} h^S(T, \mathcal{U}),$$

where the supremum takes over all finite open covers.

Let $(x_i)_{i=1}^{n} \in X^{(n)}$. A finite cover $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$ of X is said to be an *admissible cover* with respect to $(x_i)_{i=1}^{n}$ if for each $1 \le j \le k$, there exists $1 \le i_j \le n$ such that $x_{i_j} \notin \overline{U_j}$. Analogously, we define admissible partitions with respect to $(x_i)_{i=1}^{n}$.

Definition 2.1. [15, 29] An n-tuple $(x_i)_{i=1}^{n} \in X^{(n)} \setminus \Delta_n(X)$, $n \ge 2$ is called the following.

- A sequence entropy n-tuple for μ if for any admissible finite Borel measurable partition α with respect to $(x_i)_{i=1}^{n}$, there exists a sequence $S = \{m_i\}_{i=1}^{\infty}$ of $\mathbb{Z}_+$ such that $h_\mu^S(T, \alpha) > 0$. Denote by $SE_n^\mu(X, T)$ the set of all sequence entropy n-tuples for μ.
- A sequence entropy n-tuple if for any admissible finite open cover $\mathcal{U}$ with respect to $(x_i)_{i=1}^{n}$, there exists a sequence $S = \{m_i\}_{i=1}^{\infty}$ of $\mathbb{Z}_+$ such that $h^S(T, \mathcal{U}) > 0$. Denote by $SE_n(X, T)$ the set of all sequence entropy n-tuples.

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We say that $f \in L^2(X, \mathcal{B}_X, \mu)$ is *almost periodic* if $\{f \circ T^n : n \in \mathbb{Z}_+\}$ is precompact in $L^2(X, \mathcal{B}_X, \mu)$. The set of all almost periodic functions is denoted by H_c, and there exists a T-invariant σ-algebra $\mathcal{K}_\mu \subset \mathcal{B}_X$ such that $H_c = L^2(X, \mathcal{K}_\mu, \mu)$, where $\mathcal{K}_\mu$ is called the Kronecker algebra of $(X, \mathcal{B}_X, \mu, T)$. The product σ-algebra of $X^{(n)}$ is denoted by $\mathcal{B}_X^{(n)}$. Define the measure $\lambda_n(\mu)$ on $\mathcal{B}_X^{(n)}$ by letting

$$\lambda_n(\mu)\bigg(\prod_{i=1}^{n} A_i\bigg) = \int_X \prod_{i=1}^{n} \mathbb{E}(1_{A_i}|\mathcal{K}_\mu)\, d\mu.$$

Note that $SE_n^\mu(X, T) = \mathrm{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$ [15, Theorem 3.4].

3. *Proof of Theorem 1.4*

Definition 3.1. [28] For $2 \le n \in \mathbb{N}$ and a t.d.s. (X, T) with $\mu \in M(X, T)$, we say that the n-tuple $(x_1, x_2, \ldots, x_n) \in X^{(n)} \setminus \Delta_n(X)$ is

(1) a *μ-mean n-sensitive tuple* if for any open neighborhoods U_i of x_i with $i = 1, 2, \ldots, n$, there is $\delta > 0$ such that for any $A \in \mathcal{B}_X$ with $\mu(A) > 0$, there are $y_1, y_2, \ldots, y_n \in A$ and a subset F of $\mathbb{Z}_+$ with $\overline{D}(F) > \delta$ such that $T^k y_i \in U_i$ for all $i = 1, 2, \ldots, n$ and $k \in F$;

(2) a *μ-n-sensitive in the mean tuple* if for any $\tau > 0$, there is $\delta = \delta(\tau) > 0$ such that for any $A \in \mathcal{B}_X$ with $\mu(A) > 0$, there is $m \in \mathbb{N}$ and $y_1^m, y_2^m, \ldots, y_n^m \in A$ such that

$$\frac{\#\{0 \le k \le m-1 : T^k y_i^m \in B(x_i, \tau), i = 1, 2, \ldots, n\}}{m} > \delta.$$

We denote the set of all μ-mean n-sensitive tuples (respectively μ-n-sensitive in the mean tuples) by $MS_n^\mu(X, T)$ (respectively $SM_n^\mu(X, T)$). We call an n-tuple $(x_1, x_2, \ldots, x_n) \in X^{(n)}$ *essential* if $x_i \ne x_j$ for each $1 \le i < j \le n$ and at this time, we write the collection of all essential n-tuples in $MS_n^\mu(X, T)$ (respectively $SM_n^\mu(X, T)$) as $MS_n^{\mu,e}(X, T)$ (respectively $SM_n^{\mu,e}(X, T)$).

Proof of Theorem 1.4. It suffices to prove $SM_n^{\mu,e}(X, T) \subset SE_n^{\mu,e}(X, T)$. Let $(x_1, \ldots, x_n) \in SM_n^{\mu,e}(X, T)$. Take $\alpha = \{A_1, \ldots, A_l\}$ as an admissible partition of $(x_1, \ldots, x_n)$. Then for each $1 \le k \le l$, there is $i_k \in \{1, \ldots, n\}$ such that $x_{i_k} \notin \overline{A_k}$. Put $E_i = \{1 \le k \le l : x_i \notin \overline{A_k}\}$ for $1 \le i \le n$. Obviously, $\bigcup_{i=1}^n E_i = \{1, \ldots, l\}$. Set

$$B_1 = \bigcup_{k \in E_1} A_k,\ B_2 = \bigcup_{k \in E_2 \setminus E_1} A_k, \ldots, B_n = \bigcup_{k \in E_n \setminus (\bigcup_{j=1}^{n-1} E_j)} A_k.$$

Then, $\beta = \{B_1, \ldots, B_n\}$ is also an admissible partition of $(x_1, \ldots, x_n)$ such that $x_i \notin \overline{B_i}$ for all $1 \le i \le n$. Without loss of generality, we assume $B_i \ne \emptyset$ for $1 \le i \le n$. It suffices to show that there exists a sequence $S = \{m_i\}_{i=1}^\infty$ of $\mathbb{Z}_+$ such that $h_\mu^S(T, \beta) > 0$, as $\alpha \succ \beta$. Let

$$h_\mu^*(T, \beta) = \sup\{h_\mu^S(T, \beta) : S \text{ is a sequence of } \mathbb{Z}_+\}.$$

By [15, Lemma 2.2 and Theorem 2.3], we have $h_\mu^*(T, \beta) = H_\mu(\beta|\mathcal{K}_\mu)$, where $\mathcal{K}_\mu$ is the Kronecker algebra of $(X, \mathcal{B}_X, \mu, T)$. So it suffices to show $\beta \not\subseteq \mathcal{K}_\mu$.

We prove it by contradiction. Now we assume that $\beta \subseteq \mathcal{K}_\mu$. Then for each $i = 1, \ldots, n$, 1_{B_i} is an almost periodic function. By [33, Theorems 4.7 and 5.2], 1_{B_i} is a μ-equicontinuous in the mean function. That is, for each $1 \le i \le n$ and any $\tau > 0$, there is a compact $K \subset X$ with $\mu(K) > 1 - \tau$ such that for any $\epsilon' > 0$, there is $\delta' > 0$ such that for all $m \in \mathbb{N}$, whenever $x, y \in K$ with $d(x, y) < \delta'$,

$$\frac{1}{m}\sum_{t=0}^{m-1} |1_{B_i}(T^t x) - 1_{B_i}(T^t y)| < \epsilon'. \tag{3.1}$$

However, take $\epsilon > 0$ such that $B_\epsilon(x_i) \cap B_i = \emptyset$ for $i = 1, \ldots, n$. Since $(x_1, \ldots, x_n) \in SM_n^{\mu,e}(X, T)$, there is $\delta := \delta(\epsilon) > 0$ such that for any $A \in \mathcal{B}_X$ with $\mu(A) > 0$, there are $m \in \mathbb{N}$ and $y_1^m, \ldots, y_n^m \in A$ such that if we denote $C_m = \{0 \le t \le m-1 : T^t y_i^m \in B_\epsilon(x_i) \text{ for all } i = 1, 2, \ldots, n\}$, then $\#C_m \ge m\delta$. Since $B_\epsilon(x_1) \cap B_1 = \emptyset$, then $B_\epsilon(x_1) \subset \bigcup_{i=2}^n B_i$. This implies that there is $i_0 \in \{2, \ldots, n\}$ such that

$$\#\{t \in C_m : T^t y_1^m \in B_{i_0}\} \ge \frac{\#C_m}{n-1}.$$

For any $t \in C_m$, we have $T^t y_{i_0}^m \in B_\epsilon(x_{i_0})$, and then $T^t y_{i_0}^m \notin B_{i_0}$, as $B_\epsilon(x_{i_0}) \cap B_{i_0} = \emptyset$. This implies that

$$\frac{1}{m}\sum_{t=0}^{m-1} |1_{B_{i_0}}(T^t y_1^m) - 1_{B_{i_0}}(T^t y_{i_0}^m)| \ge \frac{\#C_m}{m(n-1)} \ge \frac{\delta}{n-1}. \tag{3.2}$$

Choose a measurable subset $A \subset K$ such that $\mu(A) > 0$ and $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\} < \delta'$, and $\epsilon' = \delta/2(n-1)$. Then by equation (3.1), for any $m \in \mathbb{N}$ and $x, y \in A$,

$$\frac{1}{m}\sum_{t=0}^{m-1} |1_{B_{i_0}}(T^t x) - 1_{B_{i_0}}(T^t y)| < \frac{\delta}{2(n-1)},$$

which is a contradiction with equation (3.2). Thus, $SM_n^{\mu,e}(X, T) \subset SE_n^{\mu,e}(X, T)$. □

4. *Proof of Theorem 1.5*

In §4.1, we first reduce Theorem 1.5 to just prove that it is true for the ergodic m.p.s. with a continuous factor map to its Kronecker factor, and then we finish the proof of Theorem 1.5 under this assumption. In §4.2, we show the condition that μ is ergodic is necessary.

4.1. *Ergodic case.*

Throughout this section, we will use the following two types of factor maps between two m.p.s. $(X, \mathcal{B}_X, \mu, T)$ and $(Z, \mathcal{B}_Z, \nu, S)$.

(1) *Measurable factor maps:* a measurable map $\pi : X \to Z$ such that $\mu \circ \pi^{-1} = \nu$ and $\pi \circ T = S \circ \pi$ μ-almost everywhere;

(2) *Continuous factor maps:* a topological factor map $\pi : X \to Z$ such that $\mu \circ \pi^{-1} = \nu$.

If a continuous factor map π such that $\pi^{-1}(\mathcal{B}_Z) = \mathcal{K}_\mu$, π is called a continuous factor map to its Kronecker factor.

The following result is a weaker version in [20, Proposition 3.20].

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LEMMA 4.1. *Let $(X, \mathcal{B}_X, \mu, T)$ be an ergodic m.p.s. Then there exists an ergodic m.p.s. $(\tilde{X}, \tilde{B}, \tilde{\mu}, \tilde{T})$ and a continuous factor map $\tilde{\pi} : \tilde{X} \to X$ such that $(\tilde{X}, \tilde{B}, \tilde{\mu}, \tilde{T})$ has a continuous factor map to its Kronecker factor.*

The following result shows that we only need to prove $SE_n^{\mu}(X, T) \subset MS_n^{\mu}(X, T)$ for all ergodic m.p.s. with a continuous factor map to its Kronecker factor.

LEMMA 4.2. *If $SE_n^{\tilde{\mu}}(\tilde{X}, \tilde{T}) \subset MS_n^{\tilde{\mu}}(\tilde{X}, \tilde{T})$ for all ergodic m.p.s. $(\tilde{X}, \tilde{B}, \tilde{\mu}, \tilde{T})$ with a continuous factor map to its Kronecker factor, then $SE_n^{\mu}(X, T) \subset MS_n^{\mu}(X, T)$ for all ergodic m.p.s. $(X, \mathcal{B}_X, \mu, T)$.*

Proof. By Lemma 4.1, there exists an ergodic m.p.s. $(\tilde{X}, \tilde{B}, \tilde{\mu}, \tilde{T})$ and a continuous factor map $\tilde{\pi} : \tilde{X} \to X$ such that $(\tilde{X}, \tilde{B}, \tilde{\mu}, \tilde{T})$ has a continuous factor map to its Kronecker factor. Thus, $SE_n^{\tilde{\mu}}(\tilde{X}, \tilde{T}) \subset MS_n^{\tilde{\mu}}(\tilde{X}, \tilde{T})$, by the assumption.

For any $(x_1, \ldots, x_n) \in SE_n^{\mu}(X, T) \setminus \Delta'_n(X)$, by [15, Theorem 3.7], there exists an n-tuple $(\tilde{x_1}, \ldots, \tilde{x_n}) \in SE_n^{\tilde{\mu}}(\tilde{X}, \tilde{T}) \setminus \Delta'_n(\tilde{X})$ such that $\tilde{\pi}(\tilde{x_i}) = x_i$. For any open neighborhood $U_1 \times \cdots \times U_n$ of $(x_1, \ldots, x_n)$ with $U_i \cap U_j = \emptyset$ for $i \neq j$, then $\tilde{\pi}^{-1}(U_1) \times \cdots \times \tilde{\pi}^{-1}(U_n)$ is an open neighborhood of $(\tilde{x_1}, \ldots, \tilde{x_n})$. Since $(\tilde{x_1}, \ldots, \tilde{x_n}) \in SE_n^{\tilde{\mu}}(\tilde{X}, \tilde{T}) \setminus \Delta'_n(\tilde{X}) \subset MS_n^{\tilde{\mu}}(\tilde{X}, \tilde{T}) \setminus \Delta'_n(\tilde{X})$, there exists $\delta > 0$ such that for any $A \in \mathcal{B}_X$ with $\tilde{\mu}(\tilde{\pi}^{-1}(A)) = \mu(A) > 0$, there exist $F \subset \mathbb{N}$ with $\overline{D}(F) \geq \delta$ and $\tilde{y_1}, \ldots, \tilde{y_n} \in \tilde{\pi}^{-1}(A)$ such that for any $m \in F$,

$$(\tilde{T}^m \tilde{y_1}, \ldots, \tilde{T}^m \tilde{y_n}) \in \tilde{\pi}^{-1}(U_1) \times \cdots \times \tilde{\pi}^{-1}(U_n)$$

and hence $(T^m \tilde{\pi}(\tilde{y_1}), \ldots, T^m \tilde{\pi}(\tilde{y_n})) \in U_1 \times \cdots \times U_n$. Note that $\tilde{\pi}(\tilde{y_i}) \in A$ for each $i = 1, 2, \ldots, n$. Thus we have $(x_1, \ldots, x_n) \in MS_n^{\mu}(X, T)$. □

According to the above-mentioned lemma, in the rest of this section, we fix an ergodic m.p.s. with a continuous factor map $\pi : (X, \mathcal{B}_X, \mu, T) \to (Z, \mathcal{B}_Z, \nu, R)$ to its Kronecker factor. Moreover, we fix a measure disintegration $z \to \eta_z$ of μ over π, that is, $\mu = \int_Z \eta_z \, d\nu(z)$.

The following lemma plays a crucial role in our proof. In [20, Proposition 3.11], the authors proved it for $n = 2$, but general cases are similar in idea. For readability, we move the complicated proof to Appendix A.

LEMMA 4.3. *Let $\pi : (X, \mathcal{B}_X, \mu, T) \to (Z, \mathcal{B}_Z, \nu, R)$ be a continuous factor map to its Kronecker factor. Then for each $n \in \mathbb{N}$, there exists a continuous map $\boldsymbol{x} \mapsto \lambda_{\boldsymbol{x}}^n$ from $X^{(n)}$ to $M(X^{(n)})$ such that the map $\boldsymbol{x} \mapsto \lambda_{\boldsymbol{x}}^n$ is an ergodic decomposition of $\mu^{(n)}$, where $\mu^{(n)}$ is the n-fold product of μ and*

$$\lambda_{\boldsymbol{x}}^n = \int_Z \eta_{z+\pi(x_1)} \times \cdots \times \eta_{z+\pi(x_n)} \, d\nu(z) \quad \text{for } \boldsymbol{x} = (x_1, x_2, \ldots, x_n).$$

The following two lemmas can be viewed as generalizations of Lemma 3.3 and Theorem 3.4 in [15], respectively.

LEMMA 4.4. *Let $\pi : (X, \mathcal{B}_X, \mu, T) \to (Z, \mathcal{B}_Z, \nu, R)$ be a continuous factor map to its Kronecker factor. Assume that $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ is a measurable cover of X. Then*

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for any measurable partition α finer than $\mathcal{U}$ as a cover, there exists an increasing sequence $S \subset \mathbb{Z}_+$ with $h^S_\mu(T, \alpha) > 0$ if and only if $\lambda^n_{\boldsymbol{x}}(U^c_1 \times \cdots \times U^c_n) > 0$ for all $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^{(n)}$.

Proof. ($\Rightarrow$) In contrast, we may assume that $\lambda^n_{\mathbf{x}}(U^c_1 \times \cdots \times U^c_n) = 0$ for some $\mathbf{x} = (x_1, \ldots, x_n) \in X^{(n)}$. Let $C_i = \{z \in Z : \eta_{z+\pi(x_i)}(U^c_i) > 0\}$ for $i = 1, \ldots, n$. Then

$$\mu(U^c_i \setminus \pi^{-1}(C_i)) = \int_Z \eta_{z+\pi(x_i)}(U^c_i \cap \pi^{-1}(C^c_i))\, d\nu(z) = 0.$$

Put $D_i = \pi^{-1}(C_i) \cup (U^c_i \setminus \pi^{-1}(C_i))$. Then $D_i \in \pi^{-1}(\mathcal{B}_Z) = \mathcal{K}_\mu$ and $D^c_i \subset U_i$, where $\mathcal{K}_\mu$ is the Kronecker factor of X.

For any $\mathbf{s} = (s(1), \ldots, s(n)) \in \{0, 1\}^n$, let $D_{\mathbf{s}} = \bigcap_{i=1}^n D_i(s(i))$, where $D_i(0) = D_i$ and $D_i(1) = D^c_i$. Set $E_1 = (\bigcap_{i=1}^n D_i) \cap U_1$ and $E_j = (\bigcap_{i=1}^n D_i) \cap (U_j \setminus \bigcup_{i=1}^{j-1} U_i)$ for $j = 2, \ldots, n$.

Consider the measurable partition

$$\alpha = \{D_{\mathbf{s}} : \mathbf{s} \in \{0, 1\}^n \setminus \{(0, \ldots, 0)\}\} \cup \{E_1, \ldots, E_n\}.$$

For any $\mathbf{s} \in \{0, 1\}^n \setminus \{(0, \ldots, 0)\}$, we have $s(i) = 1$ for some $i = 1, \ldots, n$, then $D_{\mathbf{s}} \subset D^c_i \subset U_i$. It is straightforward that for all $1 \le j \le n$, $E_j \subset U_j$. Thus, α is finer than $\mathcal{U}$ and by hypothesis, there exists an increasing sequence S of $\mathbb{Z}_+$ with $h^S_\mu(T, \alpha) > 0$.

However, since $\lambda^n_{\mathbf{x}}(U^c_1 \times \cdots \times U^c_n) = 0$, we deduce $\nu(\bigcap_{i=1}^n C_i) = 0$ and hence $\mu(\bigcap_{i=1}^n D_i) = 0$. Thus, we have $E_1, \ldots, E_n \in \mathcal{K}_\mu$. It is also clear that $D_{\mathbf{s}} \in \mathcal{K}_\mu$ for all $\mathbf{s} \in \{0, 1\}^n \setminus \{(0, \ldots, 0)\}$, as $D_1, \ldots, D_n \in \mathcal{K}_\mu$. Therefore, each element of α is $\mathcal{K}_\mu$-measurable, by [15, Lemma 2.2],

$$h^S_\mu(T, \alpha) \le H_\mu(\alpha|\mathcal{K}_\mu) = 0,$$

which is a contradiction.

($\Leftarrow$) Assume $\lambda^n_{\mathbf{x}}(U^c_1 \times \cdots \times U^c_n) > 0$ for any $\mathbf{x} \in X^{(n)}$. In particular, we take $\mathbf{x} = (x, \ldots, x)$ such that $\pi(x)$ is the identity element of group Z. Without loss of generality, we may assume that any finite measurable partition α which is finer than $\mathcal{U}$ as a cover is of the type $\alpha = \{A_1, A_2, \ldots, A_n\}$ with $A_i \subset U_i$, for $1 \leqslant i \leqslant n$. Let α be one of such partitions. We observe that

$$\int_Z \eta_z(A^c_1) \ldots \eta_z(A^c_n)\, d\nu(z) \ge \int_Z \eta_z(U^c_1) \ldots \eta_z(U^c_n)\, d\nu(z) = \lambda^n_{\mathbf{x}}(U^c_1 \times \cdots \times U^c_n) > 0.$$

Therefore, $A_j \notin \mathcal{K}_\mu$ for some $1 \leqslant j \leqslant n$. It follows from [15, Theorem 2.3] that there exists a sequence $S \subset \mathbb{Z}_+$ such that $h^S_\mu(T, \alpha) = H_\mu(\alpha \mid \mathcal{K}_\mu) > 0$. This finishes the proof. $\square$

LEMMA 4.5. *For any $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^{(n)}$,*

$$SE^\mu_n(X, T) = \operatorname{supp} \lambda^n_{\boldsymbol{x}} \setminus \Delta_n(X).$$

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Proof. On the one hand, let $\mathbf{y} = (y_1, \ldots, y_n) \in SE_n^{\mu}(X, T)$. We show that $\mathbf{y} \in \operatorname{supp} \lambda_{\mathbf{x}}^n \setminus \Delta_n(X)$. It suffices to prove that for any measurable neighborhood $U_1 \times \cdots \times U_n$ of $\mathbf{y}$,

$$\lambda_{\mathbf{x}}^n(U_1 \times U_2 \times \cdots \times U_n) > 0.$$

Without loss of generality, we assume that $U_i \cap U_j = \emptyset$ if $y_i \neq y_j$. Then $\mathcal{U} = \{U_1^c, U_2^c, \ldots, U_n^c\}$ is a finite cover of X. It is clear that any finite measurable partition α finer than $\mathcal{U}$ as a cover is an admissible partition with respect to $\mathbf{y}$. Therefore, there exists an increasing sequence $S \subset \mathbb{Z}_+$ with $h_\mu^S(T, \alpha) > 0$. By Lemma 4.4, we obtain that

$$\lambda_{\mathbf{x}}^n(U_1 \times U_2 \times \cdots \times U_n) > 0,$$

which implies that $\mathbf{y} \in \operatorname{supp} \lambda_{\mathbf{x}}^n$. Since $\mathbf{y} \notin \Delta_n(X)$, $\mathbf{y} \in \operatorname{supp} \lambda_{\mathbf{x}}^n \setminus \Delta_n(X)$.

On the other hand, let $\mathbf{y} = (y_1, \ldots, y_n) \in \operatorname{supp} \lambda_{\mathbf{x}}^n \setminus \Delta_n(X)$. We show that for any admissible partition $\alpha = \{A_1, A_2, \ldots, A_k\}$ with respect to $\mathbf{y}$, there exists an increasing sequence $S \subset \mathbb{Z}_+$ such that $h_\mu^S(T, \alpha) > 0$. Since α is an admissible partition with respect to $\mathbf{y}$, there exist closed neighborhoods U_i of y_i, $1 \leqslant i \leqslant n$, such that for each $j \in \{1, 2, \ldots, k\}$, we find $i_j \in \{1, 2, \ldots, n\}$ with $A_j \subset U_{i_j}^c$. That is, α is finer than $\mathcal{U} = \{U_1^c, U_2^c, \ldots, U_n^c\}$ as a cover. Since

$$\lambda_{\mathbf{x}}^n(U_1 \times U_2 \times \cdots \times U_n) > 0,$$

by Lemma 4.4, there exists an increasing sequence $S \subset \mathbb{Z}_+$ such that $h_\mu^S(T, \alpha) > 0$. $\square$

Now we are ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. We only need to prove that $SE_n^{\mu,e}(X, T) \subset MS_n^{\mu,e}(X, T)$. We let $\pi : (X, \mathcal{B}_X, \mu, T) \to (Z, \mathcal{B}_Z, \nu, R)$ be a continuous factor map to its Kronecker factor. For any $\mathbf{y} = (y_1, \ldots, y_n) \in SE_n^{\mu,e}(X, T)$, let $U_1 \times U_2 \times \cdots \times U_n$ be an open neighborhood of $\mathbf{y}$ such that $U_i \cap U_j = \emptyset$ for $1 \leq i \neq j \leq n$. By Lemma 4.5, one has $\lambda_{\mathbf{x}}^n(U_1 \times U_2 \times \cdots \times U_n) > 0$ for any $\mathbf{x} = (x_1, \ldots, x_n) \in X^{(n)}$. Since the map $\mathbf{x} \mapsto \lambda_{\mathbf{x}}^n$ is continuous, X is compact, and $U_1, U_2, \ldots, U_n$ are open sets, it follows that there exists $\delta > 0$ such that for any $\mathbf{x} \in X^{(n)}$, $\lambda_{\mathbf{x}}^n(U_1 \times U_2 \times \cdots \times U_n) \geq \delta$. As the map $\mathbf{x} \mapsto \lambda_{\mathbf{x}}^n$ is an ergodic decomposition of $\mu^{(n)}$, there exists $B \subset X^{(n)}$ with $\mu^{(n)}(B) = 1$ such that $\lambda_{\mathbf{x}}^n$ is ergodic on $X^{(n)}$ for any $\mathbf{x} \in B$.

For any $A \in \mathcal{B}_X$ with $\mu(A) > 0$, there exists a subset C of $X^{(n)}$ with $\mu^{(n)}(C) > 0$ such that for any $\mathbf{x} \in C$,

$$\lambda_{\mathbf{x}}^n(A^n) > 0.$$

Take $\mathbf{x} \in B \cap C$, by the Birkhoff pointwise ergodic theorem, for $\lambda_{\mathbf{x}}^n$-almost every (a.e.) $(x_1', \ldots, x_n') \in X^{(n)}$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} 1_{U_1 \times U_2 \times \cdots \times U_n}(T^m x_1', \ldots, T^m x_n') = \lambda_{\mathbf{x}}^n(U_1 \times U_2 \times \cdots \times U_n) \geq \delta.$$

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Since $\lambda_{\mathbf{x}}^n(A^n) > 0$, there exists $(x_1'', \ldots, x_n'') \in A^n$ such that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N}\#\{m \in [0, N-1] : (T^m x_1'', \ldots, T^m x_n'') \in U_1 \times U_2 \times \cdots \times U_n\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{m=0}^{N-1} 1_{U_1\times U_2\times\cdots\times U_n}(T^m x_1'', \ldots, T^m x_n'') \\
&= \lambda_{\mathbf{x}}^n(U_1 \times U_2 \times \cdots \times U_n) \geq \delta.
\end{aligned}$$

Let $F = \{m \in \mathbb{Z}_+ : (T^m x_1'', \ldots, T^m x_n'') \in U_1 \times U_2 \times \cdots \times U_n\}$. Then $D(F) \geq \delta$ and hence $\mathbf{y} \in MS_n^{\mu,e}(X, T)$. This finishes the proof. □

4.2. *Non-ergodic case*

LEMMA 4.6. *Let (X, T) be a t.d.s. For any $\mu \in M(X, T)$ with the form $\mu = \sum_{i=1}^m \lambda_i \nu_i$, where $\nu_i \in M^e(X, T)$, $\sum_{i=1}^m \lambda_i = 1$, and $\lambda_i > 0$, one has*

$$\bigcup_{i=1}^m SE_n^{\nu_i}(X, T) \subset SE_n^{\mu}(X, T) \tag{4.1}$$

and

$$\bigcap_{i=1}^m SM_n^{\nu_i}(X, T) = SM_n^{\mu}(X, T). \tag{4.2}$$

Proof. We first prove equation (4.1). For any $\mathbf{x} = (x_1, \ldots, x_n) \in \bigcup_{i=1}^m SE_n^{\nu_i}(X, T)$, there exists $i \in \{1, 2, \ldots, m\}$ such that $\mathbf{x} \in SE_n^{\nu_i}(X, T)$ and then for any admissible partition α with respect to $\mathbf{x}$, there exists $S = \{s_j\}_{j=1}^{\infty}$ such that $h_{\nu_i}^S(T, \alpha) > 0$. By the definition of the sequence entropy,

$$h_{\mu}^S(T, \alpha) = \limsup_{N\to\infty} \sum_{i=1}^m \lambda_i \frac{1}{N} H_{\nu_i}\Big(\bigvee_{j=0}^{N-1} T^{-s_j}\alpha\Big) \geq \lambda_i h_{\nu_i}^S(T, \alpha) > 0.$$

So $\mathbf{x} \in SE_n^{\mu}(X, T)$, which finishes the proof of equation (4.1).

Next, we show equation (4.2). For this, we only need to note that for any $A \in \mathcal{B}_X$, $\mu(A) > 0$ if and only if $\nu_j(A) > 0$ for some $j \in \{1, 2, \ldots m\}$. □

Proof of Theorem 1.6. We first claim that there is a t.d.s. (X, T) with $\mu_1, \mu_2 \in M^e(X, T)$ such that $SE_n^{\mu_1}(X, T) \neq SE_n^{\mu_2}(X, T)$. For example, we recall that the full shift on two symbols with the measure is defined by the probability vector $(1/2, 1/2)$. It has completely positive entropy and the measure has the full support. Thus, every non-diagonal n-tuple is a sequence entropy n-tuple for this measure. In particular, we consider two such full shifts $(X_1, T_1, \mu_1) = (\{0, 1\}^{\mathbb{Z}}, \sigma_1, \mu_1)$ and $(X_2, T_2, \mu_2) = (\{2, 3\}^{\mathbb{Z}}, \sigma_2, \mu_2)$, and define a new system (X, T) as $X = X_1 \bigsqcup X_2$, $T|_{X_i} = T_i$, $i = 1, 2$. Then, $\mu_1, \mu_2 \in M^e(X, T)$ and $SE_n^{\mu_1}(X, T) = X_1^{(n)} \setminus \Delta_n(X_1) \neq X_2^{(n)} \setminus \Delta_n(X_2) = SE_n^{\mu_2}(X, T)$.

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Let $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 \in M(X, T)$. By Lemma 4.6, if $SE_n^{\mu}(X, T) = SM_n^{\mu}(X, T)$, then we have

$$\bigcup_{i=1}^{2} SE_n^{\mu_i}(X, T) \subset SE_n^{\mu}(X, T) = SM_n^{\mu}(X, T) = \bigcap_{i=1}^{2} SM_n^{\mu_i}(X, T).$$

However, applying Theorem 1.3 to each $\mu_i \in M^e(X, T)$, one has

$$SE_n^{\mu_i}(X, T) = SM_n^{\mu_i}(X, T) \quad \text{for } i = 1, 2.$$

So $SE_n^{\mu_1}(X, T) = SE_n^{\mu_2}(X, T)$, which is a contradiction with our assumption. □

5. *Topological sequence entropy and mean sensitive tuples*

This section is devoted to providing some partial evidence for the conjecture that in a minimal system, every mean sensitive tuple is a topological sequence entropy tuple.

It is known that the topological sequence entropy tuple has lift property [29]. We can show that under the minimality condition, the mean sensitive tuple also has lift property. Let us begin with some notions. For $2 \le n \in \mathbb{N}$, we say that $(x_1, x_2, \ldots, x_n) \in X^{(n)} \setminus \Delta_n(X)$ (respectively $(x_1, x_2, \ldots, x_n) \in X^{(n)} \setminus \Delta_n'(X)$) is a *mean n-sensitive tuple* (respectively an *essential mean n-sensitive tuple*) if for any $\tau > 0$, there is $\delta = \delta(\tau) > 0$ such that for any non-empty open set $U \subset X$, there exist $y_1, y_2, \ldots, y_n \in U$ such that if we denote $F = \{k \in \mathbb{Z}_+ : T^k y_i \in B(x_i, \tau), i = 1, 2, \ldots, n\}$, then $\overline{D}(F) > \delta$. Denote the set of all mean n-sensitive tuples (respectively essential mean n-sensitive tuples) by $MS_n(X, T)$ (respectively $MS_n^e(X, T)$).

THEOREM 5.1. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between two t.d.s. Then,*

(1) $\pi^{(n)}(MS_n(X, T)) \subset MS_n(Y, S) \cup \Delta_n(Y)$ *for every $n \ge 2$;*
(2) $\pi^{(n)}(MS_n(X, T) \cup \Delta_n(X)) = MS_n(Y, S) \cup \Delta_n(Y)$ *for every $n \ge 2$, provided that (X, T) is minimal.*

Proof. Item (1) is easy to be proved by the definition. We only prove item (2).

Supposing that $(y_1, y_2, \ldots, y_n) \in MS_n(Y, S)$, we will show that there exists $(z_1, z_2, \ldots, z_n) \in MS_n(X, T)$ such that $\pi(z_i) = y_i$ for each $i = 1, 2, \ldots, n$. Fix $x \in X$ and let $U_m = B(x, 1/m)$. Since (X, T) is minimal, $\mathrm{int}(\pi(U_m)) \neq \emptyset$, where $\mathrm{int}(\pi(U_m))$ is the interior of $\pi(U_m)$. Since $(y_1, y_2, \ldots, y_n) \in MS_n(Y, S)$, there exists $\delta > 0$ and $y_m^1, \ldots, y_m^n \in \mathrm{int}(\pi(U_m))$ such that

$$\overline{D}(\{k \in \mathbb{Z}_+ : S^k y_m^i \in \overline{B(y_i, 1)} \text{ for } i = 1, \ldots, n\}) \ge \delta.$$

Then there exist $x_m^1, \ldots, x_m^n \in U_m$ with $\pi(x_m^i) = y_m^i$ such that for any $m \in \mathbb{N}$,

$$\overline{D}(\{k \in \mathbb{Z}_+ : T^k x_m^i \in \pi^{-1}(\overline{B(y_i, 1)}) \text{ for } i = 1, \ldots, n\}) \ge \delta.$$

Put

$$A = \prod_{i=1}^{n} \pi^{-1}(\overline{B(y_i, 1)}),$$

and it is clear that A is a compact subset of $X^{(n)}$.

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We can cover A with finite non-empty open sets of diameter less than 1, that is, $A \subset \bigcup_{i=1}^{N_1} A_1^i$ and $\operatorname{diam}(A_1^i) < 1$. Then for each $m \in \mathbb{N}$, there is $1 \leq N_1^m \leq N_1$ such that

$$\overline{D}(\{k \in \mathbb{Z}_+ : (T^k x_m^1, \ldots, T^k x_m^n) \in \overline{A_1^{N_1^m}} \cap A\}) \geq \delta/N_1.$$

Without loss of generality, we assume $N_1^m = 1$ for all $m \in \mathbb{N}$. Namely,

$$\overline{D}(\{k \in \mathbb{Z}_+ : (T^k x_m^1, \ldots, T^k x_m^n) \in \overline{A_1^1} \cap A\}) \geq \delta/N_1 \quad \text{for all } m \in \mathbb{N}.$$

Repeating the above procedure, for $l \geq 1$, we can cover $\overline{A_l^1} \cap A$ with finite non-empty open sets of diameter less than $1/(l+1)$, that is, $\overline{A_l^1} \cap A \subset \bigcup_{i=1}^{N_{l+1}} A_{l+1}^i$ and $\operatorname{diam}(A_{l+1}^i) < 1/(l+1)$. Then for each $m \in \mathbb{N}$, there is $1 \leq N_{l+1}^m \leq N_{l+1}$ such that

$$\overline{D}(\{k \in \mathbb{Z}_+ : (T^k x_m^1, \ldots, T^k x_m^n) \in \overline{A_{l+1}^{N_{l+1}^m}} \cap A\}) \geq \frac{\delta}{N_1 N_2 \cdots N_{l+1}}.$$

Without loss of generality, we assume $N_{l+1}^m = 1$ for all $m \in \mathbb{N}$. Namely,

$$\overline{D}(\{k \in \mathbb{Z}_+ : (T^k x_m^1, \ldots, T^k x_m^n) \in \overline{A_{l+1}^1} \cap A\}) \geq \frac{\delta}{N_1 N_2 \cdots N_{l+1}} \quad \text{for all } m \in \mathbb{N}.$$

It is clear that there is a unique point $(z_1^1, \ldots, z_n^1) \in \bigcap_{l=1}^{\infty} \overline{A_l^1} \cap A$. We claim that $(z_1^1, \ldots, z_n^1) \in MS_n(X, T)$. In fact, for any $\tau > 0$, there is $l \in \mathbb{N}$ such that $\overline{A_l^1} \cap A \subset V_1 \times \cdots \times V_n$, where $V_i = B(z_i^1, \tau)$ for $i = 1, \ldots, n$. By the construction, for any $m \in \mathbb{N}$, there are $x_m^1, \ldots, x_m^n \in U_m$ such that

$$\overline{D}(\{k \in \mathbb{Z}_+ : (T^k x_m^1, \ldots, T^k x_m^n) \in \overline{A_l^1} \cap A\}) \geq \frac{\delta}{N_1 N_2 \cdots N_l}$$

and so

$$\overline{D}(\{k \in \mathbb{Z}_+ : (T^k x_m^1, \ldots, T^k x_m^n) \in V_1 \times \cdots \times V_n\}) \geq \frac{\delta}{N_1 N_2 \cdots N_l}$$

for all $m \in \mathbb{N}$. For any non-empty open set $U \subset X$, since x is a transitive point, there is $s \in \mathbb{Z}$ such that $T^s x \in U$. We can choose $m \in \mathbb{Z}$ such that $T^s U_m \subset U$. This implies that $T^s x_m^1, \ldots, T^s x_m^n \in U$ and

$$\overline{D}(\{k \in \mathbb{Z}_+ : (T^k (T^s x_m^1), \ldots, T^k (T^s x_m^n)) \in V_1 \times \cdots \times V_n\}) \geq \frac{\delta}{N_1 N_2 \cdots N_l}.$$

So we have $(z_1^1, \ldots, z_n^1) \in MS_n(X, T)$.

Similarly, for each $p \in \mathbb{N}$, there exists $(z_1^p, \ldots, z_n^p) \in MS_n(X, T) \cap \prod_{i=1}^n \pi^{-1}(\overline{B(y_i, 1/p)})$. Set $z_i^p \to z_i$ as $p \to \infty$. Then $(z_1, \ldots, z_n) \in MS_n(X, T) \cup \Delta_n(X)$ and $\pi(z_i) = y_i$. □

Denote by $\mathcal{A}(MS_2(X, T))$ the smallest closed $T \times T$-invariant equivalence relation containing $MS_2(X, T)$.

COROLLARY 5.2. *Let (X, T) be a minimal t.d.s. Then $X/\mathcal{A}(MS_2(X, T))$ is the maximal mean equicontinuous factor of (X, T).*

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Proof. Let $Y = X/\mathcal{A}(MS_2(X,T))$ and $\pi : (X,T) \to (Y,S)$ be the corresponding factor map. We show that (Y,S) is mean equicontinuous. Assume that (Y,S) is not mean equicontinuous, by [25, Corollary 5.5], (Y,S) is mean sensitive. Then by [27, Theorem 4.4], $MS_2(Y,S) \neq \emptyset$. By Theorem 5.1, there exists $(x_1,x_2) \in MS_2(X,T)$ such that $(\pi(x_1),\pi(x_2)) \in MS_2(Y,S)$. Then $(x_1,x_2) \notin R_\pi := \{(x,x') \in X \times X : \pi(x) = \pi(x')\}$, which is a contradiction with $R_\pi = \mathcal{A}(MS_2(X,T))$.

Let (Z,W) be a mean equicontinuous t.d.s. and $\theta : (X,T) \to (Z,W)$ be a factor map. Since (X,T) is minimal, so is (Z,W). Then by [25, Corollary 5.5] and [27, Theorem 4.4], $MS_2(Z,W) = \emptyset$. By Theorem 5.1, $MS_2(X,T) \subset R_\theta$, where R_θ is the corresponding equivalence relation with respect to θ. This implies that (Z,W) is a factor of (Y,S) and so (Y,S) is the maximal mean equicontinuous factor of (X,T). □

In the following, we show Theorem 1.1. Let us begin with some preparations.

Definition 5.3. [18] Let (X,T) be a t.d.s.

- For a tuple $(A_1,A_2,\ldots,A_n)$ of subsets of X, we say that a set $J \subseteq \mathbb{Z}_+$ is an *independence set* for A if for every non-empty finite subset $I \subseteq J$ and function $\sigma : I \to \{1,2,\ldots,n\}$, we have $\bigcap_{k\in I} T^{-k}A_{\sigma(k)} \neq \emptyset$.
- For $n \geq 2$, we call a tuple $\mathbf{x} = (x_1,\ldots,x_n) \in X^{(n)}$ an *IT-tuple* if for any product neighborhood $U_1 \times U_2 \times \cdots \times U_n$ of $\mathbf{x}$ in $X^{(n)}$, the tuple $(U_1,U_2,\ldots,U_n)$ has an infinite independence set. We denote the set of IT-tuples of length n by $\mathrm{IT}_n(X,T)$.
- For $n \geq 2$, we call an IT-tuple $\mathbf{x} = (x_1,\ldots,x_n) \in X^{(n)}$ an essential *IT-tuple* if $x_i \neq x_j$ for any $i \neq j$. We denote the set of all essential IT-tuples of length n by $\mathrm{IT}_n^e(X,T)$.

PROPOSITION 5.4. [13, Proposition 3.2] *Let X be a compact metric topological group with the left Haar measure μ, and let $n \in \mathbb{N}$ with $n \geq 2$. Suppose that $V_1,\ldots,V_n \subset X$ are compact subsets satisfying that*

(i) $\overline{\mathrm{int}\, V_i} = V_i$ *for* $i = 1,2,\ldots,n$*;*
(ii) $\mathrm{int}(V_i) \cap \mathrm{int}(V_j) = \emptyset$ *for all* $1 \leq i \neq j \leq n$*;*
(iii) $\mu(\bigcap_{1\leq i\leq n} V_i) > 0$.

Further, assume that $T : X \to X$ is a minimal rotation and $\mathcal{G} \subset X$ is a residual set. Then there exists an infinite set $I \subset \mathbb{Z}_+$ such that for all $a \in \{1,2,\ldots,n\}^I$, there exists $x \in \mathcal{G}$ with the property that

$$x \in \bigcap_{k\in I} T^{-k}\mathrm{int}(V_{a(k)}), \quad \text{i.e. } T^k x \in \mathrm{int}(V_{a(k)}) \quad \textit{for any } k \in I. \tag{5.1}$$

A subset $Z \subset X$ is called *proper* if Z is a compact subset with $\overline{\mathrm{int}(Z)} = Z$. The following lemma can help us to complete the proof of Theorem 1.1.

LEMMA 5.5. *Let (X,T) and (Y,S) be two t.d.s., and $\pi : (X,T) \to (Y,S)$ be a factor map. Suppose that (X,T) is minimal. Then the image of proper subsets of X under π is a proper subset of Y.*

Proof. Given a proper subset Z of X, we will show $\pi(Z)$ is also proper. It is clear that $\pi(Z)$ is compact, as π is continuous. Now we prove $\overline{\mathrm{int}(\pi(Z))} = \pi(Z)$.

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It follows from the closeness of $\pi(Z)$ that $\overline{\mathrm{int}(\pi(Z))} \subset \pi(Z)$. However, for any $y \in \pi(Z)$, take $x \in \pi^{-1}(y) \cap Z$. Since $\pi^{-1}(y) \cap Z = \pi^{-1}(y) \cap \overline{\mathrm{int}(Z)}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in \mathrm{int}(Z)$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}$ satisfying

$$\lim_{n\to\infty} r_n = 0 \quad \text{and} \quad B(x_n, r_n) \subset \mathrm{int}(Z).$$

By the minimality of (X, T), we have π is semi-open, and hence $\mathrm{int}(\pi(B(x_n, r_n))) \neq \emptyset$. Thus, there exists $x_n' \in B(x_n, r_n)$ such that $\pi(x_n') \in \mathrm{int}(\pi(B(x_n, r_n))) \subset \mathrm{int}(\pi(Z))$. Since $x_n' \in B(x_n, r_n)$ and $\lim_{n\to\infty} x_n = x$, one has $\lim_{n\to\infty} x_n' = x$, and hence $\lim_{n\to\infty} \pi(x_n') = \pi(x) = y$. This implies that $y \in \overline{\mathrm{int}(\pi(Z))}$, which finishes the proof. □

Inspired by [**13**, Proposition 3.7], we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. It suffices to prove $MS_n^e(X, T) \subset IT_n^e(X, T)$. Given $\mathbf{x} = (x_1, \ldots, x_n) \in MS_n^e(X, T)$, we will show that $\mathbf{x} \in IT_n^e(X, T)$.

Since the minimal t.d.s. (X_{eq}, T_{eq}) is the maximal equicontinuous factor of (X, T), then X_{eq} can be viewed as a compact metric group with a T_{eq}-invariant metric d_{eq}. Let μ be the left Haar probability measure of X_{eq}, which is also the unique T_{eq}-invariant probability measure of (X_{eq}, T_{eq}). Let

$$X_1 = \{x \in X : \#\{\pi^{-1}(\pi(x))\} = 1\}, \quad Y_1 = \pi(X_1).$$

Then Y_1 is a dense G_δ-set as π is almost one to one.

Without loss of generality, assume that $\epsilon = \frac{1}{4} \min_{1\le i\ne j\le n} d(x_i, x_j)$. Let $U_i = \overline{B_\epsilon(x_i)}$ for $1 \le i \le n$. Then U_i is proper for each $1 \le i \le n$. We will show that $U_1, U_2, \ldots, U_n$ is an infinite independent tuple of (X, T), that is, there is some infinite set $I \subseteq \mathbb{Z}_+$ such that

$$\bigcap_{k\in I} T^{-k} U_{a(k)} \neq \emptyset \quad \text{for all } a \in \{1, 2, \ldots, n\}^I.$$

Let $V_i = \pi(U_i)$ for $1 \le i \le n$. By Lemma 5.5, V_i is proper for each $i \in \{1, 2, \ldots, n\}$. We claim that $\mathrm{int}(V_i) \cap \mathrm{int}(V_j) = \emptyset$ for all $1 \le i \neq j \le n$. In fact, if there is some $1 \le i \neq j \le n$ such that $\mathrm{int}(V_i) \cap \mathrm{int}(V_j) \neq \emptyset$, then

$$\mathrm{int}(V_i) \cap \mathrm{int}(V_j) \cap Y_1 \neq \emptyset,$$

as Y_1 is a dense G_δ-set. Let $y \in \mathrm{int}(V_i) \cap \mathrm{int}(V_j) \cap Y_1$. Then there are $x_i \in U_i$ and $x_j \in U_j$ such that $y = \pi(x_i) = \pi(x_j)$, which contradicts with $y \in Y_1$.

Choose a non-empty open set $W_m \subset X$ with $\mathrm{diam}(\pi(W_m)) < 1/m$ for each $m \in \mathbb{N}$. Since $\mathbf{x} \in MS_n^e(X, T)$, there exist $\delta > 0$ and $\mathbf{x}^m = (x_1^m, x_2^m, \ldots, x_n^m) \in W_m \times \cdots \times W_m$ such that $\overline{D}(N(\mathbf{x}^m, U_1 \times U_2 \times \cdots \times U_n)) \ge \delta$. Let $\mathbf{y}^m = (y_1^m, y_2^m, \ldots, y_n^m) = \pi^{(n)}(\mathbf{x}^m)$. Then,

$$\overline{D}(N(\mathbf{y}^m, V_1 \times V_2 \times \cdots \times V_n)) \ge \delta.$$

For $p \in \overline{D}(N(\mathbf{y}^m, V_1 \times V_2 \times \cdots \times V_n))$, $T_{eq}^p y_i^m \in V_i$ for each $i = 1, 2, \ldots, n$. As $\text{diam}(\pi(W_m)) < 1/m$, $d_{eq}(y_1^m, y_i^m) < 1/m$ for $1 \leq i \leq n$. Note that

$$d_{eq}(T_{eq}^p y_1^m, T_{eq}^p y_i^m) = d_{eq}(y_1^m, y_i^m) < \frac{1}{m} \quad \text{for } 1 \leq i \leq n.$$

Let $V_i^m = B_{1/m}(V_i) = \{y \in X_{eq} : d_{eq}(y, V_i) < 1/m\}$. Then, $T_{eq}^p y_1^m \in \bigcap_{i=1}^n V_i^m$ and

$$\overline{D}(N(y_1^m, \bigcap_{i=1}^n V_i^m)) \geq \delta.$$

Since (X_{eq}, T_{eq}) is uniquely ergodic with respect to a measure μ, $\mu(\bigcap_{i=1}^n V_i^m) \geq \delta$. Letting $m \to \infty$, one has $\mu(\bigcap_{i=1}^n V_i) \geq \delta > 0$.

By Proposition 5.4, there is an infinite $I \subseteq \mathbb{Z}_+$ such that for all $a \in \{1, 2, \ldots, n\}^I$, there exists $y_0 \in Y_1$ with the property that

$$y_0 \in \bigcap_{k \in I} T_{eq}^{-k} \text{int}(V_{a(k)}).$$

Set $\pi^{-1}(y_0) = \{x_0\}$. Then

$$x_0 \in \bigcap_{k \in I} T^{-k} U_{a(k)},$$

which implies that $(x_1, x_2, \ldots, x_n) \in IT_n(X, T)$. □

Acknowledgments. We thank the referee for a very careful reading and many useful comments, which helped us to improve the paper. Research of J.L. is supported by NNSF of China (Grant No. 12031019); C.L. is partially supported by NNSF of China (Grant No. 12090012); S.T. is supported by NNSF of China (Grants No. 11801584 and No. 12171175); and T.Y. is supported by NNSF of China (Grant No. 12001354) and STU Scientific Research Foundation for Talents (Grant No. NTF19047).

A. *Appendix. Proof of Lemma 4.3*

In this section, we give the proof of Lemma 4.3.

LEMMA A.1. *For an m.p.s. $(X, \mathcal{B}_X, \mu, T)$ with $\mathcal{K}_\mu$ its Kronecker factor, $n \in \mathbb{N}$ and $f_i \in L^\infty(X, \mu)$, $i = 1, \ldots, n$, we have*

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^M \prod_{i=1}^n f_i(T^m x_i) = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^M \prod_{i=1}^n \mathbb{E}(f_i | \mathcal{K}_\mu)(T^m x_i).$$

Proof. On the one hand, by the Birkhoff ergodic theorem, for $\mathbf{x} = (x_1, \ldots, x_n) \in X^{(n)}$, let $F(\mathbf{x}) = F(x_1, \ldots, x_n) = \prod_{i=1}^n f_i(x_i)$,

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^M \prod_{i=1}^n f_i(T^m x_i) = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^M F((T^{(n)})^m \mathbf{x}) = \mathbb{E}_{\mu^{(n)}}\bigg(\prod_{i=1}^n f_i | I_{\mu^{(n)}}\bigg)(\mathbf{x}),$$

where $I_{\mu^{(n)}} = \{A \in \mathcal{B}_X^{(n)} : T^{(n)} A = A\}$.

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On the other hand, following [15, Lemma 4.4], we have $(\mathcal{K}_\mu)^{\bigotimes n} = \mathcal{K}_{\mu^{(n)}}$. Then for $\mathbf{x} = (x_1, \ldots, x_n) \in X^{(n)}$,

$$\prod_{i=1}^{n} \mathbb{E}_\mu(f_i|\mathcal{K}_\mu)(x_i) = \mathbb{E}_{\mu^{(n)}}\bigg(\prod_{i=1}^{n} f_i|(\mathcal{K}_\mu)^{\bigotimes n}\bigg)(\mathbf{x}) = \mathbb{E}_{\mu^{(n)}}\bigg(\prod_{i=1}^{n} f_i|\mathcal{K}_{\mu^{(n)}}\bigg)(\mathbf{x}).$$

This implies that

$$\begin{aligned}\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{n}\mathbb{E}_\mu(f_i|\mathcal{K}_\mu)(T^m x_i) &= \mathbb{E}_{\mu^{(n)}}\bigg(\prod_{i=1}^{n}\mathbb{E}_\mu(f_i|\mathcal{K}_\mu)|I_{\mu^{(n)}}\bigg)(\mathbf{x})\\ &= \mathbb{E}_{\mu^{(n)}}(\mathbb{E}_{\mu^{(n)}}\bigg(\prod_{i=1}^{n} f_i|\mathcal{K}_{\mu^{(n)}}\bigg)|I_{\mu^{(n)}})(\mathbf{x})\\ &= \mathbb{E}_{\mu^{(n)}}\bigg(\prod_{i=1}^{n} f_i|I_{\mu^{(n)}}\bigg)(\mathbf{x}),\end{aligned}$$

where the last equality follows from the fact that $I_{\mu^{(n)}} \subset \mathcal{K}_{\mu^{(n)}}$. □

LEMMA A.2. *Let $(Z, \mathcal{B}_Z, \nu, R)$ be a minimal rotation on a compact abelian group. Then for any $n \in \mathbb{N}$ and $\phi_i \in L^\infty(Z, \nu)$, $i = 1, \ldots, n$,*

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{n}\phi_i(R^m z_i) = \int_Z \prod_{i=1}^{n}\phi_i(z_i + z)\,d\nu(z) \quad \text{for } \nu^{(n)}\text{-a.e. } (z_1, \ldots, z_n).$$

Proof. Since $(Z, \mathcal{B}_Z, \nu, R)$ is a minimal rotation on a compact abelian group, there exists $a \in Z$ such that $R^m z = z + ma$ for any $z \in Z$.

Let $F(z) = \prod_{i=1}^{n}\phi_i(z_i + z)$. Then $F(R^m e_Z) = F(ma)$, where e_Z is the identity element of Z. Since (Z, R) is minimal equicontinuous, $(Z, \mathcal{B}_Z, \nu, R)$ is uniquely ergodic. By an approximation argument, we have, for $\nu^{(n)}$-a.e. $(z_1, \ldots, z_n)$,

$$\begin{aligned}\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{n}\phi_i(R^m z_i) &= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{n}\phi_i(z_i + ma)\\ &= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}F(ma) = \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}F(R^m e_Z)\\ &= \int_Z F(z)\,d\nu(z) = \int_Z\prod_{i=1}^{n}\phi_i(z_i + z)d\nu(z).\end{aligned}$$

The proof is completed. □

Proof of Lemma 4.3. Let $z \mapsto \eta_z$ be the disintegration of μ over the continuous factor map π from $(X, \mathcal{B}_X, \mu, T)$ to its Kronecker factor $(Z, \mathcal{B}_Z, \nu, R)$. For $n \in \mathbb{N}$, define

$$\lambda_{\mathbf{x}}^n = \int_Z \eta_{z+\pi(x_1)} \times \cdots \times \eta_{z+\pi(x_n)}\,d\nu(z)$$

for every $\mathbf{x} = (x_1, \ldots, x_n) \in X^{(n)}$.

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We first note that for each $\mathbf{x} \in X^{(n)}$, the measures $\eta_{z+\pi(x_i)}$ are defined for ν-a.e. $z \in Z$ and therefore is well defined. To prove that $\mathbf{x} \mapsto \lambda^n_{\mathbf{x}}$ is continuous, first note that uniform continuity implies

$$(u_1, \ldots, u_n) \mapsto \int_Z \prod_{i=1}^n f_i(z + u_i)\, d\nu(z)$$

from $Z^{(n)}$ to $\mathbb{C}$ is continuous whenever $f_i \colon Z \to \mathbb{C}$ are continuous. An approximation argument then gives continuity for every $f_i \in L^\infty(Z, \nu)$. In particular,

$$\mathbf{x} \mapsto \int_Z \prod_{i=1}^n \mathbb{E}(f_i \mid \mathcal{B}_Z)(z + \pi(x_i))\, d\nu(z)$$

from $X^{(n)}$ to $\mathbb{C}$ is continuous whenever $f_i \in L^\infty(X, \mu)$, which in turn implies continuity of $\mathbf{x} \mapsto \lambda^n_{\mathbf{x}}$.

To prove that $\mathbf{x} \mapsto \lambda^n_{\mathbf{x}}$ is an ergodic decomposition, we first calculate

$$\int_{X^{(n)}} \int_Z \prod_{i=1}^n \eta_{z+\pi(x_i)}\, d\nu(z)\, d\mu^{(n)}(\mathbf{x}) = \int_Z \prod_{i=1}^n \int_X \eta_{z+\pi(x_i)}\, d\mu(x_i)\, d\nu(z),$$

which is equal to $\mu^{(n)}$ because all inner integrals are equal to μ. We conclude that

$$\mu^{(n)} = \int_{X^{(n)}} \lambda^n_{\mathbf{x}}\, d\mu^{(n)}(\mathbf{x}),$$

which shows $\mathbf{x} \mapsto \lambda^n_{\mathbf{x}}$ is a disintegration of $\mu^{(n)}$.

We are left with verifying that

$$\int_{X^{(n)}} F\, d\lambda^n_{\mathbf{x}} = \mathbb{E}_{\mu^{(n)}}(F \mid I_{\mu^{(n)}})(\mathbf{x})$$

for $\mu^{(n)}$-a.e. $\mathbf{x} \in X^{(n)}$ whenever $F \colon X^{(n)} \to \mathbb{C}$ is measurable and bounded. Recall that $I_{\mu^{(n)}}$ denotes the σ-algebra of $T^{(n)}$-invariant sets. Fix such an F. It follows from the pointwise ergodic theorem that

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^M F(T^m x_1, \ldots, T^m x_n) = \mathbb{E}_{\mu^{(n)}}(F \mid I_{\mu^{(n)}})(\mathbf{x})$$

for $\mu^{(n)}$-a.e. $\mathbf{x} \in X^{(n)}$. We therefore wish to prove that

$$\int_{X^{(n)}} F\, d\lambda^n_{\mathbf{x}} = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^M F(T^m x_1, \ldots, T^m x_n)$$

holds for $\mu^{(n)}$-a.e. $\mathbf{x} \in X^{(n)}$.

By an approximation argument, it suffices to verify that

$$\int_{X^{(n)}} f_1 \otimes \cdots \otimes f_n\, d\lambda^n_{\mathbf{x}} = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^M \prod_{i=1}^n f_i(T^m x_i)$$

holds for $\mu^{(n)}$-a.e. $\mathbf{x} \in X^{(n)}$ whenever f_i belongs to $L^\infty(X, \mu)$ for $i = 1, \ldots, n$.

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By Lemma A.1,

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{n} f_i(T^m x_i) = \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{n}\mathbb{E}(f_i \mid \mathcal{B}_Z)(T^m x_i)$$

for $\mu^{(n)}$-a.e. $\mathbf{x}\in X^{(n)}$. By Lemma A.2, for every ϕ_i in $L^\infty(Z,\nu)$,

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{n}\phi_i(R^m z_i) = \int_Z \prod_{i=1}^{n}\phi_i(z_i+z)\,d\nu(z)$$

for $\nu^{(n)}$-a.e. $\mathbf{z}\in Z^{(n)}$. Taking $\phi_i=\mathbb{E}(f_i \mid \mathcal{B}_Z)$ gives

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{n}\mathbb{E}(f_i \mid \mathcal{B}_Z)(T^m x_i) = \int_{X^{(n)}} f_1\otimes\cdots\otimes f_n\,d\lambda^n_{\mathbf{x}}$$

for $\mu^{(n)}$-a.e. $\mathbf{x}\in X^{(n)}$. □

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