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ON THE IDEAL EQUATION $I(B\cap C) = IB\cap IC$

BY

D. D. ANDERSON

ABSTRACT. Let R be an integral domain with quotient field K and let I be a nonzero ideal of R. We show (1) that I is invertible if and only if $I(\bigcap_\alpha B_\alpha) = \bigcap_\alpha IB_\alpha$ for every nonempty collection $\{B_\alpha\}$ of ideals of R and (2) that I is flat if and only if $I(B\cap C) = IB\cap IC$ for each pair of ideals B and C of R.

Let R be an integral domain with quotient field K and let I be a nonzero ideal of R. We show (1) that I is invertible if and only if $I(\bigcap_\alpha B_\alpha) = \bigcap_\alpha IB_\alpha$ for every nonempty collection $\{B_\alpha\}$ of ideals of R and (2) that I is flat if and only if $I(B\cap C) = IB\cap IC$ for each pair of ideals B and C of R.

THEOREM 1. *For a nonzero ideal I in an integral domain R, the following conditions are equivalent.*

(1) $I(\bigcap_\alpha B_\alpha) = \bigcap_\alpha IB_\alpha$ *for each nonempty collection* $\{B_\alpha\}$ *of ideals of* R.

(2) $I(\bigcap_\alpha B_\alpha) = \bigcap_\alpha IB_\alpha$ *for each nonempty collection* $\{B_\alpha\}$ *of fractional ideals of* R.

(3) I *is invertible.*

(4) I *is projective.*

Proof. (1) $\Rightarrow$ (2). We first show that (2) is true for a set $\{B_1, B_2\}$ of fractional ideals. There exists an $0\neq r\in R$ with $rB_1, rB_2\subseteq R$. Then

$$rI(B_1\cap B_2) = I(r(B_1\cap B_2)) = I(rB_1\cap rB_2) = I(rB_1)\cap I(rB_2) = r(IB_1\cap IB_2).$$

Hence $I(B_1\cap B_2) = IB_1\cap IB_2$. We now do the general case. Fix a $B_0\in\{B_\alpha\}$ and choose $0\neq r\in R$ with $rB_0\subseteq R$. Then $r(B_0\cap B_\alpha)\subseteq R$. Hence $rI(\bigcap_\alpha B_\alpha) = rI(\bigcap_\alpha (B_0\cap B_\alpha)) = I(\bigcap_\alpha r(B_0\cap B_\alpha)) = \bigcap_\alpha (Ir(B_0\cap B_\alpha)) = r\bigcap_\alpha (I(B_0\cap B_\alpha)) = r\bigcap_\alpha (IB_0\cap IB_\alpha) = r\bigcap_\alpha IB_\alpha$. Thus $I(\bigcap_\alpha B_\alpha) = \bigcap_\alpha IB_\alpha$.

(2) $\Rightarrow$ (3). $II^{-1} = I(\cap\{Ri^{-1}\mid 0\neq i\in R\}) = \cap IRi^{-1}\supseteq \cap R = R$. Hence $II^{-1} = R$, so I is invertible.

(3) $\Rightarrow$ (1). Clearly $I(\bigcap_\alpha B_\alpha)\subseteq\bigcap_\alpha IB_\alpha$. But $I^{-1}(\bigcap_\alpha IB_\alpha)\subseteq I^{-1}IB_\alpha = B_\alpha$, so that $I^{-1}(\bigcap_\alpha IB_\alpha)\subseteq\bigcap_\alpha B_\alpha$. Hence $\bigcap_\alpha IB_\alpha\subseteq I(\bigcap_\alpha B_\alpha)$.

The equivalence of (3) and (4) is well known.

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THEOREM 2. *For an ideal I in the integral domain R, the following conditions are equivalent.*

(1) $I(B \cap C) = IB \cap IC$ *for ideals B and C of R.*

(2) $I(B_1 \cap \cdots \cap B_n) = IB_1 \cap \cdots \cap IB_n$ *for fractional ideals $B_1, \ldots, B_n$ of R.*

(3) *I is a flat ideal of R.*

Proof. (1) ⇒ (2). The proof of the implication (1) ⇒ (2) of Theorem 1 gives (2) for the case $n = 2$. The result then follows by induction.

(2) ⇒ (3). Let J be an ideal of R and $0 \neq a \in R$. Then $I(J :_R a) = I(Ja^{-1} \cap R) = IJa^{-1} \cap I = (IJ :_I a)$. It follows from [1, Exercise 22, page 47] that I is flat.

(3) ⇒ (1). $I(B \cap C) = I \otimes (B \cap C) = (I \otimes B) \cap (I \otimes C) = IB \cap IC$ with the proper identification ([1, Proposition 6, page 17]).

One can consider to what extent Theorem 1 and Theorem 2 remain true if R is allowed to have zero-divisors. If I is invertible, then we still have $I(\bigcap_\alpha B_\alpha) = \bigcap_\alpha IB_\alpha$ for any collection of R-submodules of the total quotient ring of R. (This is given in [2, Exercise 17, page 80].) Conversely, if I is generated by regular elements and $I(\bigcap_\alpha B_\alpha) = \bigcap_\alpha IB_\alpha$ for each collection of regular fractional ideals of R, the same proof shows that I is invertible. If I is flat, then the proof given in Theorem 2 shows that $I(B \cap C) = IB \cap IC$ for ideals B and C of R. However, if (R, M) is a quasi-local ring with $M^2 = 0$, then clearly $M(B \cap C) = MB \cap MC$ for all ideals B and C of R (in fact, for any collection of ideals), but such an M need not be flat. Theorem 1 may also be generalized in another direction. If P is a projective R-module, then $\bigcap I_\alpha P = (\bigcap I_\alpha)P$ for any collection of ideals $\{I_\alpha\}$ of R.

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THE UNIVERSITY OF IOWA
IOWA CITY, IOWA 52242