BULL. AUSTRAL. MATH. SOC. 57N13, 57P10, 20F36
VOL. 58 (1998) [349–351]

Three topics in topology and group theory

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This thesis is comprised of three parts which treat, in turn, three quite distinct topics from low-dimensional topology and group theory.

1. EMBEDDING 3-MANIFOLDS IN 4-SPACE

The first and largest part of the thesis considers the question of which closed 3-manifolds embed in 4-space (specifically, in S^4 or an homology 4-sphere). The $\mathbb{Z}/2\mathbb{Z}$-Index Theorem is applied, in a manner suggested by Massey's proof of Whitney's conjecture on embeddings of nonorientable surfaces in S^4, to obtain a general embedding constraint for manifolds which are Seifert fibred over nonorientable surfaces. Considerable effort is also made to exhibit embeddings where possible. For a given 3-manifold M, various techniques may be employed to achieve this. One method involves using the surgery calculus for 3-manifolds to find a surgery description for M in which the link is a disjoint union of two slice sublinks and every surgery coefficient is 0. The manifold may then be embedded smoothly in S^4 via ambient surgery on such a link lying in the equatorial 3-sphere. By exhibiting embeddings for sufficiently many manifolds, the embedding constraint referred to above is shown to be sharp in cases where the set of strictly singular fibres is invariant under a change of orientation. In particular, one obtains a complete classification (originally obtained by J.A.Hillman, private communication) of the circle bundles which embed in any homology 4-sphere.

In addition to Seifert manifolds, we also consider the class of 3-manifolds which are unions $N \cup_\phi N$ where N is a circle bundle over the Möbius band, and by a further adaptation of the Index Theorem argument, and by exhibiting ambient surgery links, we determine precisely which of these embed in either a 4-sphere or an homology 4-sphere. (Up to orientation, there are exactly six which do). This result is important when considering the question for Sol^3-manifolds as is done in joint work with Hillman [1], where it is shown that there are (up to orientation) precisely ten Seifert fibred 3-manifolds with elementary amenable fundamental group and three Sol^3-manifolds which

Received 14th April, 1998
Thesis submitted to The University of Sydney, August 1997. Degree approved, February 1998. Supervisor: Dr Jonathan A. Hillman

embed in an homology 4-sphere. Twelve of these are shown, in the thesis, to embed via ambient surgery. (The remaining manifold is the Poincaré homology sphere which is known to embed topologically but not smoothly!)

2. Decomposition of PD^3-complexes

By analogy to the connected sum decomposition of 3-manifolds, we investigate the decomposition of 3-dimensional Poincaré complexes, homotopy analogues of 3-manifolds which essentially capture the algebraic properties (namely equivariant Poincaré duality) of 3-manifolds. In particular, we give a "virtual" answer to a question of C.T.C. Wall from 1967, showing that if the fundamental group of an orientable PD^3-complex has infinitely many ends then it is either a proper free product or virtually free of finite rank. It follows, using work of V.G. Turaev, that every PD^3-complex is finitely covered by one which is homotopy equivalent to a connected sum of aspherical PD^3-complexes and copies of $S^1 \times S^2$. Furthermore, we show that any torsion element of the fundamental group of an orientable PD^3-complex has finite centraliser, and thereby recover a result of Hillman. These sorts of results are of interest because they help us to distill those properties of 3-manifolds which follow from duality and group theoretic considerations alone, from those which are inherently topological.

3. Injective maps between Artin groups

The braid groups are well-known for their topological interpretation and their connection with the study of knots and links in 3-space. They also have a natural generalisation to the family of groups known as Artin groups, the study of which involves both combinatorial and geometric ideas and has thrown up some difficult and unanswered problems.

In this thesis we identify a family of interesting relationships amongst the Artin groups. We propose a large class of maps between Artin monoids which we show to be injective. For one of these so-called LCM-homomorphisms there is a natural way to realise the corresponding map on Artin groups geometrically as the map induced on fundamental groups by an inclusion of certain finite simplicial complexes. These complexes (first introduced by Salvetti) are known to be $K(G,1)$ spaces when the Artin groups are of finite type (corresponding to finite Coxeter groups), and in that case we show that the group homomorphism involved is injective. It is hoped that injectivity will hold also for homomorphisms of this kind between Artin groups of infinite type, however this remains only a conjecture.

In treating the finite type cases one uses the fact that a finite type Artin monoid is naturally a submonoid of its corresponding Artin group, a result which is not known general. In this thesis we establish injectivity of the Artin monoid for two infinite families, $\widetilde{B}_n$ and $\widetilde{C}_n$, of affine infinite type.

https://doi.org/10.1017/S0004972700032299 Published online by Cambridge University Press

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