ON FREE PRODUCTS OF CYCLIC ROTATION GROUPS

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We consider the group of rotations in three-dimensional Euclidean space, leaving the origin fixed. These rotations are represented by real orthogonal third-order matrices with positive determinant. It is known that this rotation group contains free non-abelian subgroups of continuous rank (see **1**).

In this paper we shall prove the following conjectures of J. de Groot (**1**, pp. 261–262):

THEOREM 1. *Two rotations with equal rotation angles α and with arbitrary but different rotation axes are free generators of a free group, if $\cos\alpha$ is transcendental.*

THEOREM 2. *A free product of at most continuously many cyclic groups can be isomorphically represented by a rotation group.*

More precisely: Theorem 2 is a special case of the following conjecture of J. de Groot (**1**, p. 262): A free product of at most continuously many rotation groups, each consisting of less than continuously many elements, can be isomorphically represented by a rotation group.

J. Mycielski at Wroclaw informed me that he, with S. Balcerzyk has proved a theorem, which includes our Theorem 2 as a special case; moreover, our Theorem 1 seems to intersect with a theorem proved by S. Balcerzyk.

Preliminaries. We define

$$A(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \qquad D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $i_\sigma = \pm 1$, $k_\sigma = 1, 2, 3, \ldots$ $(\sigma = 1, \ldots, s)$. Furthermore, we assume $\cos\alpha$ is transcendental. Then we have

LEMMA: *No product P_s $(s \geqslant 1)$ of the form*

$$P_s = D(\theta_0)A^{i_1k_1}(\alpha)D(\theta_1)A^{i_2k_2}(\alpha)\ldots A^{i_sk_s}(\alpha)D(\theta_s)$$

is the identity, if one of the following conditions is satisfied for $\sigma = 1, \ldots, s-1$:

(a) *θ_σ is not a multiple of π;*

(b) *θ_σ is not a multiple of 2π and the exponents of A are of alternating sign: $i_{\sigma+1} = -i_\sigma$.*

Proof: We use the formulas $(k > 0)$:

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$$\begin{aligned} \cos k\alpha &= 2^{k-1}\cos^k\alpha + \ldots, \\ \sin k\alpha &= \sin\alpha\,(2^{k-1}\cos^{k-1}\alpha + \ldots), \\ \sin^2\alpha &= 1 - \cos^2\alpha, \end{aligned}$$

where ... denote terms of lower degree in $\cos\alpha$, So we have

$$(1)\quad A^{i_\sigma k_\sigma}(\alpha)D(\theta_\sigma) = \begin{pmatrix} \cos\theta_\sigma, & -\sin\theta_\sigma, & 0 \\ q_\sigma \sin\theta_\sigma \cos\alpha + \ldots, & q_\sigma\cos\theta_\sigma\cos\alpha + \ldots, & -i_\sigma q_\sigma \sin\alpha + \ldots \\ i_\sigma q_\sigma \sin\theta_\sigma \sin\alpha + \ldots, & i_\sigma q_\sigma \cos\theta_\sigma \sin\alpha + \ldots, & q_\sigma \cos\alpha + \ldots \end{pmatrix},$$

where ... denote terms of lower degree in $\cos\alpha$ and $\sin\alpha$ and

$$q_\sigma = (2\cos\alpha)^{k_\sigma - 1}.$$

By induction with respect to σ we find that the elements of the matrices $P_\sigma = (p^\sigma{}_{ik})$ are polynomials in $\cos\alpha$, multiplied or not by a factor $\sin\alpha$. In particular the elements $p^\sigma{}_{32}$ and $p^\sigma{}_{33}$ obtain the form (we consider the leading terms only, denoting terms of lower degree by ...):

$$\begin{aligned} p^\sigma{}_{32} &= i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha + \ldots, \\ p^\sigma{}_{33} &= q_\sigma V_\sigma \cos\alpha + \ldots . \end{aligned}$$

Indeed, for $\sigma = 1$ we have $V_1 = 1$ and multiplying P_σ with the matrix (1), where σ is replaced by $\sigma + 1$, we find

$$\begin{aligned} p_{32}^{\sigma+1} &= i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha \cdot q_{\sigma+1}\cos\theta_{\sigma+1}\cos\alpha \\ &\quad + q_\sigma V_\sigma \cos\alpha \cdot i_{\sigma+1} q_{\sigma+1} \cos\theta_{\sigma+1} \sin\alpha + \ldots \\ &= i_{\sigma+1} q_{\sigma+1} q_\sigma V_\sigma \cos\theta_{\sigma+1} \cos\alpha \sin\alpha\,(i_\sigma i_{\sigma+1}\cos\theta_\sigma + 1) + \ldots \\ p_{33}^{\sigma+1} &= i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha \cdot -\, i_{\sigma+1} q_{\sigma+1} \sin\alpha \\ &\quad + q_\sigma V_\sigma \cos\alpha \cdot q_{\sigma+1} \cos\alpha + \ldots \\ &= q_\sigma q_{\sigma+1} V_\sigma \cos^2\alpha (i_\sigma i_{\sigma+1} \cos\theta_\sigma + 1) + \ldots . \end{aligned}$$

Hence,

$$V_{\sigma+1} = q_\sigma V_\sigma \cos\alpha\,(1 + i_\sigma i_{\sigma+1}\cos\theta_\sigma).$$

From this it follows that the coefficient of the leading term of $p^s{}_{33}$ does not vanish if

$$1 + i_\sigma i_{\sigma+1} \cos\theta_\sigma \neq 0 \qquad (\sigma = 1, \ldots, s-1),$$

that is, if (a) or (b) holds true.

Thus since $p^s{}_{33}$ is a polynomial in $\cos\alpha$ and $\cos\alpha$ is transcendental, the product P_s satisfying (a) or (b) obviously is unequal to the identity, by which the lemma is proved.

Proof of Theorem 1. Two rotations with rotation angles α, the axes of which intersect under an angle θ, may be represented by the matrices $A = A(\alpha)$ and $B = D(\theta)A(\alpha)D(-\theta)$. Clearly the theorem is proved if we show that A and B generate a free non-abelian group when $\cos\alpha$ is transcendental and θ is not a multiple of π.

Since all non-trivial products of the elements $A^{\pm 1}$ and $B^{\pm 1}$ have the form P_s satisfying condition (a), they are not equal to the identity by virtue of the lemma, by which the theorem is proved.

Proof of Theorem 2. J. von Neumann **(2)** proved that the real numbers x_t defined by

$$x_t = \sum_{n=0}^{\infty} 2^{2^{[nt]}-2^{n^2}} \qquad (t > 0)$$

are algebraically independent over the field of rational numbers.

We define

$$(2) \qquad \begin{cases} \phi_t = 2 \operatorname{arctg} x_t \\ \alpha = 2 \operatorname{arctg} x_1 \end{cases} \qquad (0 < t < 1).$$

Then, according to a theorem of J. de Groot **(1)**, we have:

The continuously many rotations

$$B_t = D(\phi_t)A(\alpha)D(-\phi_t) \qquad (0 < t < 1)$$

are free generators of a free rotation group.

Let (F) denote the group generated by the rotation F. We shall now prove:

The group generated by the continuously many rotations

$$F_t(\delta_t) = B_t D(\delta_t) B_t^{-1} \qquad (0 < t < 1)$$

is a free product of the cyclic groups $(F_t(\delta_t))$. *This obviously implies Theorem* 2.

Consider any non-trivial product

$$\begin{aligned} &F_{t1}^{m1}(\delta_{t1})F_{t2}^{m2}(\delta_{t2}) \ldots F_{ts}^{ms}(\delta_{ts}) \\ &\quad = D(\phi_{t1})A(\alpha)D(m_1\delta_{t1})A^{-1}(\alpha)D(\phi_{t2} - \phi_{t1})A(\alpha)D(m_2\delta_{t2}) \ldots \\ &\qquad A(\alpha)D(m_s\delta_{ts})A^{-1}(\alpha)D(-\phi_{ts}). \end{aligned}$$

We may assume that

$$m_k\delta_{tk} \qquad (k = 1, \ldots, s)$$

is not a multiple of 2π, for otherwise we have a trivial product. Furthermore, the numbers

$$\phi_{t_{k+1}} - \phi_{t_k} \qquad (k = 1, \ldots, s-1)$$

are not multiples of 2π by virtue of (2). Thus the product considered has the form P_{2s} satisfying condition (b). According to the lemma this product is unequal to the identity, by which the theorem is proved.

References

1. J. de Groot, *Orthogonal isomorphic representations of free groups*, Can. J. Math., *8* (1956), 256–262.
2. J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann., *99* (1928), 134–141.

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