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Lorenz attractors through Šil'nikov-type bifurcation. Part I†

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Abstract The main result of this paper is a construction of geometric Lorenz attractors (as axiomatically defined by J Guckenheimer) by means of an Ω-explosion The unperturbed vector field on $\mathbb{R}^3$ is assumed to have a hyperbolic fixed point, whose eigenvalues satisfy the inequalities $\lambda_1>0$, $\lambda_2<0$, $\lambda_3<0$ and $|\lambda_2|>|\lambda_1|>|\lambda_3|$ Moreover, the unstable manifold of the fixed point is supposed to form a double loop Under some other natural assumptions a generic two-parameter family containing the unperturbed vector field contains geometric Lorenz attractors

A possible application of this result is a method of proving the existence of geometric Lorenz attractors in concrete families of differential equations A detailed discussion of the method is in preparation and will be published as Part II

0 *Introduction*

In this paper we attempt to establish rigorous methods for verifying the existence of geometric Lorenz attractors in concrete examples of differential equations By a *geometric Lorenz attractor* we mean an object satisfying the axioms of J Guckenheimer [4]

Our method is based on a particular type of Ω-explosion We start with a vector field in $\mathbb{R}^3$ with a hyperbolic fixed point The eigenvalues of the linearization at this fixed point will be denoted by λ_1, λ_2 and λ_3 We assume that λ_2 and λ_3 are negative and $\lambda_1>0$ and that they satisfy the following inequality

$$|\lambda_2|>|\lambda_1|>|\lambda_3| \tag{0 1}$$

Therefore the unstable manifold of our point has dimension 1 and the stable manifold has dimension 2 The unstable manifold is assumed to be a part of the stable manifold, and as a result it forms a double loop (figure 1) (The reader should notice now that our vector fields are assumed to be symmetric with respect to the reflection through the eigendirection corresponding to the eigenvalue λ_3) In the unperturbed system the Ω-set contains the double loop of the unstable manifold It is a consequence of this assumption concerning the eigenvalues at the equilibrium

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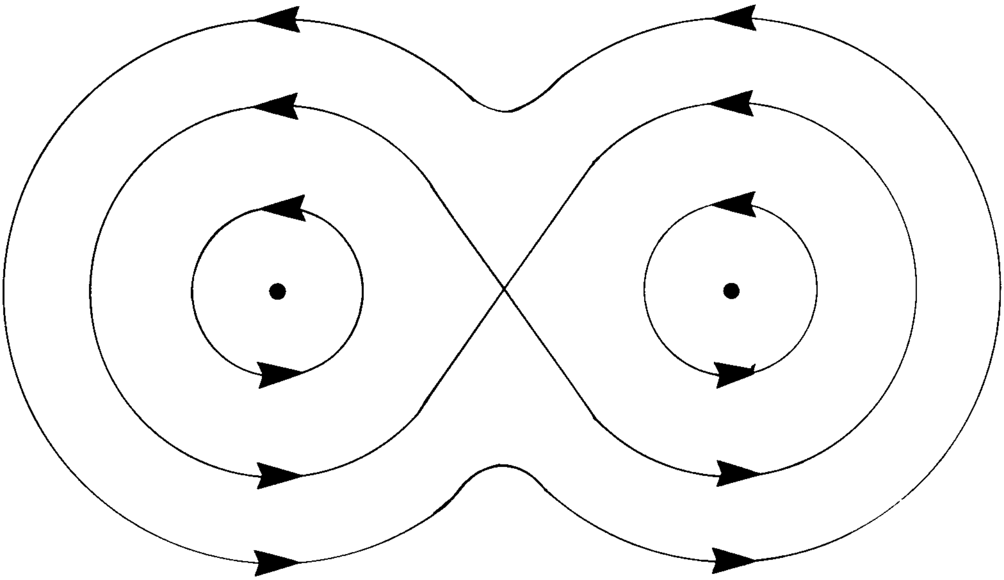

FIGURE 1

that *the double loop of the unstable manıfold of the unperturbed system ıs stable* Suppose now that our vector field ıs embedded ın a two-parameter famıly of vector fields It proves that ın a generıc famıly there are perturbatıons of our ınıtıal flow leadıng to a creatıon of a geometrıc Lorenz attractor

Our ıdea ıs related to so called Šıl'nıkov bıfurcatıon The ınequalıty (0 1) dıstınguıshes our assumptıons from those of Šıl'nıkov He assumes that any posıtıve eıgenvalue ıs larger than the absolute value of any negatıve eıgenvalue Under hıs assumptıons concernıng the eıgenvalues one can obtaın a suspensıon of the two-shıft as the Ω-set of the perturbed flow However, one cannot obtaın an attractor In thıs manner one obtaıns a system wıth transıent chaos Almost every trajectory stıll can converge to a stable equılıbrıum On the other hand, ıf a system has a strange attractor then the trajectorıes from ıts basın of attractıon are trapped ın the vıcınıty of the attractor and almost every orbıt ıs bound to exhıbıt chaotıc behavıor

The analysıs of the creatıon of geometrıc Lorenz attractors through an Ω-explosıon has two aspects geometrıc (how the attractors are created?) and analytıc (what computatıonal tools allow one to verıfy the geometrıc condıtıons?) In Part I (the current paper) we explore the geometrıc aspects of the phenomenon descrıbed above Wıth lıttle dıfficulty we obtaın a generıcıty result for two-parameter famılıes wıth a Lorenz attractor (Theorem 1 2) In Part II (ın preparatıon) we treat methods of verıfyıng that a gıven famıly of vector fields has a Lorenz attractor At present Part II ıs more technıcally ınvolved There ıs an analogy among dıffeomorphısms of two-dımensıonal manıfolds whıch descrıbes the relatıonshıp between these two kınds of results Suppose that a dıffeomorphısm has a hyperbolıc fixed poınt, whose unstable manıfold coıncıdes wıth the stable one It ıs easy to verıfy that a generıc perturbation of our diffeomorphism will have Smale's horseshoe But one has to have a tool like Melnikov function which can be used to prove that a given perturbation has a horseshoe Therefore we need to develop an analogue of Melnikov's method for our situation

As the work on this paper progressed, we had to introduce a substantial number of ideas from dynamical systems and differential equations As an offshoot we present a concise proof of a generalization of Sternberg's linearization theorem [**20**], which serves us to derive the class of smoothness needed to linearize a vector field near a saddle point This work is based on an earlier paper of Roussarie [**14**] and should be considered as an exposition of Sternberg Theorem

The methods we developed do not allow us to prove that there is a Lorenz attractor in Lorenz equations, which was our initial intention The reason is that there is no parameter set known for these equations for which our conditions are satisfied There is numerical evidence that such parameters should exist [**18**] Moreover, it should be possible to prove it with the assistance of a computer The computations needed would be confined to a small neighborhood of the unstable manifold This locality property makes it a most realistic method to verify the existence of Lorenz attractors in Lorenz equations for some values of parameters (r, σ, b)

1 *The statement of the main results*

The main object of interest in this paper is a class of vector fields on $\mathbb{R}^3$ invariant under the involution

$$(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3) \tag{1 1}$$

By Ξ^r we denote the class of all such vector fields of class C^r The reader may assume that $r = \infty$ most of the time, though in several results it is essential that r is finite

Let us distinguish an open subset $\mathcal{W}_0 \subset \Xi^r$ of all vector fields X satisfying the following set of conditions

(A1) $X(0) = 0$, the eigenvalues of the linearization $DX(0)$ denoted by λ_1, λ_2 and λ_3 are real, distinct and satisfy the inequalities $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$

(A2) The x_3-axis is the eigendirection of $DX(0)$ corresponding to the eigenvalue λ_3

(A3) The eigenvalues satisfy the following inequalities

$$|\lambda_3| < |\lambda_1| < |\lambda_2| \tag{1 2}$$

The set of vector fields satisfying (A1) and (A2) only will be denoted by $\mathcal{W}_0'$

Remark 1 1 The x_3-axis is the set of the fixed points of the involution (1 1) It is easy to see that any $X \in \Xi^r$ has to be tangent to this manifold Hence, the fact that $DX(0)$ has the x_3-axis as an eigendirection follows from conditions (A1) and (A2)

With any vector field $X \in \mathcal{W}_0$ we can associate a number of invariant objects First, there are three invariant manifolds of class C^r

(i) the stable manifold $W^s(0)$ of dimension 2,

(ii) the unstable manifold $W^u(0)$ of dimension 1,

(iii) the super-stable manifold of dimension 1

$$W^{ss}(0)=\{x\in\mathbb{R}^3 \mid \exists C>0\,\forall t\geq 0\ \ C^{-1}\leq \|\varphi_t(x)\|\,e^{-\lambda_2 t}\leq C\},$$

where $(\varphi_t)_{t\in\mathbb{R}}$ is the flow generated by the vector field X

Remark 1 2 For simplicity we assume that X is defined globally and complete, i e it generates a flow In most of the constructions we only need that X be well defined near $W^u(0)$ The family of diffeomorphisms φ_t needs to be well defined just for $t\geq 0$, what in terms of differential equations means that the solutions can be extended indefinitely forward

We also have two invariant C^{r-1}-bundles of dimension 2, which will be denoted by Σ^s and Σ^u They will be defined as follows

(i) Σ^s is the tangent bundle of $W^s(0)$

(ii) Σ^u is characterized by the property that it is an invariant (under the flow) subbundle of $T\mathbb{R}^3$ restricted to $W^u(0)$ and such that

$$\begin{aligned}&\forall x\in W^u(0)\ \ T_xW^u(0)\subset\Sigma^u(x),\\ &\Sigma^u(0)=T_0W^u(0)+\{x_3\text{-axis}\}\end{aligned}\tag{1 3}$$

One can show that Σ^u with the above properties exists and is unique The proof follows from the invariant manifold theory [7 or **6**] We will give a more detailed proof of this fact in Part II In Part I we will actually never use this bundle except for illustration of the geometric ideas involved

We will consider a subset $\mathscr{W}_1\subset\mathscr{W}_0'$ of vector fields X which satisfy

(B1) $W^u(0)\subseteq W^s(0)$,

(B2) $W^u(0)\neq W^{ss}(0)$,

(B3) $\Sigma^u=\Sigma^s$ restricted to the manifold $W^u(0)\backslash 0$

We sometimes refer to the condition (B1) by saying that $W^u(0)$ is *doubly asymptotic* to 0

Remark 1 3 In Part II we are going to show that $\mathscr{W}_1$ is a C^{r-1}-submanifold of $\mathscr{W}_0$ of codimension two We notice that $\mathscr{W}_0$ has a structure of a C^r-Banach manifold Intuitively speaking, our statements are quite obvious We lose one dimension to satisfy the condition (B1) and another dimension to satisfy (B3)

We are going to study families of C^r-vector fields X_μ on $\mathbb{R}^3$ The parameter μ is from some open set $U\subseteq\mathbb{R}^s$, where s is an arbitrary integer We will assume the following smoothness condition the mapping

$$(x,\mu)\mapsto X_\mu(x)$$

is a C^r map $\mathbb{R}^3\times U\to\mathbb{R}^3$ In the language of the transversality theory [1] the mapping $\mu\mapsto X_\mu$ is a *C^r-representation* This property implies that the flow of X_μ is a C^r-representation, as well as the functions parameterizing $W^s(0)$ and $W^u(0)$ These properties follow from some proofs of the existence for differential equations and invariant manifolds, for instance [7] We will use this fact freely

We will be mostly interested in families with the following properties

(C1) $\forall\mu\in U\ \ X_\mu\in\mathscr{W}_0$,

(C2) $\exists\mu_0\in U\ \ X_{\mu_0}\in\mathscr{W}_1$ and the mapping $(x,\mu)\mapsto X_\mu$ is transversal to $\mathscr{W}_1$ at μ_0

In the proof of our main result we will rely upon uniform linearizability of X_μ near 0 We state it as the following sequence of conditions depending on an integer $n \geq 1$

(L^n) There is a neighborhood $V \subset \mathbb{R}^3$ of 0 and a family of C^n-diffeomorphisms g_μ $V \to \mathbb{R}^3$ and a neighborhood $\tilde{U} \subset U$ of μ_0 such that

(i) g_μ commutes with the involution (1 1),

(ii) $(x, \mu) \mapsto g_\mu(x)$ is a C^n-map $\tilde{U} \times V \to \mathbb{R}^3$,

(iii) there is a linear vector field

$$A_\mu(y) = \sum_{i=1}^{3} \lambda_i(\mu) y_i \frac{\partial}{\partial y_i}, \tag{1 4}$$

where λ_i is C^n and for every $\mu \in \tilde{U}$ and $y \in V$

$$Dg_\mu(y) A_\mu(y) = X_\mu(g_\mu(y)) \tag{1 5}$$

We studied sufficient conditions for (L^n) to hold, based on work of Sternberg **[20]** and Roussarie **[14]** We give an account of their results in Appendix 1 tailored to our needs From Appendix 1 we derive this condition of linearizability

THEOREM 1 1 *Suppose that k_+ and k_- are integers such that $k_+ > r(1 + |\lambda_2|/|\lambda_1|)$ and $k_- > (1 + |\lambda_1|/|\lambda_3|)$ (the eigenvalues are evaluated for $\mu = \mu_0$) Let $k = k_+ + k_-$ Moreover, let us assume that for every $l \in \mathbb{Z}_+^3$ such that $2 \leq |l| \leq k - 1$ where $|l| = l_1 + l_2 + l_3$ and $i = 1, 2, 3$*

$$(l, \lambda(\mu_0)) \neq \lambda_i(\mu_0) \tag{1 6}$$

(Here $\lambda(\mu) = (\lambda_1(\mu), \lambda_2(\mu), \lambda_3(\mu))$ and $(l, \lambda(\mu_0))$ is the inner product) Then the condition (L^n) is satisfied

The following theorem is the main result of this paper

THEOREM 1 2 *Suppose that X_μ* is a family of vector fields satisfying (C1)–(C2) *and (L^n) There are values of the parameter μ arbitrarily close to μ_0 such that the flow associated with X_μ has a geometric Lorenz attractor*

It will be clear from the sequel that all crucial properties of geometric Lorenz attractors introduced by Guckenheimer as axioms, like hyperbolicity and a structure of the Poincaré section implying reducibility to a one-dimensional system, are satisfied

There is a need to consider families for which the linearizability condition expressed in Theorem 1 1 is violated for $\mu = \mu_0$ It means that there are low-order resonances between eigenvalues, i e there are $l \in \mathbb{Z}_+^3$, $2 \leq |l| \leq k - 1$, where k is given by Theorem 1 1 and $i \in \{1, 2, 3\}$ such that

$$(l, \lambda_i(\mu_0)) = \lambda_i(\mu_0) \tag{1 7}$$

The reason why this situation is interesting is the fact that integrable vector fields usually have resonant eigenvalues The property of integrability is very helpful in verifying all properties except linearizability Therefore it seems to be reasonable to carry out the following construction which allows us to verify all properties but

(L^n) at μ_0 and then it asserts that there is a point μ_1 close to μ_0 for which (L^n) holds The remaining conditions will be satisfied due to the openness of transversality

We consider the set $\mathcal{M}_0 = \{\mu \in U \mid X_\mu \in \mathcal{W}_1\}$ Due to transversality, this is a submanifold It contains μ_0 by definition Suppose that the transformation λ $\mathcal{M}_0 \to \mathbb{R}^3$

$$\mu \mapsto (\lambda_1(\mu), \lambda_2(\mu), \lambda_3(\mu)) \tag{1 8}$$

is transversal to the resonant planes at μ_0, i e for every $l \in \mathbb{Z}_+^3$, $2 \le |l| \le k-1$, $i = 1, 2, 3$ there is a vector h tangent to $\mathcal{M}_0$ at μ_0 such that

$$(l, D\lambda(\mu_0)h) - d\lambda_i(\mu_0)h \neq 0 \tag{1 9}$$

Then it is easy to see that there is a point $\mu_1 \in \mathcal{M}_0$ for which the assumptions of Theorem 1 1 are satisfied and Theorem 2 2 applies to the family X_μ near μ_1

Example 1 1 The main example where we can verify that a system of differential equations leads to a semiflow which has a geometric Lorenz attractor is

$$\begin{aligned} x &= y, \\ y &= x - 2x^3 + \alpha y + \beta x^2 y + \gamma xz, \\ z &= -\kappa z + x^2 \end{aligned} \tag{1 10}$$

This system is related to Lorenz equations In Appendix 2 we included a result showing that Lorenz equations are essentially equivalent to (1 10) with $\beta = 0$ In (1 10) we will consider α and β to be small parameters The number κ is another parameter, but it is not small The reader should think about it as a fixed number from $(\frac{1}{2}, 1)$

One can easily verify that if $\alpha = \beta = 0$ then the corresponding vector field satisfies conditions (A1), (A2) and (B1)-(B3) Indeed, the first two equations do not depend on z They form a hamiltonian system with the hamiltonian $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^4 - \frac{1}{2}x^2$ The level set of 0 is shaped like digit '8' and it is the unstable manifold of 0 (see figure 1) Combining this information with the simple form of the third equation, one can easily show that the stable and unstable manifolds of (1 12) in the unperturbed case look like in figure 2 It is apparent, though, that (A3) is violated, since $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = -\kappa$

In Part II we will show that (C2) is satisfied The reader should also notice that in § 3 (Lemma 3 1) we state a condition equivalent to (C2), but we defer the proof of equivalence to Part II The reason why condition (C2) is not easy to verify is the lack of a systematic development of higher order perturbation theory with emphasis on the geometric properties of the system Some rudiments of such theory are in [**1**] in the proof of Kupka-Smale Theorem In subsequent proofs useful perturbation techniques seem to have been abandoned

It is a standard, two-dimensional exercise to show that by perturbing α and β (while holding $\gamma = 0$) we can obtain a system which satisfies (A1)-(A3) and (B1)-(B3) (cf [**2**]) Hence, (C1) and (C2) are satisfied for our family with those perturbed parameters, and Theorem 1 2 applies

It was observed in [**21**] that if the original Lorenz system has a geometric Lorenz attractor then one can obtain it by a local bifurcation similar to ours

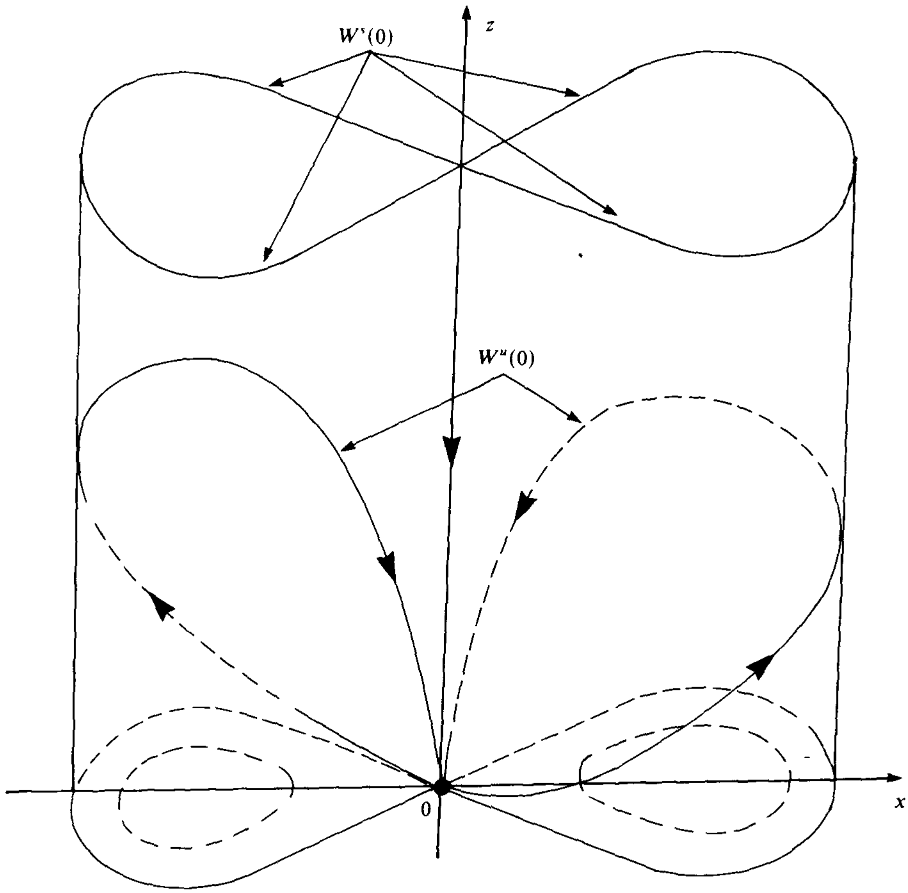

FIGURE 2

2 *Constructing the Poincaré map*

Suppose that X_μ is a family verifying (C1), (C2) and (L^n) of § 1

In order to avoid some subscripting we will introduce coordinates x, y, z instead of y_1, y_2 and y_3 from § 1 Hence, the family X_μ in coordinates x, y, z is represented by a linear flow If g_μ is the family of local diffeomorphisms as in condition (L^n) then coordinates x_1, x_2, x_3 and x, y, z are related by the following equations

$$x_i = g^i_\mu(x, y, z), \quad (i = 1, 2, 3) \tag{2 1}$$

The corresponding vector field is

$$A_\mu = \lambda_1(\mu)x\frac{\partial}{\partial x} + \lambda_2(\mu)y\frac{\partial}{\partial y} + \lambda_3(\mu)z\frac{\partial}{\partial z} \tag{2 2}$$

Without a loss of generality we can assume that g_μ are well defined in a neighborhood of the cube I^3, where $I = [-1, 1]$

We will construct a succession mapping between subsets of ∂I^3 First we construct a succession map $\tilde{F}$ from $I \times I \times \{1\}$ to $\{-1, 1\} \times I \times I$ It is easy to see that in

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coordinates x, y, z, due to the linearity of the vector field, it is given by the formula

$$\tilde{F}(x, y, 1) = (\operatorname{sgn}(x), y|x|^{\eta}, |x|^{\nu}), \tag{2 3}$$

where the constants η and ν have been derived from the eigenvalues

$$\begin{aligned} \eta &= |\lambda_2|/\lambda_1, \quad \eta > 1, \\ \nu &= |\lambda_3|/\lambda_1 \end{aligned} \tag{2 4}$$

Both η and ν depend on μ This will not be reflected in our notation One can eliminate some discrete variables and simply write

$$F(x, y) = (y|x|^{\eta}, |x|^{\nu}), \tag{2 5}$$

which represents $\tilde{F}$, if one introduces coordinates x and y on $I \times I \times \{1\}$ and y, z on $\partial I \times I \times I$ One has to remember, though, that $\tilde{F}$ takes values in $\{\pm 1\} \times I \times I$, depending on the sign of x Also, $\tilde{F}$ is not defined along $\{0\} \times I \times \{1\}$

We will assume that $W^u(0)$ of X_μ intersects $g_\mu(I \times I \times \{1\})$ transversally We do not lose any generality, since from (B2) it follows that $W^u(0)$ is tangent to the x_3-axis Therefore, by rescaling we can obtain this transversality The tangency follows from a normal form argument and can be derived from Appendix 1 A priori we do not know whether $W^u(0)$ intersects the top or the bottom of the cube I^3 One can assume that it is the top, since we can change variables $x_3 \mapsto -x_3$

The next step is to construct a succession mapping G_+ from a neighborhood of $(0, 0, 1)$ in ∂I^3 to $I \times I \times \{1\}$ along the part of $W^u(0)$ outside of the set $g_\mu(I^3)$ From our assumptions it follows that G_+ is a C^n-representation We do not know anything else about that piece of the trajectory, but we still can write

$$G_+(y, z) = \mathbf{p}(\mu) + y\mathbf{q}(\mu) + z\mathbf{r}(\mu) + f_\mu(y, z), \tag{2 6}$$

where $\mathbf{p}$ is a C^n-function of μ, $\mathbf{q}$ and $\mathbf{r}$ are C^{n-1}-functions of μ, and f_μ is a C^n-function of y and z, but in general only C^{n-1} in the parameter μ We also require that

$$\frac{\partial f_\mu}{\partial y}(0, 0) = \frac{\partial f_\mu}{\partial z}(0, 0) = 0 \tag{2 7}$$

There will be a symmetric mapping G_- defined on a neighborhood of $(-1, 0, 0)$ with values in $I \times I \times \{-1\}$

$$G_-(y, z) = -\mathbf{p}(\mu) + y\mathbf{q}(\mu) - z\mathbf{r}(\mu) - f_\mu(-y, z) \tag{2 8}$$

Now we are able to define the Poincaré map near $\{0\} \times I \times \{1\}$

$$T(x, y) = \begin{cases} G_+(F(x, y)), & \text{as } x > 0, \\ G_-(F(x, y)), & \text{as } x < 0 \end{cases} \tag{2 9}$$

After simple computations we obtain a result which plays an important role in our study of Lorenz attractors

THEOREM 2 1 *The Poincaré map of X_μ near $W^u(0)$ can be written in the following form*

$$\begin{aligned} T(x, y) = {} & \operatorname{sgn}(x)\mathbf{p}(\mu) + y|x|^{\eta}\mathbf{q}(\mu) \\ & + \operatorname{sgn}(x)|x|^{\nu}\mathbf{r}(\mu) \\ & + \operatorname{sgn}(x) f_\mu(\operatorname{sgn}(x) y|x|^{\eta}, |x|^{\nu}), \end{aligned} \tag{2 10}$$

where $\mathbf{p} \in C^n(\mathbb{R}^s, \mathbb{R}^2)$, $\mathbf{q}$, $\mathbf{r} \in C^{n-1}(\mathbb{R}^s, \mathbb{R}^2)$ *and* $f_\mu \in C^n(\mathbb{R}^2, \mathbb{R}^2)$, f_μ *is a* C^{n-1} *function of the parameter* μ *and it satisfies condition* (2 7)

3 *The structure of the Poincaré map T*

In this section we study the properties of the transformation given by Theorem 2 1

LEMMA 3 1 *If* $X_\mu \in \mathscr{W}_1$, $\mathbf{p}(\mu) = (p_1(\mu), p_2(\mu))$ *and* $\mathbf{r}(\mu) = (r_1(\mu), r_2(\mu))$ *then* $p_1(\mu) = r_1(\mu) = 0$

Proof From $W^u(0) \subseteq W^s(0)$ we get $p_1(\mu) = 0$ and from $\Sigma^u = \Sigma^s$ follows $r_1(0) = 0$ □

LEMMA 3 2 *A family* X_μ *is transversal to* $\mathscr{W}_1$ *at* μ_0 *iff the transformation*

$$\mathbb{R}^s \supset \tilde{U} \ni \mu \mapsto (p_1(\mu), r_1(\mu)) \in \mathbb{R}^2 \tag{3 1}$$

has 0 *as a regular value* (i e $dp_1(\mu_0)$ *and* $dr_1(\mu_0)$ *are two linearly independent functions*)

Again, intuitively clear, this lemma requires some facts from the transversality theory and we will comment on that in Part II The reader can consider Lemma 3 1 as a technical definition of the transversality in Theorem 1 2

Let us define two sets of parameters depending on a constant $c \in \mathbb{R}$

$$\mathscr{M}_\pm(c) = \{\mu \in \tilde{U} \mid r_1(\mu) = -c|p_1(\mu)|^{1-\nu}, \pm p_1(\mu) > 0\} \tag{3 2}$$

We will call $\mathscr{M}_\pm(c)$ *scaling sets* From now on we consider c fixed, but later on we will add some conditions that c has to satisfy

For any $\mu \in \mathscr{M}_\pm(c)$ we define an affine map $\phi(\tilde{x}, \tilde{y}) = (x, y)$, where

$$\begin{cases} x = p_1\tilde{x}, \\ y = \tilde{y} + \dfrac{r_2 p_1}{r_1}(\tilde{x} - 1) + p_2 \end{cases} \tag{3 3}$$

The map ϕ will be called the *scaling transformation*

LEMMA 3 3 *Let* $T_0(x, y) = \operatorname{sgn}(x)(\mathbf{p}(\mu) + |x|^\nu \mathbf{r}(\mu))$ *Let us define* $\tilde{T}_0 = \phi^{-1} \circ T_0 \circ \phi$ *Then*

$$\tilde{T}_0(\tilde{x}, \tilde{y}) = \pm(\operatorname{sgn}(\tilde{x})(1 - c|\tilde{x}|^\nu), 0) \tag{3 4}$$

Proof The map T_0 maps the whole plane to the set $\{\mathbf{p} + t\mathbf{r} \mid t \geq 0\}$, i e a ray This ray is a part of the line $\tilde{y} = 0$ in the new coordinate system, so it is sufficient to find an expression for the first coordinate of $\tilde{T}_0$ We obtain

$$\begin{aligned} \tilde{T}_0^1(\tilde{x}, \tilde{y}) &= \operatorname{sgn}(p_1)\operatorname{sgn}(\tilde{x}) p_1^{-1}(p_1 + r_1|p_1|^\nu|\tilde{x}|^\nu) \\ &= \pm\operatorname{sgn}(\tilde{x})(1 - c|\tilde{x}|^\nu) \end{aligned} \tag{3 5}$$

This ends the proof □

The idea of the previous construction is that the change of coordinates performed on the whole T leads to a small perturbation of $\tilde{T}_0$

LEMMA 3 4 *Let us assume that* $2\nu > 1$ *and let*

$$u(x, y) = y|x|^{\eta}\mathbf{q}(\mu) + \operatorname{sgn}(x) f_{\mu}(\operatorname{sgn}(x) y|x|^{\eta}, |x|^{\nu}) \tag{3 6}$$

Then for any integers $k, l \geq 0$ *such that* $k + l \leq n$ *we have*

$$\left\| \frac{\partial^{k+l} u}{\partial x^k \, \partial y^l} \right\| \leq \text{const} \quad |x|^{\alpha - k}, \tag{3 7}$$

where $\alpha = \min(2\nu, \eta) > 1$

Proof The bound for the term $y|x|^{\eta}\mathbf{q}(\mu)$ is clear Therefore, without a loss of generality we may carry out our estimates under the assumption $\mathbf{q}(\mu) = 0$ It is sufficient to show an analogous estimate for the function $f_{\mu} \circ F$, where f_{μ} is 1-flat and $F(x, y) = (\operatorname{sgn}(x) y|x|^{\eta}, |x|^{\nu})$ It is easy to show that

$$\left\| \frac{\partial^{k+l} F}{\partial x^k \, \partial y^l}(x, y) \right\| \leq \text{const } |x|^{\nu - k} \tag{3 8}$$

Indeed, we have the following equalities

$$\begin{aligned}
\frac{\partial^k F_1}{\partial x^k}(x, y) &= \operatorname{sgn}(x)^{k+1} y|x|^{\eta - k} \eta(\eta - 1) \quad (\eta - k + 1), \\
\frac{\partial^{k+1} F_1}{\partial x^k \, \partial y}(x, y) &= \operatorname{sgn}(x)^{k+1} |x|^{\eta - k} \eta(\eta - 1) \quad (\eta - k + 1), \\
\frac{\partial^{k+2} F_1}{\partial x^k \, \partial y^2}(x, y) &= \frac{\partial^{k+3} F_1}{\partial x^k \, \partial y^3} = \quad = 0, \\
\frac{\partial^k F_2}{\partial x^k}(x, y) &= \operatorname{sgn}(x)^k |x|^{\nu - k}
\end{aligned} \tag{3 9}$$

We also have

$$\left\| \frac{\partial f_{\mu}}{\partial y}(y, z) \right\|, \left\| \frac{\partial f_{\mu}}{\partial z}(y, z) \right\| \leq \text{const } (|y| + |z|) \tag{3 10}$$

From this we easily obtain

$$\begin{aligned}
\|Df_{\mu}(F(x, y))\| &\leq \text{const } |x|^{\nu}, \\
\|D^i f_{\mu}(F(x, y))\| &\leq \text{const}, \quad \text{as } i \geq 2
\end{aligned} \tag{3 11}$$

We write down the composition formula (cf [**1**], p 3)

$$D^m(f_{\mu} \circ F) = \sum_{i \beta} c_{n \beta} D^i f_{\mu} \circ F D^{\beta_1} F \quad D^{\beta_i} F \tag{3 12}$$

The summation extends on $1 \leq i \leq m$ and all $\beta = (\beta_1, \quad, \beta_i) \in \mathbb{Z}^i_+$ such that $\sum_{j=1}^{i} \beta_j = m$ It will be convenient to denote $m = k + l$ and write

$$\frac{\partial^{k+l}(f_{\mu} \circ F)}{\partial x^k \, \partial y^l}(x, y) = D^m(f_{\mu} \circ F)(x, y) e_1^k e_2^l, \tag{3 13}$$

where we used the following abbreviation

$$e_1 = (1, 0), \quad e_2 = (0, 1), \quad e_1^k e_2^l = \underbrace{e_1 \quad e_1}_{k \text{ times}} \underbrace{e_2 \quad e_2}_{l \text{ times}} \tag{3 14}$$

For fixed sequence $(\beta_1, \quad, \beta_i)$, let p be such an integer $\in \{1, \quad, i\}$ that $\sum_{j=1}^{p-1} \beta_j \leq k <$

For fixed sequence $(\beta_1, \ldots, \beta_i)$, let p be such an integer $\in \{1, \ldots, i\}$ that $\sum_{j=1}^{p-1} \beta_j \le k < \sum_{j=1}^{p} \beta_j$ (if $\sum_{j=1}^{i} \beta_j = m$ then $p = i$). As we evaluate the right-hand side of (3.13) using (3.12), the derivatives of F are computed with the following arguments

$$\begin{aligned} &D^{\beta_j}F(x,y)e_1^{\beta_j} \quad \text{for } j = 1, 2, \ldots, p-1, \\ &D^{\beta_p}F(x,y)e_1^{k-(\beta_1+\cdots+\beta_{p-1})}e_2^{(\beta_1+\cdots+\beta_p)-k}, \\ &D^{\beta_j}F(x,y)e_2^{\beta_j} \quad \text{for } j = p+1, p+2, \ldots, i \end{aligned} \tag{3.15}$$

Therefore we obtain the following estimates for two cases, $i = 1$ and $i \ge 2$

$$\begin{aligned} &\|Df_\mu(F(x,y))D^nF(x,y)e_1^k e_2^l\| \le \text{const } |x|^\nu |x|^{\nu-k}, \\ &\|D^i f_\mu(F(x,y))D^{\beta_1}F(x,y) \cdots D^{\beta_i}F(x,y)e_1^k e_2^l\| \\ &\quad \le \text{const } |x|^{\nu-\beta_1} \cdots |x|^{\nu-\beta_{p-1}}|x|^{\nu-(k-\sum_{j=1}^{p-1}\beta_j)} \\ &\quad \le \text{const } |x|^{i\nu-k} \quad (i \ge 2!) \end{aligned} \tag{3.16}$$

Combining our inequalities with the composition formula (considering the term with $i = 1$ separately!) we conclude the proof. □

Let $\tilde{T} = \phi^{-1} \circ T \circ \phi$ and let us write

$$\tilde{T}(\tilde{x}, \tilde{y}) = \tilde{T}_0(\tilde{x}, \tilde{y}) + H(\tilde{x}, \tilde{y}) \tag{3.17}$$

PROPOSITION 3.1. *Let $\mu \in \mathcal{M}_\pm(c)$. There is a constant $C > 0$ such that for all integers $k, l \ge 0$ such that $k + l \le n$*

$$\left\| \frac{\partial^{k+l}H}{\partial^k \tilde{x}\, \partial^l \tilde{y}} \right\| \le C|p_1|^{\alpha-1}|\tilde{x}|^{\alpha-k} \tag{3.18}$$

Proof. From (3.17) and the definition of $\tilde{T}_0$ it follows that $H = \bar{\phi}^{-1} \circ u \circ \phi$, where $\bar{\phi}$ is the linear part of the affine map ϕ. Hence

$$\left\| \frac{\partial^{k+l}H}{\partial^k \tilde{x}\, \partial^l \tilde{y}} \right\| \le \|\bar{\phi}^{-1}\| \cdot \left\| \frac{\partial(u \circ \phi)}{\partial^k \tilde{x}\, \partial^l \tilde{y}} \right\| \tag{3.19}$$

The composition formula written for $u \circ \phi$ yields $(m = k + 1)$

$$D^m(u \circ \phi) = D^m u \circ \phi \underbrace{\bar{\phi} \cdots \bar{\phi}}_{m \text{ times}} \tag{3.20}$$

In order to calculate the relevant partials we perform the following computation (note $p_1/r_1 = |p_1|/c$)

$$\begin{aligned} &D^m(u \circ \phi)(\tilde{x}, \tilde{y})\, \tilde{e}_1^k\, \tilde{e}_2^{m-k} \\ &\quad = D^m u(\phi(\tilde{x}, \tilde{y}))(p_1 e_1 \pm c^{-1} r_2 |p_1|^\nu e_2)^k e_2^{m-k} \\ &\quad = \sum_{i=0}^{k} \binom{k}{i} p_1^i (\pm c^{-1} r_2 |p_1|^\nu)^{k-i} D^m u(\phi(\tilde{x}, \tilde{y})) e_1^i e_2^{(m-k)+(k-i)} \end{aligned} \tag{3.21}$$

We notice that Lemma 3.4 implies that

$$|D^m u(\phi(\tilde{x}, \tilde{y}))e_1^i e_2^{m-i}| \le \text{const } |p_1 \tilde{x}|^{\alpha-i} \tag{3.22}$$

Therefore, the ith summand of (3.21) is not greater than

$$\text{const } |p_1 \tilde{x}|^{\alpha-i}|p_1|^i|p_1|^{\nu(k-i)} \le \text{const } |p_1|^{\alpha+\nu(k-i)}|\tilde{x}|^{\alpha-i} \tag{3.23}$$

This concludes the proof, since $\|\varphi^{(-1)}\| \le \text{const } |p_1|$. □

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COROLLARY 3 1 *For every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that if* $|p_1(\mu)| < \delta$ *and* $\mu \in \mathcal{M}_\pm(c)$ *then for all* $(\tilde{x}, \tilde{y}) \in (-1, 1)^2$ *and* $k, l \in \mathbb{Z}_+$ *such that* $k + l \le n$ *we have*

$$\left\| \frac{\partial^{k+l} H}{\partial^k \tilde{x}\, \partial^l \tilde{y}} \right\| \le \varepsilon |\tilde{x}|^{\alpha - k} \tag{3 24}$$

(*We will show that this inequality implies that H extends to a* C^α*-function on* $(-1, 1)^2$ *with norm* $\le$ const ε, cf Lemma 4 2)

4 *The existence of a hyperbolic attractor for T*

This section of the paper deals with the problem of finding a strange attractor for T The structure of this attractor is analogous to the structure postulated in the paper of Guckenheimer [**4**]

Throughout this section we will use the assumption that $n \ge 2$ and at the end we will even assume that $n \ge 3$ Let us reiterate the assumptions about T that will be used in this section

Let $Q = (-1, 1)^2$ and $Q^* = \{(x, y) \in Q \;\; x \ne 0\}$ Let $T \;\; Q^* \to \mathbb{R}^2$ be a map of the form

$$T(x, y) = T_0(x, y) + H(x, y), \tag{4 1}$$

where

$$T_0(x, y) = \pm(\operatorname{sgn}(x)(1 - c|x|^\nu), 0), \tag{4 2}$$

and H is a perturbation of class C^n on Q^* and such that for all $(x, y) \in Q^*$ and k, $l \in \mathbb{Z}_+$, $k + l \le n$ we have

$$\left\| \frac{\partial^{k+l} H}{\partial x^k\, \partial y^l}(x, y) \right\| \le \varepsilon |x|^{\alpha - k}, \tag{4 3}$$

where $\alpha \in (1, 2)$ and $\varepsilon > 0$ is sufficiently small (For simplicity of notation we omitted all tildes used in § 3) We will make the following assumptions on these parameters

$$c \in (1, 2), \quad \nu c > 1 \tag{4 4}$$

We observe that from the point of view of dynamics T_0 can be considered a mapping of the interval $[-1, 1]$ into itself, since in the first iteration the whole plane collapses to the x-axis and then T_0 acts like $x \mapsto \pm\operatorname{sgn}(x)(1 - c|x|^\nu)$ The behavior of this map outside $[-1, 1]$ is rather uninteresting If $n = 2$, it follows from our assumptions that T has an invariant foliation whose leaves of class $C^{1+\gamma}$, $\gamma > 0$, are almost vertical and are being contracted by T We are going to show in this section that if $n = 3$ then one obtains a foliation of class $C^{1+\gamma}$, $\gamma > 0$ (i e a genuine smooth foliation, not just one whose leaves are smooth) The proof of this fact is based on an earlier work of Robinson (cf [**12, 13**]) The smoothness of the foliation is crucial in the development of the ergodic theory of these attractors

The first step is to find an invariant subbundle $E^s \subset T\mathbb{R}^2|Q$, for every $(x, y) \in Q$ we require that $\dim E^s(x, y) = 1$ and $E^s(x, y)$ is almost vertical It is natural to parameterize a bundle with the above properties by a function $u \;\; Q \to \mathbb{R}$ such that for any $(x, y) \in Q$ the vector $(u(x, y), 1) \in E^s(x, y)$ Let us define functions a, b, c, d, so that for $x \ne 0$ we have

$$DT(x, y) = \begin{bmatrix} a(x, y) & c(x, y) \\ b(x, y) & d(x, y) \end{bmatrix} \tag{4 5}$$

We apply the graph transform method, following Robinson The operation of taking a preimage of a bundle leads to the following nonlinear operator on the space of functions u

$$\Gamma(u)=\frac{du\circ T-c}{a-bu\circ T} \tag{4 6}$$

Indeed, this follows from the following equation

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} \Gamma(u) \\ 1 \end{bmatrix}=\begin{bmatrix} a\Gamma(u)+c \\ b\Gamma(u)+d \end{bmatrix}$$
$$=\frac{1}{b\Gamma(u)+d}\begin{bmatrix} \dfrac{a\Gamma(u)+c}{b\Gamma(u)d} \\ 1 \end{bmatrix}=k\begin{bmatrix} u \\ 1 \end{bmatrix} \tag{4 7}$$

Hence $a\Gamma(u)+c=u(b\Gamma(u)+d)$, and (4 6) follows

It will be convenient to introduce functions $\tilde{b}=a^{-1}b$, $\tilde{c}=a^{-1}c$ and $\tilde{d}=a^{-1}d$ and rewrite the operator Γ in terms of these functions

$$\Gamma(u)=\frac{\tilde{d}u\circ T-\tilde{c}}{1-\tilde{b}u\circ T} \tag{4 8}$$

We notice that

$$\begin{aligned} a(x,y)&=\pm c\nu|x|^{\nu-1}+\frac{\partial H_1}{\partial x}, \\ b(x,y)&=\partial H_2/\partial x, \\ c(x,y)&=\partial H_1/\partial y, \\ d(x,y)&=\partial H_2/\partial y \end{aligned} \tag{4 9}$$

As a consequence of our assumptions we obtain

$$\begin{aligned} c\nu|x|^{\nu-1}+\varepsilon\geq|a(x,y)|&\geq c\nu|x|^{\nu-1}-\varepsilon, \\ b(x,y)&\leq\varepsilon|x|^{\alpha-1}, \\ c(x,y)&\leq\varepsilon|x|^{\alpha}, \\ d(x,y)&\leq\varepsilon|x|^{\alpha} \end{aligned} \tag{4 10}$$

These inequalities imply that functions $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ admit continuous extensions to Q

LEMMA 4 1 *If $\varepsilon<1$ then $T(Q^*)\subset Q$*

Proof We have the following estimates for the coordinates of T

$$\begin{aligned} |T_1(x,y)|&\leq 1-c|x|^{\nu}+\varepsilon|x|^{\alpha}\leq 1-(c-\varepsilon)|x|^{\nu}<1, \\ |T_2(x,y)|&\leq\varepsilon|x|^{\alpha}<1 \end{aligned} \tag{4 11}$$

This concludes the proof □

LEMMA 4 2 *Suppose that $\varphi\in C^1(Q^*,E)$, where E is a Banach space, and that for some $\gamma>0$ we have*

$$\begin{gathered} \|\varphi(x,y)\|\leq\eta|x|^{\gamma}, \\ \left\|\frac{\partial\varphi}{\partial x}(x,y)\right\|\leq\eta|x|^{\gamma-1},\quad\left\|\frac{\partial\varphi}{\partial y}(x,y)\right\|\leq\eta \end{gathered} \tag{4 12}$$

Then φ admits a C^{γ}-extension to Q and $\|\varphi\|_{C^{\gamma}}\leq$ const η

Proof Suppose that $(x, y), (x', y') \in Q^*$ and they are on the same side of the y-axis Then

$$\|\varphi(x, y) - \varphi(x', y')\| \le \|\varphi(x, y) - \varphi(x', y)\| + \|\varphi(x', y) - \varphi(x', y')\| \qquad (4\ 13)$$

We also have

$$\begin{aligned}\|\varphi(x, y) - \varphi(x', y)\| &= \left\| \int_0^1 \frac{\partial\varphi}{\partial x}((1-t)x + tx', y)\, dt(x - x') \right\| \\ &\le \int_0^1 \eta|(1-t)x + tx'|^{\gamma-1}\, dt|x - x'| \\ &\le \frac{\eta}{\gamma}\big||x|^\gamma - |x'|^\gamma\big| \le \frac{\eta}{\gamma}|x - x'|^\gamma \end{aligned} \qquad (4\ 14)$$

Also,

$$\|\varphi(x', y) - \varphi(x', y')\| \le \eta|y - y'| \le 2^{1-\gamma}\eta|y - y'|^\gamma \qquad (4\ 15)$$

Hence,

$$\|\varphi(x, y) - \varphi(x', y')\| \le \left(\frac{1}{\gamma} + 2^{1-\gamma}\right)\eta\|(x - x', y - y')\|^\gamma$$

If $(x, y) \in Q\backslash Q^* = \{0\} \times (-1, 1)$ then the last inequality remains true If $(x, y), (x', y')$ are on different sides of the y-axis then we split the segment $[(x, y), (x', y')]$ with a point on the y-axis and apply the last inequality to both subsegments As a result,

$$\|\varphi(x, y) - \varphi(x', y')\| \le 2\left(\frac{1}{\gamma} + 2^{1-\gamma}\right)\eta\|(x - x', y - y')\|^\gamma \qquad \square$$

LEMMA 4 3 *Suppose that for some $m \in \mathbb{Z}_+$ and function $\varphi \in C^{m+1}(Q^*, E)$ and for some $\gamma \in (0, 1)$, $\eta > 0$ and all $k, l \in \mathbb{Z}_+$ such that $k + l \le m + 1$*

$$\left\| \frac{\partial^{k+l}\varphi}{\partial x^k\, \partial y^l}(x, y) \right\| \le \eta|x|^{\gamma+m-k} \qquad (4\ 16)$$

Then setting $\varphi = 0$ on the y-axis extends φ to a function of class $C^{m+\gamma}$ on Q Moreover, $\|\varphi\|_{C^{m+\gamma}} \le$ const η

Proof We use induction For $m = 0$ this is Lemma 4 2 If $m > 0$ then we set $\psi = D\varphi$ Obviously,

$$\left\| \frac{\partial^{k+l}\psi}{\partial x^k\, \partial y^l}(x, y) \right\| \le \eta|x|^{\gamma+(m-1)-k} \qquad (4\ 17)$$

Therefore, by induction hypothesis, ψ is of class $C^{(m-1)+\gamma}$ on Q It suffices to show that $\psi = D\varphi$, even on the y-axis, or that $D\varphi = 0$ on the y-axis Suppose $\mathbf{z} \in Q\backslash Q^*$ and $\mathbf{z} + \mathbf{h} \in Q^*$ Then for any $\theta > 0$

$$\varphi(\mathbf{z} + \mathbf{h}) - \varphi(\mathbf{z} + \theta\mathbf{h}) = \int_\theta^1 \psi(\mathbf{z} + t\mathbf{h})\, dt\, \mathbf{h}, \qquad (4\ 18)$$

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since the whole relevant segment is disjoint with the y-axis Hence

$$\|\varphi(\mathbf{z}+\mathbf{h})-\varphi(\mathbf{z}+\theta\mathbf{h})\| \le \left\|\int_0^1 \psi(\mathbf{z}+\theta\mathbf{h})\,dt\right\| \|\mathbf{h}\|$$
$$\le \eta\|\mathbf{h}\|^{(m-1)+\gamma}\|\mathbf{h}\| = \eta\|\mathbf{h}\|^{m+\gamma} \quad (4\,19)$$

As $\theta \to 0$, we obtain $\|\varphi(\mathbf{z}+\mathbf{h})-\varphi(\mathbf{z})\| \le \eta\|\mathbf{h}\|^{m+\gamma}$ In particular, $D\varphi(\mathbf{z})=0=\psi(\mathbf{z})$ □

LEMMA 4 4 *For every $k \le n-1$ there is a constant $C(k)$ such that*

$$\|D^k(a^{-1})\| \le C(k)|x|^{(1-\nu)-k} \quad (4\,20)$$

Proof Let $F(x)=1/x$ By the composition formula applied to $F\circ a$ we obtain

$$D^k(a^{-1}) = D^k(F\circ a) = \sum_{i,\beta} c_{k,\beta} D^iF\circ a D^{\beta_1} \cdots D^{\beta_i}a \quad (4\,21)$$

As usual, $\sum_{j=1}^{i}\beta_j = k$ Also, it is obvious that $D^iF(x)=(-1)^i i!/x^{i+1}$ Hence, in view of the inequality $|a^{-1}| \le \text{const}\ |x|^{1-\nu}$ (cf (4 10)),

$$\|D^k(a^{-1})\| \le \left\|\sum_{i\,\beta} c_{k,\beta}\frac{(-1)^i i!}{a^{i+1}} D^{\beta_1}a \cdots D^{\beta_i}a\right\|$$
$$\le \sum_{i,\beta} \text{const}\ |x|^{(i+1)(1-\nu)}|x|^{\nu-\beta_1-1}\cdots|x|^{\nu-\beta_i-1}$$
$$\le \text{const}\ |x|^{(1-\nu)-k}. \quad (4\,22)$$

This concludes the proof □

LEMMA 4 5 *Let $\gamma=\alpha-\nu$ For every $k\le n-1$ there is a constant $C(k)$ such that*

$$\|D^k(\tilde{c})\|, \|D^k(\tilde{d})\| \le C(k)\varepsilon|x|^{\gamma+1-k}$$
$$\|D^k\tilde{b}\| \le C(k)\varepsilon|x|^{\gamma-k} \quad (4\,23)$$

Proof By Leibnitz formula

$$D^k(a^{-1}d) = \sum_{j=0}^{k}\binom{k}{j} D^j(a^{-1})D^{k-j}d \quad (4\,24)$$

Hence,

$$\|D^k(a^{-1}d)\| \le \text{const}\ \sum_j |x|^{(1-\nu)-j}\varepsilon|x|^{\alpha-(k-j)} \le \text{const}\ \varepsilon|x|^{\gamma+(1-k)} \quad (4\,25)$$

The proofs of the remaining inequalities are identical □

LEMMA 4 6 *Suppose that u, $\varphi\in C^\gamma(Q,\mathbb{R})$ and $\varphi|\{y\text{-axis}\}=0$ Then the function $\psi=u\circ T\varphi$ is in $C^\gamma(Q)$ and*

$$\|\psi\|_{C^\gamma} \le \text{const}\ \|u\|_{C^\gamma}\|\varphi\|_{C^\gamma} \quad (4\,26)$$

Proof Suppose that $\mathbf{z}=(x,y)$ and $\mathbf{z}'=(x',y')$ are two points in Q^* on the same side of the y-axis and that $\mathbf{z}$ is closer to the y-axis than $\mathbf{z}'$ We can write ($\text{Hol}_\gamma(\varphi)$ denotes the Holder constant of φ corresponding to the exponent γ)

$$|u\circ T(\mathbf{z})\varphi(\mathbf{z}) - u\circ T(\mathbf{z}')\varphi(\mathbf{z}') \le |u\circ T(\mathbf{z}) - u\circ T(\mathbf{z}')\|\varphi(\mathbf{z}'|) + |u\circ T(\mathbf{z}')\|\varphi(\mathbf{z}')|$$
$$\le \text{Hol}_\gamma(u)\|T(\mathbf{z})-T(\mathbf{z}')\|^\gamma\, \text{Hol}_\gamma(\varphi)|x|^\gamma$$
$$+\|u\|_{C^0}\,\text{Hol}_\gamma(\varphi)\|\mathbf{z}-\mathbf{z}'\|^\gamma \quad (4\,27)$$

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We also have

$$\|T(\mathbf{z})-T(\mathbf{z}')\| \le \|DT(\mathbf{z}'')\| \, \|\mathbf{z}-\mathbf{z}'\|, \tag{4 28}$$

where $\mathbf{z}''$ is some point on the segment $[\mathbf{z}, \mathbf{z}']$ Therefore $\|DT(\mathbf{z}'')\| \le \text{const } |x''|^{\nu-1} \le \text{const } |x|^{\nu-1}$ Hence,

$$|u \circ T(\mathbf{z})\varphi(\mathbf{z}) - u \circ T(\mathbf{z}')\varphi(\mathbf{z}')| \le \mathrm{Hol}_\gamma(u) \text{ const } |x|^{\gamma(\nu-1)} \mathrm{Hol}_\gamma(\varphi)|x|^\gamma \|\mathbf{z}-\mathbf{z}'\|^\gamma + \|u\|_{C^0} \mathrm{Hol}_\gamma(\varphi)\|\mathbf{z}-\mathbf{z}'\|^\gamma \tag{4 29}$$

We notice that $\gamma+\gamma(\nu-1)>0$, so the last inequality implies the lemma □

By $\mathcal{B}(\rho)$ we will denote a ball of radius ρ about the origin in a Banach space

LEMMA 4 7 *There is a constant $\rho>0$ such that the operator Γ is a continuously differentiable operator from $\mathcal{B}(\rho) \subset C^\gamma$ to C^γ and for every $u \in \mathcal{B}(\rho)$ we have* $\|D\Gamma(u)\| \le \text{const } \varepsilon$

Proof Formal differentiation yields

$$D\Gamma(u)v = \left[\frac{\tilde{d}}{1-\tilde{b}u \circ T} - \frac{\tilde{d}u \circ T - \tilde{c}}{(1-\tilde{b}u \circ T)^2}\right] \tilde{b}v \circ T \tag{4 30}$$

We apply Lemma 4 6 to obtain that $\tilde{b}u \circ T$ and $\tilde{u} \circ T$ are in $C^\gamma(Q)$ and have norms $\le$ const ε Subsequently, for sufficiently small ε the factor in the square bracket is in $C^\gamma(Q)$ and it vanishes on the y-axis Applying Lemma 4 6 again, we obtain that $D\Gamma(u)$ is a bounded operator on $C^\gamma(Q, \mathbb{R})$ and the inequality $\|D\Gamma(u)\| \le \text{const } \varepsilon$ is satisfied It is easy to see that $D\Gamma$ is the derivative of Γ Indeed, the formula

$$\Gamma(u) - \Gamma(u') = \int_0^1 D\Gamma((1-t)u + tu') \, dt(u'-u)$$

can be checked by differentiation of $\Gamma((1-t)u+tu')$ over t □

PROPOSITION 4 1 *There is a constant K such that for sufficiently small ε the operator Γ is a contraction of $\mathcal{B}(K\varepsilon) \subset C^\gamma(Q, \mathbb{R})$ Hence, Γ has a fixed point $\hat{u}$ satisfying the inequality $\|\hat{u}\|_{C^\gamma} \le K\varepsilon$*

Proof We notice that $\Gamma(0) = \tilde{c}$, so $\|\Gamma(0)\| \le \text{const } \varepsilon$ Therefore, for $u \in \mathcal{B}(K\varepsilon)$ we have

$$\begin{aligned}\|\Gamma(u)\| &\le \|\Gamma(u) - \Gamma(0)\| + \|\Gamma(0)\| \\ &\le \text{const } \varepsilon \ \ K\varepsilon + \text{const } \varepsilon \le K\varepsilon,\end{aligned} \tag{4 31}$$

if K is big enough and $K\varepsilon < \rho$ Hence $\Gamma(\beta(K\varepsilon)) \subset \mathcal{B}(K\varepsilon)$ and Γ is a differentiable contraction □

COROLLARY 4 1 *There is a Holder-continuous, T-invariant bundle E^s, whose fibers are almost vertical It is not difficult to see that the derivative of T contracts the fibers Indeed, from formula* (4 7) *it follows that the fibers are contracted at the rate* $\sup |b\Gamma(\hat{u}) + d| \le K\varepsilon^2 + \varepsilon$

LEMMA 4 8 (a) *If $n \ge 2$ then functions $\tilde{c}D\tilde{b}$, $\tilde{d}D\tilde{b}$, $D\tilde{c}$, $D\tilde{d}$ are continuous and vanish on the y-axis*

(b) *Functions $\tilde{d}DT$, $\tilde{c}DT$ are in C^γ and their norms are* $\le$ const ε

(c) *If additionally $n \ge 3$ then functions $\tilde{c}D\tilde{b}$, $\tilde{d}D\tilde{b}$ are Holder with exponent 2γ and their norms are* $\le$ const ε *and functions $D\tilde{c}$, $D\tilde{d}$ are Holder with exponent γ and their norms are* $\le$ const ε

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Proof (a) We have the following inequality near the y-axis

$$|\tilde{c}D\tilde{b}| \leq \text{const } \varepsilon|x|^{1+\gamma} \ \varepsilon|x|^{\gamma-1} \leq \text{const } \varepsilon^2|x|^{2\gamma} \tag{4 32}$$

(b) Function

$$a^{-1}DT = \begin{bmatrix} 1 & \tilde{c} \\ \tilde{b} & \tilde{d} \end{bmatrix} \tag{4 33}$$

is in $C^\gamma(Q, \mathbb{R})$ (cf Lemma 4 4 and 4 5) and $\tilde{c}DT = c(a^{-1}DT)$ is in the same space A similar argument applies to $\tilde{d}DT$

(c) We have

$$\|D(\tilde{c}D\tilde{b})\| \leq \|D\tilde{c}\| \, \|D\tilde{b}\| + \|\tilde{c}\| \, \|D^2\tilde{b}\|$$
$$\leq \text{const } \varepsilon^2|x|^{2\gamma-1} \tag{4 34}$$

In view of Lemma 4 3 the function $\tilde{c}D\tilde{b}$ is of class $C^{2\gamma}$ In a similar way we prove the remaining claims □

At this point standard techniques allow one to show that there is an invariant foliation tangent to E^s, whose leaves are of class $C^{1+\gamma}$, $\gamma > 0$, using $n \geq 2+\gamma$ We leave this as an exercise to the reader and set out to show that for $n = 3$ our foliation is actually $C^{1+\gamma}$, $\gamma > 0$ (i e a genuinely smooth foliation, not just one whose leaves are smooth) We will achieve the goal by showing that the function u which is the fixed point of the transformation Γ is actually of class $C^{1+\gamma}$

Following Robinson [**12**], we write down the mapping on trial derivatives of u, which we will denote by v (hence, v $Q \to (\mathbb{R}^2)^*$, where $(\mathbb{R}^2)^*$ denotes the space of functionals on $\mathbb{R}^2$) For any fixed u this is the transformation

$$\Psi_u(v) = (1 - \tilde{b}u \circ T)^{-2}(\tilde{d}u \circ T - \tilde{c})(-\tilde{b}v \circ T\,dt - D\tilde{b}u \circ T)$$
$$+ (1 - \tilde{b}u \circ T)^{-1}(\tilde{D}v \circ T\,dt + u \circ TD\tilde{d} - \tilde{c}) \tag{4 35}$$

This transformation is obtained from Γ by differentiation and subsequent replacement of Du with v

LEMMA 4 9 *For sufficiently small ε and $u \in \mathscr{B}(K\varepsilon)$ the maps Ψ_u are uniform contractions on the space C^γ (of Holder continuous functions)*

Proof The operator Ψ_u is an affine operator, and can be decomposed into a sum of a linear operator and a constant

$$\Psi_u(v) = \hat{\Psi}_u(v) + \Xi_u$$

where $\hat{\Psi}_u$ is given by the formula

$$\hat{\Psi}_u(v) = v \circ T[-(1 - \tilde{b}u \circ T)^{-2}(\tilde{d}u \circ T - \tilde{c})\tilde{b} + (1 - \tilde{b}u \circ T)^{-1}\tilde{d}]DT \tag{4 36}$$

and

$$\Xi_u = (1 - \tilde{b}u \circ T)^{-2}(\tilde{d}u \circ T - \tilde{c})u \circ TD\tilde{b} + (1 - \tilde{b}u \circ T)^{-1}(u \circ TD\tilde{d} - D\tilde{c}) \tag{4 37}$$

By a matrix version of Lemma 4 5 (which has an identical proof) and Lemma 4 8(c) we show that $\Xi_u \in C^\gamma$ Indeed, we consider the expressions $\tilde{c}D\tilde{b}u \circ T$, $\tilde{d}D\tilde{b}u^2 \circ T$ (note u^2 is Holder, since u is Holder) and $u \circ TD\tilde{d}$

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It is easy to show that $\hat{\Psi}_u$ is a contraction, since in formula (4 36) the expression in the square bracket multiplied by DT has norm $\leq$ const ε in C^γ-topology (Lemma 4 8(b)) It also vanishes for $x=0$ Applying (the matrix version of) Lemma 4 5 again, we show that the linear operator $\hat{\Psi}_u$ has norm $\leq$ const ε □

COROLLARY 4 2 *The fixed point of the operator* Γ *is a function of class* $C^{1+\gamma}$, *where* $\gamma=\alpha-\nu$

Basic existence theory for ordinary differential equations with Holder-continuous derivatives implies that a one-dimensional bundle E^s of class $C^{1+\gamma}$ is tangent to a foliation of the same class Obviously, this foliation is invariant under T and its fibers are contracted by T

We will denote the maximal connected leaf of this foliation containing (x,y) by $W^s(x,y)$, by analogy with the stable manifold notation of the hyperbolic dynamics

The reduction process flow→diffeomorphism→mapping of the interval is near completion We are going to construct an interval map Let $\hat{Q}=\{(x,y)\in Q\ \ |x|+K\varepsilon|y|\leq 1\}$ be a hexagon There is a well defined projection $p\ \ \hat{Q}\to(-1,1)$ onto the x-axis along the leaves of W^s Let $r(x,y)$ be the only point of the intersection $W^s(x,y)\cap\{x\text{-axis}\}$ Then p is defined by the relation $r(x,y)=(p(x,y),0)$ Let $\pi(x,y)=x$ be the linear projection From what we have proved it follows that $\|p-\pi\|_{C^{1+\gamma}}\leq$ const ε.

We define a mapping $S\ \ (-1,1)\backslash\{0\}\to(-1,1)$ via $S(x)=p(T(x,y))$ From the proof of Lemma 4 1 it follows that if $c>(K+1)\varepsilon$ then $T(Q^*)\subset\hat{Q}$, so S is well defined Let $S_0(x)=\pm(1-c|x|^\nu)$ In some sense we should be able to say that S is close to S_0 The precise statement is in the following

PROPOSITION 4 2 (a) *For every* $x\in(-1,1)\backslash\{0\}$ *we have*

$$|S(x)-S_0(x)|\leq \text{const }\ \varepsilon|x|^{1+\delta}, \tag{4 38}$$

where $\delta=\min(1-\nu,\gamma\nu)$

(b) *The function* $S'(x)/S_0'(x)$ *admits a* C^δ*-extension* (=1 *at* 0) *to* $(-1,1)$, *and in addition*

$$\left\|\frac{S'}{S_0'}-1\right\|_{C^\delta}\leq \text{const }\ \varepsilon \tag{4 39}$$

Proof It is easy to derive (a) from (b), so we will concentrate on the proof of (b) We notice that $S_0'(x)=\mp c\nu|x|^{\nu-1}$ and that

$$\mp\frac{1}{c\nu}|x|^{1-\nu}D(p\circ T)=Dp\circ T\left(\mp\frac{1}{c\nu}|x|^{1-\nu}DT\right)$$

$$=Dp\circ T\begin{bmatrix}1\mp\frac{1}{c\nu}|x|^{1-\nu}\frac{\partial H_1}{\partial x} & \mp\frac{1}{c\nu}|x|^{1-\nu}\frac{\partial H_1}{\partial y}\\ \mp\frac{1}{c\nu}|x|^{1-\nu}\frac{\partial H_2}{\partial x} & \mp\frac{1}{c\nu}|x|^{1-\nu}\frac{\partial H_2}{\partial y}\end{bmatrix} \tag{4 40}$$

The function $Dp\circ T$ is Holder with exponent $\gamma\nu$ as a composition of two Holder mappings, and moreover

$$\|Dp\circ T-[1,0]\|_{C^{\gamma\nu}}\leq \text{const }\ \varepsilon$$

The matrix in formula (4 40) is certainly in C^γ and is closer than const ε to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The Lemma follows from the formula $S'(x) = D(p \circ T)(x, 0)\, e_1$ and from the previous estimates □

PROPOSITION 4 3 *Under the assumption that* $\nu c > 1$ *the mapping* $S\ (-1, 1) \to (-1, 1)$ *has an absolutely continuous invariant measure with respect to the Lebesgue measure The density of this measure is bounded*

Proof One needs to apply the result of Keller [**8**] The only assumption that needs verification is that the function $g(x) = 1/|S'(x)|$ extends to a Holder function on I We have $g(x) = |S_0'(x)|/h(x)$, where $h(x) = |S'(x)/S_0(x)|$ is Holder and from Lemma 4 10 it follows that h is close to 1 Therefore, g is Holder, as well □

Remark 4 1 Using the fact that the density of the invariant measure belongs to a certain functional space, one easily shows that the measure is unique in the class of probabilistic measures First one shows that the support of the measure is a finite union of segments Then one uses the fact that at least one of the segments has to contain a discontinuity As a consequence we obtain that the invariant measure is ergodic If in addition $\nu c > \sqrt{2}$ then for sufficiently small ε the measure will be mixing

COROLLARY 4 3 (i) *The mapping T can be studied by means of the kneading theory* (cf [**5, 22, 11**]) *Since our mapping is symmetric with respect to* $x \mapsto -x$, *only one kneading invariant is needed to classify T topologically*

(ii) *C Robinson's paper* [**13**] *is devoted to the ergodic aspects of the theory of geometric Lorenz attractors The invariant measure for the quotient map by applying the result of Keller* [**8**] *can be found in Robinson's paper*

We recall that in order to be amenable to our analysis, the family X_μ *that we started with, should satisfy the linearizability condition of class* C^3 *We conjecture (after Robinson) that the linearizability assumption can be dropped A possible way to eliminate it is by applying the technique of normal forms, in a way similar to that of Leontovich* [**9**]

Exercise 4 1 Obtain an invariant measure for T, which projects down to the absolutely continuous invariant measure of S (cf Lemma 1 3 in [**3**] or a similar construction in [**15**])

Exercise 4 2 Using a logarithmic bound on the return time to the Poincaré section, construct an invariant measure for the flow which is finite and corresponds to the invariant measure of T via the special flow (suspension) construction

Appendix A1 *An extension of results of Sternberg and Roussarie*

Roussarie [**14**] showed that two C^∞-vector fields which have identical normal forms at 0, which is assumed to be an equilibrium point of saddle type, are C^∞-equivalent in a neighborhood of 0 This theorem strengthens the results of Sternberg [**20**] concerning linearizability Here we are going to derive a related result which applies to the situation when one needs a C^r-linearizability for a finite r This theorem generalizes the results of Sternberg and Roussarie and its value, besides a simple proof, is an explicit estimate of the smoothness required to produce a C^r-equivalence

Our technique is mostly due to Roussarie and it is a version of Moser's homotopy method We use notation close to Roussarie's

By $\mathcal{V}^r(n)$ we denote the vector space of germs of C^r-vector fields at 0 in $\mathbb{R}^n$ As usual, $\mathcal{E}(n)$ denotes the algebra of C^∞ germs of functions on $\mathbb{R}^n$ at 0 By $\mathcal{M}$ we denote the maximal ideal of $\mathcal{E}(n)$ More generally, by $\mathcal{M}(f_1, f_2, \quad, f_k)$ we will denote the ideal generated by arbitrary functions $f_1, f_2, \quad, f_k \in \mathcal{E}(n)$ In order to allow parameters, we will also consider a subspace of $\mathcal{V}^r(n+s)$ defined as follows

$$\mathcal{V}(n,s) = \mathcal{M}(x_1, x_2, \quad, x_n) \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \quad, \frac{\partial}{\partial x_n}\right\} \tag{A1 1}$$

For convenience we will denote the coordinates of $\mathbb{R}^{n+s}$ by $x_1, x_2, \quad, x_n, y_1, y_2, \quad, y_s$ For a given vector field $X \in \mathcal{V}^r(n,s)$ by X_y we will denote the vector field $pr_1 \circ X(x,y)$, where pr_1 means the projection on the first n coordinates Hence $X_y \in \mathcal{V}^r(n)$ for every $y \in \mathbb{R}^s$ sufficiently close to 0 By $\mathcal{G}^r(n)$ we denote the group of C^r diffeomorphisms $g \ (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ By $\mathcal{G}^r(n,s)$ we denote the subset of $\mathcal{G}^r(n+s)$ consisting of those g which fix the subspaces $\mathbb{R}^n \times \{y\}$ for every y If $r = \infty$ then we skip the superscript r in all our notations

For each of the above objects we introduce a homotopy version, i e an extra coordinate $\tau \in [0,1]$ So, $\mathcal{V}_\tau(n)$ will denote the space of C^∞-vector fields on $[0,1] \times \mathbb{R}^n$, identified if they coincide on a neighborhood of $[0,1] \times \{0\}$ In a similar way we define $\mathcal{E}_\tau(n)$, $\mathcal{V}^r_\tau(n,s)$ etc

We call the germs $X, X' \in \mathcal{V}(n,s)$ C^r-equivalent if there is $g \in \mathcal{G}^r(n,s)$ such that

$$Ad_g(X) = X', \tag{A1 2}$$

where $Ad_g(X) = (Dg \quad X) \circ g^{-1}$ In other words,

$$Dg(z)X(z) = X'(g(z)) \tag{A1 3}$$

for every $z = (x,y) \in \mathbb{R}^{n+s}$ near 0

By $\lambda_1(y), \lambda_2(y), \quad, \lambda_n(y)$ we denote the eigenvalues of $DX_y(0)$ From now on we assume that $\Re\lambda_i(0) \neq 0$ and $\Re\lambda_i(0) > 0$ for $i = 1, 2, \quad, p$ and $\Re\lambda_i(0) < 0$ for $i = p+1, p+2, \quad, n$. So, 0 is a saddle-type equilibrium We will also assume that $DX_y(0)$ renders the splitting $\mathbb{R}^p \times \mathbb{R}^{n-p}$ invariant and that the eigenvalues of $DX_y(0)$ restricted to $\mathbb{R}^p$ are $\lambda_1(y), \quad, \lambda_p(y)$ By the Stable Manifold Theorem we can assume that $\mathbb{R}^p \times \{0\} = W^u(0, X_y)$ and $\{0\} \times \mathbb{R}^{n-p} = W^s(0, X_y)$ for all y Moreover, we can assume that $DX_y(0)$ is nearly diagonal

Let us introduce two constants depending just on the eigenvalues

$$\begin{aligned} A_+ &= \max_{p<i\leq n} |\Re\lambda_i(0)| / \min_{1\leq i\leq p} |\Re\lambda_i(0)|, \\ A_- &= \max_{1\leq i\leq p} |\Re\lambda_i(0)| / \min_{p<i\leq n} |\Re\lambda_i(0)| \end{aligned} \tag{A1 4}$$

In other words, $A_+ = $ (maximal contraction)/(minimal expansion)

THEOREM A 1 1 *Let $k_\pm$ be two integers such that for some integer $r \geq 1$*

$$k_\pm > r(1+A_\pm) \tag{A1 5}$$

Let $X' \in \mathcal{V}(n, s)$ be such that

$$X' - X \in \mathcal{M}(x_1, \quad , x_p)^{k_+}\mathcal{V}(n, s) + \mathcal{M}(x_{p+1}, \quad , x_n)^{k_-}\mathcal{V}(n, s) \tag{A1 6}$$

Then X' is C^r equivalent to X

Proof Let $X_\tau = X + \tau(X' - X) \in \mathcal{V}_\tau(n, s)$ We will look for a homotopy $Y_\tau \in \mathcal{V}^r_\tau(n, s)$ which satisfies the following equation

$$[X_\tau, Y_\tau] = -X_\tau = -(X' - X) \tag{A1 7}$$

One will obtain the C^r equivalence of X and X' by integrating the following nonautonomous ODE

$$\frac{\partial g_\tau}{\partial \tau} = Y_\tau \circ g_\tau, \quad g_0 = \mathrm{id} \tag{A1 8}$$

Suppose that we have found a solution Y_τ to the above equation We set out to prove that $g_\tau \in \mathcal{G}^r_\tau(n, s)$ is a C^r-equivalence of X_0 with X_τ, so $g = g_1$ is an equivalence of X with X' We need to check that $Ad_{g_\tau}(X_0) = X_\tau$, or equivalently, that $Ad_{g_\tau^{-1}}(X_\tau) = X_0$ for every $\tau \in [0, 1]$ It suffices to show that

$$\frac{\partial}{\partial \tau} Ad_{g_\tau^{-1}}(X_\tau) = 0 \tag{A1 9}$$

(We follow the convention that $[X, Y] = DX \quad Y - DY \quad X$) Let $h_\delta = g_{\tau+\delta} \circ g_\tau^{-1}$ Obviously,

$$\left.\frac{\partial}{\partial \delta}\right|_{\delta=0} h_\delta = Y_\tau, \qquad \left.\frac{\partial}{\partial \delta}\right|_{\delta=0} h_\delta^{-1} = -Y_\tau \tag{A1 9a}$$

We also have

$$\begin{aligned}
\frac{\partial}{\partial \tau} Ad_{g_\tau^{-1}}(X_\tau) &= \left.\frac{\partial}{\partial \delta}\right|_{\delta=0} Ad_{g_{\tau+\delta}^{-1}}(X_\tau) \\
&= \left.\frac{\partial}{\partial \delta}\right|_{\delta=0} Ad_{g_\tau^{-1}} Ad_{h_\delta^{-1}}(X_{\tau+\delta}) \\
&= Ad_{g_\tau^{-1}}\left(\left.\frac{\partial}{\partial \delta}\right|_{\delta=0} Ad_{h_\delta^{-1}}(X_{\tau+\delta})\right) \\
&= Ad_{g_\tau^{-1}}\left(\left.\frac{\partial}{\partial \delta}\right|_{\delta=0} Ad_{h_\delta^{-1}}(X_\tau) + \left.\frac{\partial}{\partial \delta}\right|_{\delta=0} X_{\tau+\delta}\right) \\
&= Ad_{g_\tau^{-1}}(-[Y_\tau, X_\tau] + X_\tau)
\end{aligned} \tag{A1 9b}$$

Now from (A1 7) we obtain that the last expression is 0

An equation of the form $[X_\tau, Y_\tau]=Z_\tau$ admits a formal solution

$$Y_\tau=\begin{cases} -\int_0^\infty Ad_{\varphi_u}(Z_\tau)\,du, \\ \int_0^\infty Ad_{\varphi_{-u}}(Z_\tau)\,du, \end{cases} \tag{A1 10}$$

where φ_u is the flow generated by X_τ Indeed,

$$\begin{aligned} \frac{\partial}{\partial u} Ad_{\varphi_u}(Y_\tau) &= \frac{\partial}{\partial\delta}\bigg|_{\delta=0} Ad_{\varphi_{u+\delta}}(Y_\tau)=\frac{\partial}{\partial\delta}\bigg|_{\delta=0} Ad_{\varphi_u}(Ad_{\varphi\delta}(Y_\tau)) \\ &= Ad_{\varphi_u}\left(\frac{\partial}{\partial\delta}\bigg|_{\delta=0} Ad_{\varphi\delta}(Y_\tau)\right)=Ad_{\varphi_u}([X_\tau, Y_\tau]) \\ &= Ad_{\varphi_u}(Z_\tau) \end{aligned} \tag{A1 11}$$

and because $\varphi_0=\mathrm{id}$ we have $Ad_{\varphi_0}(Y_\tau)=Y_\tau$ Therefore by integrating (A1 11) from 0 to ∞ and assuming that $Ad_{\varphi_u}(Y_\tau)\to 0$ as $u\to\infty$, we obtain the first of formulas (A1 10) The second formula can be obtained by reversing the time Our task is to make the formal solutions (A1 10) into C^r-solutions by adding appropriate conditions on X_τ and Z_τ

In the first step we choose representatives of X and X' in such fashion that $X_y(x)-DX_v(0)x$ has a compact support and is small in C^1-topology A standard way to proceed is to choose a function $\chi\in C^\infty(\mathbb{R}^n)$ supported on the unit ball and $\equiv 1$ in a neighborhood of 0 We replace the homotopy X_τ with

$$\tilde{X}_\tau=D_xX_\tau(0,y)x+\chi(x/\varepsilon)\quad(X_\tau(x,y)-D_xX_\tau(0,y)x) \tag{A1 12}$$

We easily find that the support of $\tilde{X}_{\tau,y}$ is in $\mathscr{B}(\varepsilon)\times\mathbb{R}^s$($\mathscr{B}(\varepsilon)$ the ball of radius ε centered at the origin) The germs X_τ and $\tilde{X}_\tau$ are identical and one shows easily that

$$\|D_x(\tilde{X}_\tau-D_xX_\tau(0,y)x)\|_{C^0}\le \text{const}\quad\varepsilon \tag{A1 13}$$

Therefore we can assume that

$$\|X_\tau(x,y)-D_xX_\tau(0,y)x\|_{C^1} \tag{A1 14}$$

is arbitrarily small We will defer the proof of Theorem A1 1 until several auxiliary results become available

PROPOSITION A1 1 *Let C_1 be an arbitrary constant bigger than*

$$\max_{1\le i\le p}|\Re\lambda_i(0)| \tag{A1 15}$$

For sufficiently small ε in (A1 12) *and for every $r\ge 1$ there is $K_r>0$ such that*

$$\|D^r\varphi_u\|_{C^0}\le K_r\exp(C_1ru) \tag{A1 16}$$

Proof Let us start with $r=1$ We have

$$\frac{\partial}{\partial u}D\varphi_u=DX_\tau\circ\varphi_uD\varphi_u \tag{A1 17}$$

We may assume that $\|DX_\tau\|_{C^0} \le C_1$, so

$$\left\| \frac{\partial}{\partial u} D\varphi_u \right\| \le C_1 \| D\varphi_u \| \tag{A1 18}$$

This inequality implies (A1 16) with $r=1$ by a standard argument

Then we proceed by induction For $r>1$ we can write the composition formula

$$\frac{\partial}{\partial u} D^r\varphi_u = D^r(X_\tau \circ \varphi_u) = \sum_{l\,\alpha} c_{l\,\alpha} D^l X_\tau \circ \varphi_u D^{\alpha_1}\varphi_u \quad D^{\alpha_l}\varphi_u, \tag{A1 19}$$

where the summation extends over all l and α such that $0 \le l \le r$, $|\alpha| = l$ and

$$\sum_{j=1}^{l} \alpha_j = r \tag{A1 20}$$

The constants $c_{l,\alpha}$ are universal We can rewrite (A1 19) as follows

$$\frac{\partial}{\partial u} D^r\varphi_u = DX_\tau \circ \varphi_u \quad D^r\varphi_u + \mathscr{P}(D\varphi_u, D^2\varphi_u, \quad , D^{r-1}\varphi_u), \tag{A1 21}$$

where $\mathscr{P}$ stands for a 'differential polynomial' Let us notice that $D^r\varphi_0 = 0$ for $r>1$ We can write the solution to (A1 21) explicitly as

$$D^r\varphi_u = \int_0^u D\varphi_{u-v}\mathscr{P}(D\varphi_v, D^2\varphi_v, \quad , D^{r-1}\varphi_v)\, dv \tag{A1 22}$$

By induction hypothesis we find that

$$\begin{gathered} \|\mathscr{P}(D\varphi_v, \quad , D^{r-1}\varphi_v)\| \le \tilde{K}_r \exp(C_1 r v), \\ \|D\varphi_{u-v}\| \le \exp(C_1(u-v)) \end{gathered} \tag{A1 23}$$

Hence we have

$$\|D^r\varphi_u\| \le \tilde{K}_r \int_0^u \exp(C_1 u + (r-1)C_1 v)\, dv \le \frac{\tilde{K}_r}{(r-1)C_1} \exp(C_1 r u) \tag{A1 24}$$

The proof has been completed □

The following lemma shows that $Y_\tau \in C^0$

LEMMA A1 1 *Suppose that k is an integer, $k > A_+$ and*

$$Z_r \in \mathscr{M}(x_1, \quad , x_p)^k \mathscr{V}(n, s) \tag{A1 25}$$

Then the first of the formulas (A1 10) defines $Y_\tau \in \mathscr{V}^0(n, s)$

Proof Let $\rho(z) = (x_1^2 + \quad + x_p^2)^{1/2}$ (the distance from $\{0\} \times \mathbb{R}^{n-p} \times \mathbb{R}^s$) Let

$$X = \sum_{i=1}^{n} X_i(z) \frac{\partial}{\partial x_i} \tag{A1 26}$$

From the Taylor expansion of X_i for $i \le p$ we obtain the following representation

$$X_i = \sum_{j=1}^{p} X_{i,j}(z) x_j \tag{A1 27}$$

Let $X(\rho^2)$ be the Lie derivative of the function ρ^2 along the vector field X We obtain

$$X(\rho^2) = 2\sum_{i=1}^{p} x_i X_i(z) = 2\sum_{i,j=1}^{p} X_{i,j}(z) x_i x_j \tag{A1 28}$$

Since the matrix $[X_{i,j}(0, y)] = D_x X_y(0)$ is nearly diagonal, we can write that

$$2C_2\rho^2 \le X(\rho^2) \le 2C_1\rho^2, \qquad (A1\ 29)$$

where C_1 and C_2 are numbers satisfying the inequalities

$$C_2 < \min_{1\le i\le p} |\Re\lambda_i(0)|, w$$
$$C_1 > \max_{p<i\le n} |\Re\lambda_i(0)| \qquad (A1\ 30)$$

Moreover, we assume that C_1 and C_2 are sufficiently close to the above bounds, so that $k > C_1/C_2 > A_+$ This may require a linear change of coordinates to assure that $D_x X_y(0)$ is sufficiently close to the diagonal matrix It may also require using ε small enough in (A1 12)

We have the following estimate

$$\exp(C_2 u)\rho(z) \le \rho(\varphi_u(z)) \le \exp(C_1 u)\rho(z) \qquad (A1\ 31)$$

Now we can write

$$\begin{aligned}\|Ad_{\varphi u}(Z_\tau)\| &\le \|(D\varphi_u \circ \varphi_{-u})Z_\tau\| \\ &\le \exp(C_1 u)\rho(\varphi_{-u}(z))^k C_3 \le \exp(C_1 u)[\rho(z)\exp(-C_2 u)]^k C_3 \qquad (A1\ 32)\\ &\le C_3 \exp((C_1 - kC_2)u)\rho(z)^k,\end{aligned}$$

where $C_3 = \sup \|Z_\tau\|$ If $k > C_1/C_2$ then the first of the integrals (A1 10) converges uniformly for all z and therefore defines a continuous vector field □

In a similar fashion we will establish that Y_τ is smooth

PROPOSITION A1 2 *Let Y_τ^a be defined by the formula*

$$Y_\tau^a = -\int_0^a Ad_{\varphi_u}(Z_\tau)\,du \qquad (A1\ 33)$$

Let $k > r(1 + A_+)$ and

$$Z_\tau \in \mathcal{M}(x_1, \quad , x_p)^k \mathcal{V}(n, s) \qquad (A1\ 34)$$

Then Y_τ^a converges to Y_τ in C^r-topology, as $a \to \infty$

Proof It is obvious that Y_τ^a is in $\mathcal{V}^\infty(n, s)$ We can write

$$D^m Y_\tau^a = -\int_0^a D^m Ad_{\varphi u}(Z_\tau)\,du \qquad (A1\ 35)$$

As an immediate consequence Y_τ is C^∞ on the complement of $\mathbb{R}^p \times \{0\} \times \mathbb{R}^s$ We need to show that the last integral converges uniformly as $m \le r$ and $a \to \infty$

By another application of the Composition Theorem we obtain

$$\begin{aligned}D^m Ad_{\varphi_u}(Z_r) &= D^m[(DX_\tau \quad Z_\tau)\circ\varphi_{-u}] \\ &= \sum_{l,\alpha} c_{n,\alpha} D^l(DX_\tau \quad Z_\tau)\circ\varphi_{-u} D^{\alpha_1}\varphi_{-u} \qquad D^{\alpha_l}\varphi_{-u}, \qquad (A1\ 36)\end{aligned}$$

where the summation is over $l \le m$ and $\alpha = (\alpha_1, \quad , \alpha_l)$ such that $|\alpha| = l$ and $\sum_{j=1}^l \alpha_j = m$ Also by Leibnitz's Rule we get

$$D^l(DX_\tau \quad Z_\tau)\circ\varphi_{-u} = \sum_{j=0}^{l} \binom{l}{j} D^{j+1}X_\tau \circ \varphi_{-u} D^{l-j} Z_\tau \circ \varphi_{-u} \qquad (A1\ 37)$$

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Besides, from (A1 34) we get

$$\|D^{l-j}Z_\tau\| \le \text{const} \quad \rho(z)^{k-(l-j)} \tag{A1 38}$$

Therefore

$$\|D^{l-j}Z_\tau \circ \varphi_{-u}\| \le \text{const}\ \rho(z)^{k-l} \exp(-(k-l)C_2 u) \tag{A1 39}$$

Let $C_1 > \max_{p<i\le n} |\Re\lambda_i(0)|$ and $C_2 < \min_{0\le i\le p} |\Re\lambda_i(0)|$ be as in the proof of Lemma A1 1 By Proposition A1 1 applied to $-X_\tau$ we obtain

$$\|D^j\varphi_{-u}\| \le \text{const} \quad \exp(C_1 j u) \tag{A1 40}$$

Now from (A1 37) and (A1 39) we obtain

$$\|D^l(DX_\tau\ \ Z_\tau)\circ\varphi_{-u}\| \le \text{const} \quad \rho(z)^{k-l} \exp(-(k-l)C_2 u) \tag{A1 41}$$

since in (A1 37) all $D^{j+1}X_\tau$ are bounded as a consequence of (A1 12) By applying (A1 40) and (A1 41) to (A1 36) we get

$$\|D^r Ad_{\varphi_u}(Z_\tau)\| \le \text{const} \quad \rho(z)^{k-r} \exp\{[r(C_2+C_1)-kC_2]u\} \tag{A1 42}$$

Therefore, if $k > r(1+C_1/C_2)$ then the integral (A1 35) converges uniformly, as $a \to \infty$ □

COROLLARY A1 1 *Under the assumptions of Proposition* A1 2 $Y^\tau \in \mathcal{V}^r(n, s)$

Now we are ready to complete the proof of Theorem A1 1 We need to split $Z_\tau = X_\tau = X' - X = Z'_\tau + Z''_\tau$, so that

$$\begin{aligned} Z'_\tau &\in \mathcal{M}(x_1, \quad , x_p)^{k_+}\mathcal{V}^r(n, s), \\ Z''_\tau &\in \mathcal{M}(x_1, \quad , x_p)^{k_-}\mathcal{V}^r(n, s), \end{aligned} \tag{A1 43}$$

and obtain solutions to the equations $[X_\tau, Y'_\tau] = Z'_\tau$ and $[X_\tau, Y''_\tau] = Z''_\tau$ We define the solution to $[X_\tau, Y_\tau] = Z_\tau$ as $Y_\tau = Y'_\tau + Y''_\tau$ Then we solve (A1 8) to get the C^r-equivalence □

COROLLARY A1 2 *Let $X \in \mathcal{V}(n, s)$ and $k = k_+ + k_-$ Let X' be the Taylor polynomial of X of degree $k-1$ Then X and X' are C^r-equivalent*

Proof We can write

$$X' - X = \sum_{j=1}^{n} \sum_{|\alpha|=k} x^\alpha Q_{\alpha,j}(z) \frac{\partial}{\partial x_j}, \tag{A1 44}$$

where $Q_{\alpha\,j} \in C^r(\mathbb{R}^{n+s})$ Let $\alpha' = (\alpha_1, \quad , \alpha_p)$ and $\alpha'' = (\alpha_{p+1}, \quad , \alpha_n)$ We split the sum (A1 44) into two parts, a term will be in the first or second part, depending on whether $|\alpha'| \ge k_+$ or $|\alpha''| > k_-$ (exactly one of those two possibilities holds, since $|\alpha'| + |\alpha''| = k$) □

In conjunction with the technique of normal forms these theorems give a powerful method for deciding C^r-linearizability of saddles In this fashion Theorem 1 1 follows easily from the last Corollary and Poincaré-Dulac Theorem (cf [2]) A subtle point is that the corresponding diffeomorphisms both in the Poincaré-Dulac Theorem and in our Corollary commute with the symmetry (1 1), if we apply them to a vector field observing that symmetry This is due to the nature of that symmetry, since observing it is equivalent to vanishing some terms in the Taylor expansion of X and the derived diffeomorphisms

Remark A1 1. (a) It is sufficient to assume in Theorem A1 1 that X and X' are of class C^{r+1} and that

$$X' - X \in \mathcal{M}(x_1, \quad , x_p)^{k_+}\mathcal{V}^r(n, s) + \mathcal{M}(x_{p+1}, \quad , x_n)^{k_-}\mathcal{V}(n, s) \qquad \text{(A1 45)}$$

(b) Subsequently, we can assume in Corollary A1 2 that X is of class C^{k+r} instead of C^∞

Example A1 1 Let us consider the following system of equations on the plane

$$\begin{aligned} x &= x, \\ y &= -\lambda y + x^m y^{n+1}, \end{aligned} \qquad \text{(A1 46)}$$

where λ is nearly m/n If $\lambda = m/n$ then we have a resonance and cannot C^∞-linearize the system On the other hand, one can write down a $C^{m-\varepsilon}$-linearization, which smoothly depends on λ

$$g(x, y) = (x, y(1 + nx^m y^n \theta)^{-1/n}), \qquad \text{(A1 47)}$$

where θ is the 'Leontovich variable' (cf [9])

$$\theta = \begin{cases} (|x|^{n\lambda - m} - 1)/(n\lambda - m), & \text{as } m - n\lambda \neq 0, \\ \ln |x| & \text{otherwise} \end{cases} \qquad \text{(A1 48)}$$

Example A1 2 In [12] C Robinson considers the eigenvalues of the Lorenz system (with the original values of parameters) $\lambda_1 = -11/2 + 1/2\sqrt{1201}$, $\lambda_2 = -11/2 - 1/2\sqrt{1201}$, $\lambda_3 = -8/3$ It is easy to see that $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is resonant and that the lowest order resonance is

$$8\lambda_1 + 8\lambda_2 - 33\lambda_3 = 0 \qquad \text{(A1 49)}$$

Also, $A_+ = |\lambda_2|/\lambda_1$ and $A_- = \lambda_1/|\lambda_3|$ Numerically, $A_+ \approx 1\ 93$ and $A_- \approx 4\ 43$ For $r = 1$ we get $k_+ > 1 + A_+ \approx 2\ 93$ and $k_- > 1 + A_- \approx 5\ 43$ Hence, the best choice is $k_+ = 3$ and $k_- = 6$ This yields $k = 9$ and X needs to be $C^{k+r} = C^{10}$ in order to be C^1-linearizable Similarly for $r = 2$ we need $k_\pm > 2(1 + A_\pm)$, which yields $k_+ > 5\ 86$ and $k_- > 10\ 86$ Hence we put $k_+ = 6$, $k_- = 11$, $k = 17$ and we obtain that X has to be of class C^{19} in order to be C^2-linearizable These estimates are slightly better than those of Robinson (C^{14} and C^{20})

One can see easily that the highest class C^r for which we still have a C^r-linearization without any assumptions on X is less than the order of the resonance (A1 49), i e $(8 + 8 + 33) - 1 = 48$ X admits a C^{47}-linearization if it is of class at least C^{541}!

Exercise A1 1 Rewrite this Appendix, so that it applies to diffeomorphisms

Exercise A1 2 Verify that in the case of C^∞-vector fields with the same normal form at 0 we get a C^∞-equivalence (Sternberg, Roussarie)

Appendix A2 *A reparameterization of the Lorenz system*

By simple calculations one can show the following theorem, which can be found in a similar form in [21]

THEOREM A2 1 *Suppose that $\sigma(r-1)>0$ and we change variables in the Lorenz system according to*

$$X=\frac{1}{2\sqrt{\sigma(r-1)}}x,$$

$$Y=\frac{1}{2(r-1)}(y-x),$$

$$Z=\frac{1}{4\sqrt{\sigma(r-1)}}\left(1-\frac{b}{2\sigma}\right)^{-1}\left(z-\frac{x^2}{2\sigma}\right), \tag{A2 1}$$

$$\frac{d}{ds}=\frac{1}{\sqrt{\sigma(r-1)}}\frac{d}{dt}$$

Then the system becomes

$$\begin{cases} X'=Y, \\ Y'=X-2X^3+\alpha Y+\gamma XZ, \\ Z'=-\kappa Z+X^2, \end{cases} \tag{A2 2}$$

where

$$\alpha=-\frac{1+\sigma}{\sqrt{\sigma(r-1)}},$$

$$\gamma=-\frac{4\sqrt{\sigma(r-1)}}{r-1}\left(1-\frac{b}{2\sigma}\right), \tag{A2 3}$$

$$\kappa=\frac{b}{\sqrt{\sigma(r-1)}}$$

Moreover, if α, γ and κ are such that

$$4\alpha-\gamma+2\kappa\neq 0, \quad \gamma-2\kappa\neq 0$$

then there are (r, σ, b), solving (A2 3) and such that $\sigma(r-1)>0$, given by the following formulas

$$\sigma=\frac{\gamma-2\kappa}{4\alpha-\gamma+2\kappa},$$

$$r=1+\frac{16}{(4\alpha-\gamma+2\kappa)(\gamma-2\kappa)}, \tag{A2 4}$$

$$b=-\frac{4\kappa}{4\alpha-\gamma+2\kappa}$$

Hence, if $(\alpha, \gamma)\to 0$, while $\kappa\neq 0$ is fixed, then

$$(r, \sigma, b)\to\left(1-\frac{4}{\kappa^2}, -1, -2\right)$$

We notice that for $\kappa\in(\frac{1}{2}, 1)$ we have $r\in(-15, -3)$ As it follows from § 0 and § 3, these values of κ played a special role in our considerations

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Remark A2 1 It ıs clear that our result does not ımply that there are geometrıc Lorenz attractors ın the Lorenz system for some values of r, σ and b The reader should also be aware that one can not bıfurcate geometrıc Lorenz attractors near the segment of parameters $r \in (-15, -3)$, $\sigma = -1$, $b = -2$ ın the way descrıbed ın thıs paper The reason ıs, roughly speakıng, that we are short of one parameter to prove transversalıty of our famıly By addıng another term ın equatıons (A1 12) (and a new parameter β) we achıeved the followıng goal one can perturb the system wıthout dısruptıng the double unstable manıfold loop (figure 2) The loop becomes attractıve and another perturbatıon can be applıed to obtaın a geometrıc Lorenz attractor

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