Note on a trigonometrical proof of the orthocentre property of a triangle.

The following argument is simple and instructive, but I do not remember seeing it before.

We start with the identity

$$\sin^2\alpha + \cos^2\alpha = \sin^2\beta + \cos^2\beta$$

from which it follows that, for all values of α, β, γ,

$$(\cos\alpha - \cos\beta)(\cos\alpha + \cos\beta + \cos\gamma - \cos\gamma)$$
$$+ (\sin\alpha - \sin\beta)(\sin\alpha + \sin\beta + \sin\gamma - \sin\gamma)$$

vanishes.

Hence if the rectangular Cartesian coordinates of four points A, B, C, P are, respectively,

$$(\cos\alpha,\ \sin\alpha),$$
$$(\cos\beta,\ \sin\beta),$$
$$(\cos\gamma,\ \sin\gamma),$$

and $$(\cos\alpha + \cos\beta + \cos\gamma,\ \sin\alpha + \sin\beta + \sin\gamma),$$

then the line AB is perpendicular to CP. Similarly BC is perpendicular to AP, and CA to BP. Hence P is the orthocentre of the triangle ABC.

Conversely every triangle has an orthocentre, since we can apply the above argument to any given triangle by taking the circumcentre 0 of the triangle as origin, and the radius of the circumcircle as unit of measurement.

Again, since the coordinates of G the centroid of ABC are

$$\{\tfrac{1}{3}(\cos\alpha + \cos\beta + \cos\gamma),\ \tfrac{1}{3}(\sin\alpha + \sin\beta + \sin\gamma)\},$$

it follows that G lies in OP and that $3\,OG = OP$.

H. W. TURNBULL.

https://doi.org/10.1017/S1757748900001845 Published online by Cambridge University Press