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Onsager's 'ideal turbulence' theory

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In 1945–1949, Lars Onsager made an exact analysis of the high-Reynolds-number limit for individual turbulent flow realisations modelled by incompressible Navier–Stokes equations, motivated by experimental observations that dissipation of kinetic energy does not vanish. I review here developments spurred by his key idea that such flows are well described by distributional or 'weak' solutions of ideal Euler equations. 1/3 Hölder singularities of the velocity field were predicted by Onsager and since observed. His theory describes turbulent energy cascade without probabilistic assumptions and yields a local, deterministic version of the Kolmogorov 4/5th law. The approach is closely related to renormalisation group methods in physics and envisages 'conservation-law anomalies', as discovered later in quantum field theory. There are also deep connections with large-eddy simulation modelling. More recently, dissipative Euler solutions of the type conjectured by Onsager have been constructed and his 1/3 Hölder singularity proved to be the sharp threshold for anomalous dissipation. This progress has been achieved by an unexpected connection with work of John Nash on isometric embeddings of low regularity or 'convex integration' techniques. The dissipative Euler solutions yielded by this method are wildly non-unique for fixed initial data, suggesting 'spontaneously stochastic' behaviour of high-Reynolds-number solutions. I focus in particular on applications to wall-bounded turbulence, leading to novel concepts of spatial cascades of momentum, energy and vorticity to or from the wall as deterministic, space–time local phenomena. This theory thus makes testable predictions and offers new perspectives on large-eddy simulation in the presence of solid walls.

Key words: turbulence theory, Navier–Stokes equations, vortex shedding

1. Introduction

This entire essay shall be concerned with a theory of 'ideal turbulence' which was proposed by Lars Onsager (figure 1) in a 1949 paper entitled 'Statistical Hydrodynamics'.

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Figure 1. Lars Onsager (1903–1976). Photograph published originally in *Svenska Dagbladet* on 6 December 1968 shortly after the announcement of Onsager's award of the Nobel Prize in Chemistry, reproduced on license from ZUMA Press.

The proposal was set forth by Onsager in his characteristic laconic style in the final paragraph of that paper, which I quote here in full:

> It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such 'ideal' turbulence cannot obey any LIPSCHITZ condition of the form
>
> (26) $$|\boldsymbol{v}(\boldsymbol{r}' + \boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{r}')| < (\text{const.})r^n$$
>
> for any order n greater than $1/3$; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description; for example, the formulation (15) in terms of FOURIER series will do. The detailed conservation of energy (17) does not imply conservation of the total energy if the number of steps in the cascade is infinite, as expected, and the double sum of $Q(\boldsymbol{k}, \boldsymbol{k}')$ converges only conditionally. (Onsager 1949)

In fact, the germ of these remarks were contained in a short abstract on fluid turbulence that Onsager published four years earlier, where he noted that '…various experiments indicate that the viscosity has a negligible effect on the primary process; hence one may inquire about the laws of turbulent dissipation in an ideal fluid' (Onsager 1945*a*). Although he never published an argument justifying his 'it is possible to show' assertion, it is now known that Onsager had indeed derived a mathematical identity which implies his conclusion and which he communicated by letter to Theodore von Kármán and Chia-Chiao Lin in 1945 (Eyink & Sreenivasan 2006). Although Onsager's innovative ideas on this subject were long overlooked and conflated with the related theory of Kolmogorov (1941*a*,*b*), it is now understood that Onsager's work essentially refined and extended the concepts of Kolmogorov, anticipating ideas in large-eddy simulation (LES) modelling, modern field-theoretic notions of conservation-law anomalies and renormalisation-group invariance, and the concept of weak Euler solutions in the mathematical theory of partial differential equations.

I first became aware of Onsager's proposal in 1990 when I was a postdoc at Rutgers University and Uriel Frisch, visiting for the spring semester, delivered a series of seminars on hydrodynamic turbulence. Frisch had rediscovered some of Onsager's key ideas on his own (Sulem & Frisch 1975) and he only learned of Onsager's prior work himself from Robert Kraichnan in 1972 (Frisch, private communication). Onsager's $1/3$ Hölder claim was discussed briefly at the end of Frisch's Rutgers lectures, which followed closely an expository article he published that same year (Frisch & Orszag 1990). (Interestingly, the reference to Onsager's 1949 paper and the $1/3$ Hölder claim were cut from the published article, presumably because of length restrictions.) I looked up Onsager's paper and was immediately impressed by his remarks because, unlike Kolmogorov's arguments for 1/3 power-law scaling based upon statistical assumptions and dimensional analysis, Onsager claimed that the 1/3 exponent could be derived on a purely dynamical basis for individual solutions of the fluid equations. In fact, we now know that Onsager's 1/3 result is not equivalent to that of Kolmogorov (1941*a*,*b*), e.g. being completely compatible with inertial-range intermittency, which Onsager had already anticipated in 1945 (Eyink & Sreenivasan 2006). After consulting with two Rutgers experts on the mathematics of Navier–Stokes equations, Giovanni Gallavotti and Vladimir Scheffer, I was surprised to learn that Onsager's claims were unknown to the partial differential equation (PDE) community in 1990 and that nothing was established about their validity. The situation is now very different and Onsager's ideas have become the focus of major developments in the mathematical theory of PDEs, connecting surprisingly with ideas of John Nash on a completely different problem of isometric embeddings of Riemannian manifolds (De Lellis & Székelyhidi 2019). These developments by many people have in turn attracted renewed attention to Onsager's ideas in the fluid mechanics community, being indeed the subject of an earlier Perspectives essay by Dubrulle (2019).

The purpose of the present work is to explain Onsager's ideas in a pedagogical and straightforward manner. Although I shall follow somewhat the chronological development of the theory, the emphasis in this work is on science and not history. Most of the research on Onsager's theory so far has been by mathematicians and this may have led to the impression among many that the subject is an esoteric branch of pure mathematics. Although the rigorous development leads indeed to some non-trivial mathematical issues, the theory is directly motivated by experimental observations and by intuitive physical ideas. Some of these points have been explained in my earlier unpublished short note (Eyink 2018*b*) and in my online turbulence course notes (Eyink 2007). In fact, I believe that Onsager's point of view is one of the easiest to teach to beginning students of turbulence. The theory does not cover all turbulent phenomena, as it concerns the regime of very high Reynolds numbers, but it is not restricted to the idealised limit of infinite Reynolds number only (as has been sometimes misunderstood). It sheds little light, therefore, on the important problem of transition to turbulence. However, one of the great virtues of Onsager's approach is the naturalness with which it extends concepts of high-Reynolds-number turbulence from the traditional problem of incompressible fluid turbulence to more general cases of compressible fluids, relativistic fluids, plasma kinetics and quantum superfluids. See § 3.1.3.4.

There are, by now, a very large number of researchers who have contributed to this subject, many at a deeper mathematical level than I have. There are already several illuminating and insightful reviews written by mathematicians, such as De Lellis & Székelyhidi (2013, 2019) and Buckmaster & Vicol (2020), but the present essay is not designed as such a comprehensive review. Instead, my purpose is to explain the subject from the personal perspective of a mathematical physicist working for some years in this area. As with any currently active field, all researchers may not agree with my proffered interpretations and points of view. I endeavour, however, to give technically correct explanations of results, even those to which I have not contributed myself. I do not want to present the subject as a *fait accompli*, which it is not, but instead as a living scientific theory with many critical questions still open. The main reason that I was excited to write this essay is that I believe that some of the most fundamental problems in this area remain unsolved and call for the combined efforts, not only of mathematicians, but also of fluid mechanicians, computational scientists, turbulence modellers and physicists, both theorists and experimentalists. The earlier essay of Dubrulle (2019) lucidly explained how analytical tools developed in the Onsager theory could be applied to direct analysis of empirical data. I will be concerned here instead with the complementary issue of how the 'ideal turbulence' theory connects with the numerical modelling method of LES, a topic only briefly treated in that earlier work (Dubrulle 2019, § 8.6). Although the two subjects have developed historically with little interaction, they are in fact quite intimately connected. For example, both LES and 'ideal turbulence' aspire to describe individual realisations of a turbulent flow. In this essay, I shall focus in particular on the issue of turbulence-wall interactions, which currently concerns deeply both the LES and the mathematical PDE communities and which poses some of the greatest current challenges in turbulence research. As we shall see, Onsager explored this problem himself and his attempts anticipate some recent developments.

Since this essay is rather long, it may be useful to summarise briefly its contents and to explain the organisation, so as to provide the reader with some guide for the journey. The bare synopsis of the contents is as follows:

Our tour through this subject begins in § 2 with a summary of some of the experimental phenomena and the basic theoretical assumptions generally made to explain them. The first half of the essay, § 3, then treats turbulence away from solid walls, such as wakes and jets, which was the main subject of Onsager's original work. As I discuss, Onsager himself made an exact and essentially rigorous analysis of this problem, deriving various results on energy cascade, singularities, scale-locality, etc. § 3.1 on this subject is titled 'Onsager's Result', although he never published his own derivations, and his results had to be recovered and extended by the work of others. We shall examine Onsager's unpublished material on turbulence, which are now available, along with his unpublished results on many other subjects, in the online Onsager Archive hosted by the Norwegian University of Science and Technology: https://www.ntnu.edu/onsager/lars-onsager-archive. In addition, however, Onsager made more ambitious proposals on the existence of dissipative Euler solutions and their emergence in the infinite-Reynolds-number limit, which were motivated by observations but for which he almost certainly had no analytical arguments. Only much later was dramatic progress made on these questions and, as I review in § 3.2 entitled 'Onsager's Conjecture', this has involved rather sophisticated mathematical tools. The second half of my essay, § 4, deals with the subject of wall-bounded turbulence, which is of crucial importance for most terrestrial turbulent flows and a keen interest of Onsager's, but which has only recently been seriously tackled by his methods. This section closely parallels the previous ones, with § 4.1 reviewing particular features of the high-Reynolds-number limit for turbulent-wall interactions, and early ideas of Taylor and Onsager on the subject. Then § 4.2 describes in detail how the analysis pioneered by Onsager applies to wall-bounded turbulence and, in recent work of many researchers, leads to a picture of both spatial and scale cascades of momentum, energy and vorticity, but understood as deterministic and space–time local processes. The concluding § 5 of this essay offers some final remarks about the empirical status and future importance of Onsager's theory.

2. Background

Before I can discuss any theory, I must briefly review the 'various experiments' on turbulent energy dissipation mentioned by Onsager (1945*a*) that motivated his analysis. The specific work cited by Onsager (1949) was the experimental study of Hugh L. Dryden (1943) on wake turbulence behind a grid using hotwire anemometry, which reported that the decay rate of kinetic energy $Q = -(\mathrm{d}/\mathrm{d}t)\frac{3}{2}u'^2$ satisfies the scaling law

$$Q \sim A\frac{u'^3}{L} \tag{2.1}$$

with $A \doteq 0.2056$ at sufficiently high Reynolds number $Re = u'L/\nu$, where u' is the root-mean-square (r.m.s.) streamwise velocity fluctuation, L is the velocity integral length and ν is the kinematic viscosity of the fluid. The scaling law (2.1) seems to have been first hypothesised by Taylor (1935), who already argued in Taylor (1917) that kinetic energy can be 'dissipated in fluid of infinitesimal viscosity, when the turbulent motion takes place in three dimensions'. Onsager inferred from the results of Dryden (1943) that the coefficient $A(Re)$ becomes constant for $Re \gg 1$, but the first systematic evidence was obtained by Sreenivasan (1984) based on a compilation of data from several experiments with different types of grids. See figure 2(*a*). Note that the same hypothesis of the asymptotic Re-independence of non-dimensionalised energy dissipation rate was made also by Kolmogorov (1941*a*,*b*).

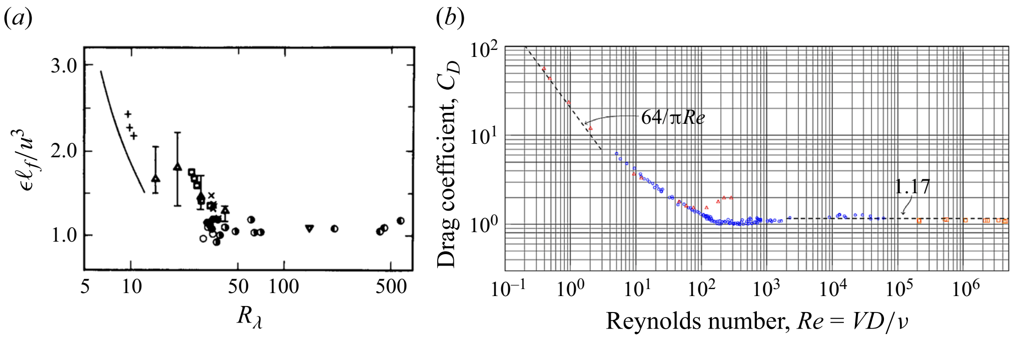

Figure 2. Some key pieces of empirical evidence that turbulent energy dissipation is anomalous (non-vanishing) at high Reynolds numbers. (*a*) Quantity $A = QL/u'^3$ for bi-plane square-mesh grids, compiled from several experiments, ϵ denoting my Q and ℓ_f my L. Reproduced from Sreenivasan (1984), with permission of AIP Publishing. (*b*) Drag coefficient for flow past a circular disk oriented normal to the flow. Open red triangles from Hoerner (1965), open circles from Roos & Willmarth (1971), orange squares from Shoemaker (1926). Reproduced with permission from https://kdusling.github.io/teaching/Applied-Fluids.

Although this observation is basic to several theories of high-Reynolds turbulence and is thus sometimes referred to as the 'zeroth law of turbulence', the experimental situation is in fact more complex and interesting. It is indeed true that the result is observed to hold well in wake flows, such as those past bluff or streamlined bodies. One evidence for this is the common observation that the drag coefficient of the body

$$C_D = \frac{F_d}{\frac{1}{2}\rho U_\infty^2 A} \tag{2.2}$$

tends to a constant value for $Re \gg 1$, where F_d is the drag force, ρ is fluid mass density, U_∞ is external flow velocity and A is the frontal area of the body. See figure 2(*b*) for the example of a circular disk and Frisch (1995, § 5.2) for the connection with dissipation in the wake. However, there is a striking dichotomy for internal flows such as flows through pipes or channels, likewise Taylor–Couette, Rayleigh–Bénard and von Kármán flows, and as well for flat-plate boundary layers, in all of which the strict independence from Reynolds number depends upon whether the solid boundary is hydraulically smooth or hydraulically rough. For mathematical readers, I must clarify that this distinction has nothing to do with the mathematical smoothness of the boundary and 'hydraulic roughness' means simply that the solid boundary has small ripples, ridges, etc. with some characteristic roughness height k that impede the flow. Thus, a hydraulically rough surface may in some cases be modelled by a C^∞ manifold. The general observation in these flows is that the dimensionless dissipation rate becomes asymptotically independent of Reynolds number only when the solid boundary is hydraulically rough. This fact was noted as early as the 18th century by the French engineer Antoine de Chézy who attributed the seasonal variation of drag in the Parisian water canals to the growth and decline of algae and moss on the sidewalls. A more quantitative observation was made in straight circular pipe flows by Nikuradse (1933) who studied the friction factor

$$\lambda = \frac{-\dfrac{\partial P}{\partial x}(2R)}{\frac{1}{2}\rho \bar{U}^2}, \tag{2.3}$$

where $-(\partial P/\partial x)$ is the applied pressure gradient, R is the pipe radius and $\bar{U}$ is the mean flow velocity. He found that $\lambda(Re)$ slowly decayed to zero as $Re \to \infty$ for a smooth-wall pipe but, with sand-grains glued to the wall, instead tended to a positive constant that depended on the grain height k. These same qualitative observations have been confirmed in a great variety of internal flows, e.g. see Cadot *et al.* (1997). Onsager, by the way, was certainly aware of such observations as indeed he cited in his 1949 paper the work of Montgomery (1943), who considered drag laws in pipe flows with both smooth and rough walls, and who discussed the classic works of Nikuradse (1933) and others.

The starting point of the theoretical analysis of Onsager (1949) was the incompressible Navier–Stokes equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}, \quad \nabla \boldsymbol{\cdot} \boldsymbol{u} = 0, \tag{2.4}$$

where $p = P/\rho$ is kinematic pressure, in line with the common assumption that all turbulence phenomena at low Mach numbers can be described within this continuum approximation. Most of my essay will present results that have been obtained from this standard point of view. It is not, however, entirely obvious that this equation should be adequate to describe turbulent energy dissipation fully, since the fundamental fluctuation-dissipation relation of statistical physics implies that molecular dissipation phenomena and thermal fluctuations are intrinsically intertwined and must always occur together. Onsager was, of course, deeply familiar with the statistical theory of thermal fluctuations. The so-called 'Onsager principle', which he proposed in his 1931 work on reciprocal relations (Onsager 1931*a*,*b*) and which was worked out by Onsager & Machlup (1953) for the linear regime, is probably the most elegant form of the fluctuation-dissipation relation, expressing the probability of a time-history to arise by thermal fluctuations directly in terms of the time-integrated dissipation. The prediction that thermal fluctuations should fundamentally modify the turbulent dissipation range starting at the Kolmogorov length scale $\eta = \nu^{3/4}/\varepsilon^{1/4}$ (with ε the mean dissipation rate of kinetic energy per unit mass) was apparently published first in this journal by Betchov (1957). However, it was not until the work of Landau & Lifshitz (1959), and independently Betchov (1961), that a stochastic version of the Navier–Stokes equation was formulated that incorporates the fluctuation-dissipation relation. For a low-Mach incompressible fluid this equation for fluctuating velocity field $\tilde{\boldsymbol{u}}$ takes the form

$$\partial_t \tilde{\boldsymbol{u}} + (\tilde{\boldsymbol{u}} \boldsymbol{\cdot} \nabla)\tilde{\boldsymbol{u}} = -\nabla \tilde{p} + \nu \Delta \tilde{\boldsymbol{u}} + \nabla \boldsymbol{\cdot} \tilde{\boldsymbol{\tau}}, \quad \nabla \boldsymbol{\cdot} \tilde{\boldsymbol{u}} = 0, \tag{2.5}$$

where $\tilde{\boldsymbol{\tau}}$ is a Gaussian random stress field with mean zero and covariance

$$\langle \tilde{\tau}_{ij}(\boldsymbol{x}, t)\tilde{\tau}_{kl}(\boldsymbol{x}', t')\rangle = \frac{2\nu k_B T}{\rho}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)\delta^3(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t'), \tag{2.6}$$

which represents the thermal momentum transport. Here T is absolute temperature and k_B is Boltzmann's constant. Recently, evidence has emerged from numerical simulation of the stochastic equation (2.5) and related models which confirm Betchov's prediction (Bandak *et al.* 2022; Bell *et al.* 2022). I shall therefore critically examine also in this essay the question whether the deterministic equations (2.4) are sufficient to explain all of the experimental observations on turbulent energy dissipation.

Since it has been traditionally assumed that the Navier–Stokes equations are an adequate model of incompressible fluid turbulence, direct numerical simulation (DNS) of those equations has also been used as a tool to study turbulent energy dissipation. I am not aware of any systematic DNS study of the Reynolds-dependence of dissipation in 3-D wall-bounded flows (but see Nguyen van yen *et al.* (2018) for a 2-D flow). Instead, most of the attention has been focused on turbulence in a periodic box stirred by a large-scale body force, which has been considered a close analogue of the decaying turbulence behind a grid. A great advantage of numerical simulations is that the entire flow field is accessible and thus local viscous energy dissipation per mass

$$\varepsilon(\boldsymbol{x}, t) = 2\nu S_{ij}(\boldsymbol{x}, t) S_{ij}(\boldsymbol{x}, t), \tag{2.7}$$

with strain rate tensor $S_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$, can be calculated exactly for Navier–Stokes solutions, limited only by numerical resolution and machine precision. Corresponding to the prefactor A studied in grid turbulence is the dimensionless dissipation rate

$$D = \frac{\varepsilon}{u'^3/L} = \frac{2}{Re}\hat{S}_{ij}\hat{S}_{ij}, \tag{2.8}$$

where the hat denotes the strain tensor calculated from the dimensionless variables $\hat{\boldsymbol{u}} = \boldsymbol{u}/u'$, $\hat{\boldsymbol{x}} = \boldsymbol{x}/L$, $\hat{t} = t/(L/u')$. Numerical results of Sreenivasan (1998) and Kaneda *et al.* (2003) are consistent with the hypothesis that the volume- and time-averaged dimensionless dissipation rate $\bar{D}(Re)$ are asymptotically independent of Re. Of course, no empirical data could ever strictly verify this hypothesis, as it would be impossible to rule out an extremely slow decay.

3. Turbulence away from walls

Except for brief mention of pipe flow, Onsager (1949) restricted his attention to the ‘simplest type of turbulence’ which is the ‘nearly homogeneous and isotropic turbulence [produced] by means of a grid in a streaming gas’. I shall therefore consider this topic first in this essay, since most subsequent work has been done on this problem. I emphasise at the outset, however, that no statistical assumptions of homogeneity or isotropy are required for the results mentioned in the final paragraph of Onsager’s paper. The conclusions thus apply more generally to turbulence away from walls, such as in ubiquitous wake flows that are neither homogeneous nor isotropic. It is a bit ironic that Onsager suggested a completely deterministic approach to the analysis of turbulent flow in a paper entitled ‘Statistical Hydrodynamics’!

3.1. *Onsager’s result*

In the quote that began this essay, Onsager (1949) wrote that ‘it is possible to show that’, which was his characteristic catchphrase to assert that HE had shown something and to invite others to prove it as well. He never published his own calculations (which was also customary), but, as I shall discuss below, he had worked out the essential steps of a proof that energy will be conserved when the velocity field satisfies a Hölder–Lipschitz condition ‘for any order n greater than 1/3’. The first published result was given by Eyink (1994), who attempted to transform the brief argument using Fourier series sketched by Onsager (1949) into a rigorous proof. The idea was to show that the triply infinite series of Fourier coefficients which arises from the time-derivative of kinetic energy is absolutely summable on the assumption of Hölder regularity with exponent $> 1/3$. This ultimately requires a bound on the spectral energy flux $\Pi(k)$, similarly as in the work of Sulem & Frisch (1975). It was quickly obvious that the $1/3$ claim would hold if the spectral flux were dominated by ‘local’ wavevector triads with all three wavevectors of magnitudes roughly k. This approach is plagued by difficulties, however, not least because there is no necessary and sufficient condition for Hölder continuity in terms of absolute Fourier coefficients.

By 1992, I finally managed to work out a proof of Onsager's $1/3$ claim, but invoking a condition on absolute Fourier coefficients stronger than Hölder continuity.

At the same time, it appeared from the analysis that most contributions to the energy flux could be bounded even with a weaker assumption on the velocity of 'Besov regularity', which corresponds to Hölder regularity in space-mean sense, closely related to standard structure functions (Eyink 1995*a*). After developing my first proof, I discussed the problem with Weinan E at the Institute for Advanced Study in 1992 and, the following year E and his collaborators, Peter Constantin and Edriss Titi, found an extremely simple proof not only of Onsager's original claim but also of the natural Besov result (Constantin, Weinan & Titi 1994). After being informed privately of this development by E, I realised that their method of proof was closely related to LES modelling and renormalisation group, and that it could be simply explained in those terms (Eyink 1995*b*). It is this argument that I present first in this section, before considering the alternative technical derivation given by Onsager himself. Note that I refer to the energy conservation statement, however, as 'Onsager's result' rather than by the term 'Onsager's conjecture' used by Constantin *et al.* (1994), which was at a time when Onsager's own mathematical argument was unknown. I prefer to use the term 'Onsager's conjecture' for the deeper statement that inviscid energy dissipation is possible when the Euler velocity field has Hölder regularity $\leq 1/3$, which was only proved much later.

3.1.1. *Ideal turbulence, renormalisation group and large-eddy simulation*

The most basic conclusion that can be inferred from the empirical observations of *Re*-independence in (2.1) or (2.8) is that the non-dimensionalised velocity gradients, if interpreted as ordinary classical derivatives, must diverge as $|\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}| \to \infty$ when $Re \to \infty$. This was the starting point of Onsager's own line of argument, who began the short abstract in Onsager (1945*a*) with the observation: 'The dissipation of energy by turbulence is regarded as primarily a "violet catastrophe".' In the language of modern field theory, turbulence exhibits ultraviolet (UV) divergences of velocity gradients in the limit $Re \to \infty$. As a consequence, the equations of motion (2.4) (or alternatively (2.5)) cannot remain valid in a naïve sense in the infinite-*Re* limit. Just as in field theory, to obtain a dynamical description of turbulence as $Re \to \infty$, one must somehow regularise these UV divergences. The simple but effective approach of Constantin *et al.* (1994) was to perform what is called 'low-pass filtering' in engineering, 'spatial coarse-graining' in physics and 'mollifying' in mathematics. This method involves the use of a smooth kernel G for space dimension d that satisfies:

(i) $G(\boldsymbol{r}) \geq 0$;
(ii) $G(\boldsymbol{r}) \to 0$ rapidly for $|\boldsymbol{r}| \to \infty$;
(iii) $\int \mathrm{d}^d r\, G(\boldsymbol{r}) = 1$.

It is understood that G is centred at $\boldsymbol{r} = \boldsymbol{0}$, $\int \mathrm{d}^d r\, \boldsymbol{r}\, G(\boldsymbol{r}) = \boldsymbol{0}$ and that $\int \mathrm{d}^d r |\boldsymbol{r}|^2 G(\boldsymbol{r}) \approx 1$. We can then set

$$G_\ell(\boldsymbol{r}) \equiv \ell^{-d} G(\boldsymbol{r}/\ell) \tag{3.1}$$

so that all of the above properties hold, except that now $\int \mathrm{d}^d r |\boldsymbol{r}|^2 G_\ell(\boldsymbol{r}) \approx \ell^2$, with $\ell > 0$ the regularisation length scale. Finally, one defines a coarse-grained velocity at length scale ℓ by the following formula involving a convolution integral:

$$\bar{\boldsymbol{u}}_\ell(\boldsymbol{x},t) = (G_\ell * \boldsymbol{u})(\boldsymbol{x},t) = \int \mathrm{d}^d r\, G_\ell(\boldsymbol{r})\, \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t), \tag{3.2}$$

spatially averaging over the eddies of size $< \ell$.

This coarse-grained field is roughly analogous to a 'block spin' in critical phenomena and field theory. An identical operation is also employed in the 'filtering approach' to turbulence advocated by Leonard (1974) and Germano (1992), but with a different motivation than regularisation of divergences. For Onsager's theory, the important point is that the coarse-graining operation (3.2) regularises gradients, so that $\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell$ remains finite as $\nu \to 0$ for any fixed length $\ell > 0$. This may be shown using the simple integration-by-parts identity

$$\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell(\boldsymbol{x},t) = -\frac{1}{\ell}\int \mathrm{d}^d r (\boldsymbol{\nabla} G)_\ell(\boldsymbol{r})\, \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t), \tag{3.3}$$

which by Cauchy–Schwartz inequality yields the bound $|\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell(\boldsymbol{x},t)| \le (1/\ell) \sqrt{C_\ell \int \mathrm{d}^d r |\boldsymbol{u}(\boldsymbol{r},t)|^2}$ with constant $C_\ell = \int \mathrm{d}^d r |(\boldsymbol{\nabla} G)_\ell(\boldsymbol{r})|^2$. Thus, the coarse-grained gradient is bounded as long as the total kinetic energy remains finite as $\nu \to 0$ (which is necessarily true for freely decaying turbulence with no stirring). The price of this regularisation, as for quantum field-theory divergences, is that a new, arbitrary regularisation scale ℓ has been introduced.

Because divergences have been eliminated, one may now seek a dynamical description in terms of the coarse-grained field defined in (3.2). The equation which is satisfied is easily found to be

$$\partial_t \bar{\boldsymbol{u}}_\ell + \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{(\boldsymbol{u}\boldsymbol{u})}_\ell = -\boldsymbol{\nabla}\bar{p}_\ell + \nu\Delta\bar{\boldsymbol{u}}_\ell, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell = 0, \tag{3.4}$$

because the coarse-graining operation commutes with all space- and time-derivatives, and one may inquire about its limit as $Re \to \infty$. For this purpose, one should non-dimensionalise the above equation and introduce hats everywhere, but doing so simply replaces the physical viscosity with the dimensionless viscosity $\hat{\nu} = 1/Re$. Thus, as customary in the mathematical literature, one may omit the hats on non-dimensionalised variables and consider the limit $Re \to \infty$ as a 'zero-viscosity limit' limit $\nu \to 0$. I shall do so hereafter, when this causes no confusion. Because the quantity $\Delta\bar{\boldsymbol{u}}_\ell$ in (3.4) remains bounded by a similar estimate as (3.3), one can easily show rigorously that $\nu\Delta\bar{\boldsymbol{u}}_\ell \to 0$ as $\nu \to 0$ for fixed ℓ. If one indexes the solutions $\boldsymbol{u}^\nu$ of the Navier–Stokes equation (2.4) by viscosity ν, then the above results suggest that a limiting field $\boldsymbol{u} = \lim_{\nu\to 0}\boldsymbol{u}^\nu$ will satisfy the equation

$$\partial_t \bar{\boldsymbol{u}}_\ell + \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{(\boldsymbol{u}\boldsymbol{u})}_\ell = -\boldsymbol{\nabla}\bar{p}_\ell, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell = 0. \tag{3.5}$$

Since one is dealing with velocity fields in function spaces, a careful statement of this hypothesis requires a suitable notion of convergence $\boldsymbol{u}^\nu \to \boldsymbol{u}$. It is not hard to show that strong L^2 convergence suffices, which is the condition that $\lim_{\nu\to 0}\|\boldsymbol{u}^\nu - \boldsymbol{u}\|_2 = 0$ where

$$\|\boldsymbol{u}\|_p := \left[\frac{1}{T}\int_0^T \mathrm{d}t \frac{1}{|\varOmega|}\int_\varOmega \mathrm{d}^d x |\boldsymbol{u}(\boldsymbol{x},t)|^p\right]^{1/p}, \quad p \ge 1. \tag{3.6}$$

As I shall discuss later (see § 3.2.3), such strong L^2-convergence is not guaranteed *a priori*, although it is in fact implied by some relatively mild assumptions on the energy spectrum which can be tested empirically. For the moment, I shall simply assume such convergence so that (3.5) directly follows, but I return to this issue later.

An important observation is that the validity of the coarse-grained equations (3.5) for all lengths $\ell > 0$ is equivalent to the condition in mathematical PDE theory that the velocity $\boldsymbol{u}$ is a weak solution of the incompressible Euler equations. Readers may be more familiar with the standard formulation of weak solutions by smearing the dynamical equations with smooth space–time test functions, so that the equations hold in the sense of distributions or generalised functions. A simple proof that 'coarse-grained solutions', satisfying (3.5) for all $\ell > 0$, are equivalent to standard weak solutions is given in § 2 of Drivas & Eyink (2018). Furthermore, these notions of weak solution are equivalent to what Onsager (1949) meant when he wrote that 'the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description; for example, the formulation (15) in terms of FOURIER series will do'. Indeed, the incompressible Euler equations written as an infinite-dimensional set of ordinary differential equations (ODEs) for the Fourier coefficients of the velocity field yield also standard weak solutions, as discussed by Eyink (1994) and De Lellis & Székelyhidi (2013). All of these equivalent formulations of weak solutions make sense even when spatial derivatives of velocity no longer exist in the classical sense, but only in the sense of distributions.

It may appear odd to some readers that (3.5) can be interpreted as 'Euler equations', since the filtered equations (3.5) are generally regarded as unclosed whereas the Euler equations are closed PDEs. It is indeed true that (3.5) are not closed equations for the coarse-grained velocity $\bar{\boldsymbol{u}}_\ell$ itself. This fact is usually underlined by introducing the turbulent/subscale stress

$$\boldsymbol{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u}) := \overline{(\boldsymbol{u}\boldsymbol{u})}_\ell - \bar{\boldsymbol{u}}_\ell \bar{\boldsymbol{u}}_\ell, \tag{3.7}$$

so that (3.5) can be rewritten as

$$\partial_t \bar{\boldsymbol{u}}_\ell + \boldsymbol{\nabla} \boldsymbol{\cdot} [\bar{\boldsymbol{u}}_\ell \bar{\boldsymbol{u}}_\ell + \boldsymbol{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u})] = -\boldsymbol{\nabla} \bar{p}_\ell, \quad \boldsymbol{\nabla} \boldsymbol{\cdot} \bar{\boldsymbol{u}}_\ell = 0 \tag{3.8}$$

with the non-closed term $\boldsymbol{\tau}_\ell$ distinguished. However, the coarse-grained velocity $\bar{\boldsymbol{u}}_\ell = G_\ell * \boldsymbol{u}$ and the subscale stress $\boldsymbol{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u})$ are both explicit functions of the fine-grained velocity field $\boldsymbol{u}$, as emphasised by my notation, and thus (3.8) are explicit conditions on that field. I may note that it is quite standard to consider that the large-scales $> \ell$ in a turbulent flow must satisfy the Euler fluid equations. For example, Landau & Lifshitz (1959, § 31), wrote that: 'We therefore conclude that, for the large eddies which are the basis of any turbulent flow, the viscosity is unimportant and may be equated to zero, so that the motion of these eddies obeys Euler's equation.' What needs to be stressed is that the proper interpretation of such commonplace remarks is not that the coarse-grained velocity $\bar{\boldsymbol{u}}_\ell$ satisfies the Euler equations in the usual naïve sense, but instead that $\boldsymbol{u}$ is described by a weak Euler solution at those scales, in the sense that (3.8) holds. Although this point is elementary, it is frequently misunderstood and the source of common errors.

There is another very important conceptual point which must be emphasised about the coarse-grained equations (3.8). Some critics have argued that these equations are unphysical and not appropriate as the basis for a fundamental theory of turbulence because both the length scale ℓ and the filter kernel G are arbitrary. For example, Tsinober (2009) wrote 'The filter decomposition is formally more general than the Reynolds decomposition. However, the former is one among many decompositions, so to say, of a technical nature' (p. 379) and also 'After all Nature may and likely does not know about *our* decompositions' (p. 114). These are very shrewd remarks. In fact, I agree completely with these concerns of Tsinober (2009) that Nature should not care about such

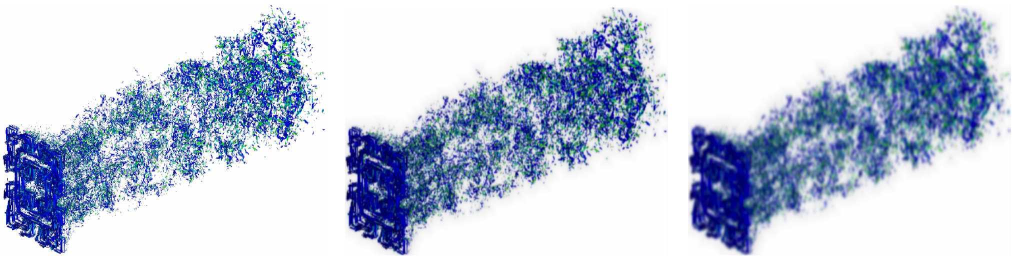

Figure 3. Schematic illustration of spatial coarse-graining for turbulent flow past a grid: (*a*) fine-grained flow resolved to the dissipation scale η; (*b*) flow coarse-grained at a length scale $\ell > \eta$; (*c*) flow coarse-grained further at scale $\ell' > \ell$. At each stage, eddies smaller than the coarsening scale are unresolved and ignored. Panel (*a*) is reproduced from Laizet *et al.* (2012) and panels (*b*,*c*) modified by image filtering, with permission from Elsevier.

arbitrary choices. Interestingly, this same problem has arisen in another area of physics, relativistic quantum field theory, which is also plagued with similar UV divergences. In that case also, arbitrary regularisations are required to eliminate the divergences and these introduce a new arbitrary length scale, equivalent to ℓ or, more commonly, a related energy scale $\mu = c\hbar/\ell$, with c the speed of light and $\hbar$ Planck's constant. In the renormalised field theory, fundamental parameters of the theory such as coupling constants $\lambda(\mu)$ become dependent on this arbitrary scale, in the same manner that $\boldsymbol{\tau}_\ell$ becomes dependent upon ℓ. Elementary particle physicists in the 1950s were also worried that predictions of the theory should not depend upon such arbitrary choices and the concept of renormalisation group (RG) invariance arose as the commonsense demand that any observable consequence of the theory should be independent of μ. For a very clear discussion, see Gross (1976, § 4.1). What makes RG invariance interesting and important is that non-trivial consequences can be deduced precisely by varying μ and demanding that physical results be μ-independent. What we shall see is that Onsager anticipated such arguments in the 1940s and that his 1/3 Hölder claim for turbulent velocities is an exact non-perturbative consequence of such RG invariance.

To underscore the nature of the argument, I emphasise that coarse-graining is a purely passive operation – 'removing one's spectacles' – which changes no physical process. The effect of the coarse-graining operation in (3.2) is illustrated by figure 3, which shows the velocity field observed at successively coarser spatial resolutions ℓ. Although the dynamics of the velocity field resolved at the Kolmogorov scale $\ell = \eta$ is described rather well by the Navier–Stokes equation (2.4), the coarse-grained velocity field $\bar{\boldsymbol{u}}_\ell$ at scales $\ell \gg \eta$ is described by the highly non-Newtonian equation (3.8). While the description of the dynamics changes with resolution ℓ, nevertheless objective facts cannot depend upon the 'eyesight' of the observer. Thus, an energy dissipation rate which is non-vanishing in the limit as $\nu \to 0$ implies that kinetic energy will decrease over a fixed interval of time $[0, t]$, as observed by experiment, and this fact cannot depend upon the arbitrary scale ℓ. An observation at resolution ℓ can only miss some kinetic energy of smaller eddies, since by convexity

$$\tfrac{1}{2}|\bar{\boldsymbol{u}}_\ell(\boldsymbol{x}, t)|^2 \le \tfrac{1}{2}\overline{\left(|\boldsymbol{u}(\boldsymbol{x}, t)|^2\right)}_\ell, \tag{3.9}$$

and it then follows that $E_\ell(t) := (1/2)\int \mathrm{d}^d x |\bar{\boldsymbol{u}}_\ell(\boldsymbol{x}, t)|^2 \le (1/2)\int \mathrm{d}^d x |\boldsymbol{u}(\boldsymbol{x}, t)|^2 = E(t)$. If kinetic energy continues to decay even in the limit as $\nu \to 0$, then such persistent energy decay must also be seen by the 'myopic' observer who observes fluid features only at space-resolution ℓ. As I now show, however, the persistent energy decay observed at the fixed length scale ℓ with $\nu \to 0$ is not due to molecular viscosity acting directly at those scales.

The local kinetic energy balance at length scales ℓ in the inertial-range obtained for the limit $\nu \to 0$ is calculated straightforwardly from (3.8) to be

$$\partial_t \left(\tfrac{1}{2}|\bar{\boldsymbol{u}}_\ell|^2\right) + \boldsymbol{\nabla} \boldsymbol{\cdot} \left[\left(\tfrac{1}{2}|\bar{\boldsymbol{u}}_\ell|^2 + \bar{p}_\ell\right)\bar{\boldsymbol{u}}_\ell + \boldsymbol{\tau}_\ell \boldsymbol{\cdot} \bar{\boldsymbol{u}}_\ell\right] = -\varPi_\ell, \tag{3.10}$$

where the quantity on the right-hand side of the equation is given (with $\boldsymbol{A} : \boldsymbol{B} = \sum_{ij} A_{ij} B_{ij}$) by

$$\varPi_\ell(\boldsymbol{x}, t) = -\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell(\boldsymbol{x}, t) : \boldsymbol{\tau}_\ell(\boldsymbol{x}, t), \tag{3.11}$$

and represents the 'deformation work' of the large-scale strain acting against small-scale stress, or the 'energy flux' from resolved scales $> \ell$ to unresolved scales $< \ell$. The mechanism of loss of energy by the inertial-range eddies is thus 'energy cascade', a term first used in this connection by Onsager (1945*a*). A key observation is that the stress-tensor $\boldsymbol{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u})$ may be rewritten in terms of velocity-increments $\delta\boldsymbol{u}(\boldsymbol{r}; \boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t) - \boldsymbol{u}(\boldsymbol{x}, t)$, as

$$\boldsymbol{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u}) = \langle \delta\boldsymbol{u}\delta\boldsymbol{u}\rangle_\ell - \langle\delta\boldsymbol{u}\rangle_\ell \langle\delta\boldsymbol{u}\rangle_\ell, \tag{3.12}$$

where $\langle f\rangle_\ell(\boldsymbol{x}, t) := \int \mathrm{d}^d r\, G_\ell(r) f(\boldsymbol{r}; \boldsymbol{x}, t)$. This formula was originally obtained by Constantin *et al.* (1994) in a slightly different form, and as above by Eyink (1995*b*) as a physical re-interpretation of their result. Equation (3.12) is easy to verify by direct calculation, but it can be simply understood as due to the invariance of the second-order cumulant $\boldsymbol{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u})$ to shifts of $\boldsymbol{u}$ by vectors that are 'non-random' with respect to the average $\langle\cdot\rangle_\ell$ over displacements $\boldsymbol{r}$, i.e. that are independent of $\boldsymbol{r}$. This allows $\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t)$ in the definition (3.7) of $\boldsymbol{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u})$ to be replaced with $\delta\boldsymbol{u}(\boldsymbol{r}; \boldsymbol{x}, t)$, yielding (3.12). Similarly, one may rewrite (3.3) for coarse-grained velocity-gradients in terms of increments as

$$\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell(\boldsymbol{x}, t) = -\frac{1}{\ell}\int \mathrm{d}^d r (\boldsymbol{\nabla} G)_\ell(\boldsymbol{r}) \delta\boldsymbol{u}(\boldsymbol{r}; \boldsymbol{x}, t), \tag{3.13}$$

using the fact that $\int \mathrm{d}^d r (\boldsymbol{\nabla} G)_\ell(\boldsymbol{r}) = \boldsymbol{0}$.

As an immediate application of these formulae, one can rederive the prediction of Onsager (1949) that Hölder singularities $h \leq 1/3$ are required in the velocity field for energy dissipation to persist in the limit $\nu \to 0$. Indeed, assuming for some constant $C > 0$ that

$$|\delta\boldsymbol{u}(\boldsymbol{r}; \boldsymbol{x}, t)| \leq C|\boldsymbol{r}|^h, \tag{3.14}$$

then it is straightforward to show using (3.11), (3.12) and (3.13) that

$$\varPi_\ell(\boldsymbol{x}, t) = O\left(\ell^{3h-1}\right). \tag{3.15}$$

As is clear from (3.10), persistent energy decay at resolution length ℓ can only occur if $\int \mathrm{d}^d x\, \varPi_\ell(\boldsymbol{x}, t) > 0$ as $\nu \to 0$. However, the resolution scale ℓ is completely arbitrary. For any fixed ℓ, one can take ν sufficiently small so that the 'ideal equations' (3.5) or (3.8) hold to any desired accuracy at that scale, and then sequentially further decreasing ℓ, one can correspondingly decrease ν. If the Hölder regularity (3.14) held for all $(\boldsymbol{x}, t)$ with $h > 1/3$, then clearly by (3.15), it would follow that $\int \mathrm{d}^d x \varPi_\ell(\boldsymbol{x}, t) \to 0$ as $\ell \to 0$. This is a contradiction, since the rate of decay of energy must be independent of the arbitrary length scale of resolution ℓ as $\ell \rightarrow 0$. Just as done by Onsager (1949), we thus infer that there must appear Hölder singularities $h \leq 1/3$ in the limit as $\nu \rightarrow 0$, or $Re \rightarrow \infty$. As I shall discuss in § 3.1.2, Onsager had given a very similar argument to that above in his unpublished work.

As an aside, I remark that the RG character of this argument can be made even more explicit by a somewhat different technical approach of Johnson (2020, 2022), where it is observed for a Gaussian kernel $G(\boldsymbol{r}) = \exp(-r^2/2)/(2\pi)^{3/2}$ that $(\partial/\partial\ell^2)\bar{\boldsymbol{u}}_\ell = \frac{1}{2}\Delta\bar{\boldsymbol{u}}_\ell$ and thus

$$\frac{\partial}{\partial\ell^2}\boldsymbol{\tau}_\ell = \frac{1}{2}\Delta\boldsymbol{\tau}_\ell + (\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell)^\top\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell. \tag{3.16}$$

In this approach, the subscale stress that 'renormalises' the bare Navier–Stokes dynamics can in fact be obtained by solving an equation that evolves $\boldsymbol{\tau}_\ell$ in the scale parameter ℓ, analogous to the RG flow equations in high-energy physics and critical phenomena (Wilson 1975; Gross 1976). Solving (3.16) with the initial data $\boldsymbol{\tau}_\ell|_{\ell=0} = \boldsymbol{0}$, Johnson (2020) obtained

$$\boldsymbol{\tau}_\ell = \int_0^{\ell^2} \mathrm{d}\lambda^2\, \overline{((\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\lambda)^\top\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\lambda)}_{\sqrt{\ell^2-\lambda^2}}. \tag{3.17}$$

Assuming the Hölder condition (3.14) then gives $\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\lambda = O(\lambda^{h-1})$ as before and, from the identity (3.17), $\boldsymbol{\tau}_\ell = O(\ell^{2h})$. Thus, the same bound (3.15) is deduced once again and this implies the 1/3 Hölder claim of Onsager. See also Isett & Oh (2016) for a similar approach.

The paper of Constantin *et al.* (1994) derived in fact a stronger result, by replacing the Hölder regularity originally assumed by Onsager (1949) with a weaker assumption of Besov regularity. As discussed by Eyink (1995*b*), Besov regularity can be understood in terms of deterministic pth-order 'velocity-structure functions' defined for absolute velocity increments and for space-averages over the flow domain Ω :

$$S_p(\boldsymbol{r}) = \frac{1}{|\Omega|}\int_\Omega \mathrm{d}^d x |\delta\boldsymbol{u}(\boldsymbol{r};\boldsymbol{x})|^p. \tag{3.18}$$

Although power laws in the separation distance $|\boldsymbol{r}|$ are generally observed empirically for turbulent fluid flows, Besov regularity requires only an upper bound

$$S_p(\boldsymbol{r}) \leq C_p|\boldsymbol{r}|^{\zeta_p}, \quad |\boldsymbol{r}| \leq 1. \tag{3.19}$$

If this inequality holds along with the modest assumption of finite pth-order moments

$$\frac{1}{|\Omega|}\int_\Omega \mathrm{d}^d x |\boldsymbol{u}(\boldsymbol{x})|^p < \infty, \tag{3.20}$$

then the velocity field $\boldsymbol{u}$ is said to belong to the Besov space $B_p^{\sigma,\infty}(\Omega)$ with $\sigma = \sigma_p := \zeta_p/p$. Note that the optimal constant $C_p = \sup_{|\boldsymbol{r}|\leq 1} S_p(\boldsymbol{r})/|\boldsymbol{r}|^{\zeta_p}$ in the inequality (3.19) is related to the so-called 'Besov semi-norm' in the mathematical literature by the formula $|||\boldsymbol{u}|||_{B_p^{\sigma,q}} = C_p^{1/p}$, so that the abstract semi-norm is given by the prefactor in expected power scaling laws (Drivas & Eyink 2019). Note further that the Hölder condition (3.14) is equivalent to the Besov condition (3.19) for $p = \infty$, with $\sigma_\infty = h$.

The result of Constantin *et al.* (1994) was that energy dissipation non-vanishing in the limit as $\nu \rightarrow 0$ requires $\zeta_p \leq p/3$ for all $p \geq 3$ or, equivalently, $\sigma_p \leq 1/3$ for $p \geq 3$.

The argument generalises that given previously for $p = \infty$ and uses the simple bound from the Hölder inequality valid for any $p \geq 3$,

$$\left| \frac{1}{|\Omega|} \int_{\Omega} \mathrm{d}^d x \, \Pi_\ell(\boldsymbol{x}, t) \right| \leq \|\Pi_\ell\|_{p/3} \leq \|\boldsymbol{\nabla} \bar{\boldsymbol{u}}_\ell\|_p \|\boldsymbol{\tau}_\ell\|_{p/2} = O\left(\ell^{3\sigma_p - 1}\right), \tag{3.21}$$

since $\|\boldsymbol{\nabla} \bar{\boldsymbol{u}}_\ell\|_p = O(\ell^{\sigma_p - 1})$ and $\|\boldsymbol{\tau}_\ell\|_{p/2} = O(\ell^{2\sigma_p})$. For mathematical details, see Constantin *et al.* (1994), Eyink (1995*b*) or Eyink (2007). Just as before, if the Besov regularity (3.19) held with $\sigma_p > 1/3$, then it would follow that $\int \mathrm{d}^d x, \Pi_\ell(\boldsymbol{x}, t) \to 0$ as $\ell \to 0$. This would yield the same contradiction as previously, since the rate of decay of energy must be independent of the arbitrary length scale of resolution ℓ as $\ell \to 0$. Generalising the statement of Onsager (1949), one can infer that in fact $\sigma_p \leq 1/3$ for all $p \geq 3$ in the limit as $\nu \to 0$.

Onsager's ideas anticipate the 'multifractal model' proposed by Parisi & Frisch (1985) for the turbulent velocity field; see also Frisch (1995) for a comprehensive exposition. In fact, it is worth recalling the first sentence of Parisi & Frisch (1985): 'A simple way of explaining power law structure function is to invoke singularities of the Euler equations considered as limit of the Navier–Stokes equations as the viscosity tends to zero.' Parisi & Frisch (1985) proposed that the turbulent velocity field possesses a spectrum of Hölder exponents $h \in [h_{min}, h_{max}]$ with each exponent h occurring on a set $S(h) \subset \Omega$ with Hausdorff dimension $D(h)$. The velocity scaling exponents are then obtained by the formula

$$\zeta_p = \inf_h \{hp + (3 - D(h))\}, \tag{3.22}$$

thus accounting for the observed deviations from the prediction $\zeta_p = p/3$ of Kolmogorov (1941*a*,*b*). The velocity scaling exponent relevant in this context can be understood as the 'maximal Besov exponent' $\sigma_p = \zeta_p/p$ for any $p \geq 0$ with

$$\zeta_p := \liminf_{|\boldsymbol{r}| \to 0} \frac{\log S_p(\boldsymbol{r})}{\log |\boldsymbol{r}|}, \tag{3.23}$$

a concept which is meaningful even without any power-law scaling (Eyink 1995*a*). As we shall see later, Onsager had arrived at similar conclusions already by 1945 (but without the modern concept of fractals). Note that $h_p = \mathrm{d}\zeta_p/\mathrm{d}p$ is the Hölder exponent h that yields the infimum in (3.22) for each $p \geq 0$. It then follows from (3.22) or even directly from the concavity of ζ_p in p (Frisch 1995; Eyink 2007) that $h_p \leq \sigma_p$ for all $p \geq 0$. Thus, the theorem of Constantin *et al.* (1994) implies also that $h_p \leq 1/3$ for $p \geq 3$ and the original result of Onsager (1949) is equivalent to the statement that $h_\infty = h_{\min} \leq 1/3$. It is important to emphasise that these are predictive statements which have received subsequent support from empirical determination of the multifractal dimension spectrum $D(h)$ both from numerical simulations (Kestener & Arneodo 2004) and from hot-wire experiments (Lashermes *et al.* 2008). As I shall discuss later, the 'ideal turbulence' theory makes connection also with the alternative multifractal theory for the energy dissipation rate (Kolmogorov 1962; Mandelbrot 1974, 1989; Meneveau & Sreenivasan 1991).

Another successful prediction made by Onsager (1945*a*, 1949) concerned the locality of the energy cascade, which is the statement that the nonlinear flux Π_ℓ defined in (3.11) as a cubic function of the velocity field $\boldsymbol{u}$ is determined predominantly by velocity modes or 'eddies' of scale near ℓ. In the words of Onsager (1945*a*): 'The modulation of a given Fourier component of the motion is mostly due to those others which belong to wavenumbers of comparable magnitude.' In his 1945 letter to von Kármán and Lin (reproduced in Eyink & Sreenivasan 2006, appendix B), Onsager stated more precisely that 'With a hypothesis slightly stronger than (14) the motion which belongs to wave-numbers of the same order of magnitude as $\underline{k}$ itself will furnish the greater part of the effective rate of shear', where the mentioned condition (14) asserts that $\boldsymbol{u}$ is square-integrable but that $\nabla \boldsymbol{u}$ is not. A condition of exactly this type is the Hölder condition (3.14), which may be shown for any $0 < h < 1$ to imply locality. In fact, scale locality of energy cascade holds assuming only the pth-order Besov condition (3.19) for any $0 < \sigma_p < 1$ (Eyink 2005; Cheskidov *et al.* 2008). These predictions have been confirmed in a number of numerical studies, for example, Domaradzki & Carati (2007), Aluie & Eyink (2009) and Cardesa *et al.* (2015).

It is crucial that these locality properties hold instantaneously and deterministically for individual flow realisations (Kraichnan 1974). The cited mathematical analyses and numerical simulations verify that averaging over velocity realisations improves the degree of locality but also that averaging is not necessary for local-in-scale interactions to dominate in energy transfer. Locality for individual realisations is necessary for another important feature of turbulence, the universality of small-scale statistics. The usual argument for such universal statistics has to do with the scale invariance of the Euler equations and the scale-locality of the energy transfer by which small scales are excited. It is by the latter property that, in the words of Onsager (1949), 'we are led to expect a cascade such that the wavenumbers increase typically in a geometric series, by a factor of the order 2 per step'. If each step in the cascade is chaotic and also of comparable nature to the previous steps, because of scale-invariance, then it is reasonable to expect that the details of large-scale flow features will be lost and superseded by motions intrinsic to the local Euler dynamics. It is further quite obvious that scale-locality for energy transfer only in the mean sense would be inadequate for such universality, because then long-range communication between large and small scales could exist instantaneously which is cancelled only by averaging over time or initial data.

This universality is one of the many features that connects Onsager's 'ideal turbulence' with the LES method of modelling turbulence. In LES, the large-scale motions (large eddies) of turbulent flow are computed directly and only small-scale (sub-grid scale; SGS) motions are modelled, resulting in a significant reduction in computational cost compared with DNS (Piomelli 1999; Meneveau & Katz 2000). The justification for the LES scheme is that the turbulent small-scales are expected to be statistically similar for every flow and thus may be universally modelled, while the non-universal large scales are explicitly computed at far lower cost. LES originated in the pioneering work of Smagorinsky (1963) and Lilly (1967), and it has developed historically with little connection to Onsager's theory of 'ideal turbulence', although the two are quite intimately related. At the most superficial level, the analysis used by Constantin *et al.* (1994) to prove Onsager's 1/3 result is the same as the filtering approach advocated by Leonard (1974) and Germano (1992), and widely used in LES. Thus, the filtered equations (3.8) used to derive Onsager's theorem are the same as those employed in LES. However, filtering/coarse-graining is just one specific form of regularisation and other schemes are possible (e.g. Onsager used also Fourier series as a regulariser). The specific regularisation by spatial filtering is not intrinsic to either method. More important is that both approaches aim to describe individual flow realisations, unlike Reynolds-averaged Navier–Stokes (RANS) modelling which resigns itself to calculation of time- or ensemble-averaged velocities only. Furthermore, Onsager's theory justifies the truism that an LES model should run even with molecular viscosity set to zero and thus should describe turbulent dissipation at infinite Reynolds number.

Figure 4. Mixing of cream in a mug of coffee, moderately turbulent from mild stirring. Reproduced on license from Shutterstock.

In concluding this section, I want to emphasise the conceptual and practical importance of a deterministic description of turbulent flow for individual flow realisations. In much of the scientific literature on fluid turbulence, it is often reflexively assumed that any mathematical description must be statistical in nature. However, this instinctive reaction does not permit one to discuss even such a commonplace flow as your morning cup of coffee (see figure 4). This flow has a decent Reynolds number that may be estimated of order $Re = UD/\nu = 3 \times 10^4$ (assuming stirring velocity $U = 20$ cm s^{-1}, cup diameter $D = 6$ cm and kinematic viscosity $\nu = 0.4$ mm^2 s^{-1}) and is thus modestly turbulent. However, there is no ensemble in sight! The coffee is stirred only a few times to mix the cream and thus the flow is decaying, so that one cannot average over time. Also, the size of the largest eddies is of the order of the cup diameter and thus one cannot average over many integral lengths of the flow. Nevertheless, the fluid flow in the cup is turbulent each time you stir it and one must be able to describe the specific, individual flow. This need is common in many areas of science, such as geophysics and astrophysics, where some specific hurricane or some specific supernova must be understood. These remarks are not intended to deny the intrinsic stochasticity of turbulent flow. It is also true that when you stir your morning cup of coffee, each time, you will see a different pattern of the cream, no matter how carefully you try to repeat your actions each day. However, in many discussions of turbulence, the applications of probabilistic methods are entirely gratuitous. It is only by eliminating the superficial and unnecessary resorts to a statistical description that one can uncover where probability theory is truly essential.

3.1.2. *Local deterministic 4/5th law and dissipative anomaly*

One of the landmark results of the statistical approach to turbulence is the celebrated 4/5th law derived by Kolmogorov (1941*a*), who invoked statistical hypotheses of local homogeneity and isotropy of the flow. In a remarkable work developing Onsager's ideas, Duchon & Robert (2000) have shown that the 4/5th law holds not only deterministically for individual flow realisations, not requiring any statistical hypotheses, but also that it holds in a much stronger space–time local form. As we shall see, these developments were also foreshadowed by Onsager's own unpublished work. Here I shall sketch the essential ideas in the work of Duchon & Robert (2000) and discuss their various ramifications. For a much more detailed explanation, see the online course notes of Eyink (2007, § III.C).

The analysis of Duchon & Robert (2000) again attempts to understand the zeroth-law of turbulence or the 'inertial-dissipation' of kinetic energy. The basic problem remains that

UV divergences must appear in the inviscid limit, so that the fluid equations can no longer be understood as PDEs in the naïve sense. The starting point of Duchon & Robert (2000) is a balance equation for a point-split kinetic energy density $\frac{1}{2}\boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)$:

$$\begin{aligned}\partial_t\left(\tfrac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{u}'\right)+\boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\tfrac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{u}'\right)\boldsymbol{u}+\tfrac{1}{2}(p\boldsymbol{u}'+p'\boldsymbol{u})+\tfrac{1}{4}|\boldsymbol{u}'|^2\delta\boldsymbol{u}-\nu\boldsymbol{\nabla}\left(\tfrac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{u}'\right)\right]\\ =\tfrac{1}{4}\boldsymbol{\nabla}_{\boldsymbol{r}}\boldsymbol{\cdot}[\delta\boldsymbol{u}|\delta\boldsymbol{u}|^2]-\nu\boldsymbol{\nabla}\boldsymbol{u}:\boldsymbol{\nabla}\boldsymbol{u}'+\tfrac{1}{2}(\boldsymbol{f}\boldsymbol{\cdot}\boldsymbol{u}'+\boldsymbol{f}'\boldsymbol{\cdot}\boldsymbol{u}),\end{aligned} \tag{3.24}$$

where I have introduced the abbreviated notation

$$\left.\begin{aligned}\boldsymbol{u}&=\boldsymbol{u}(\boldsymbol{x},t),\quad p=p(\boldsymbol{x},t),\quad \boldsymbol{f}=\boldsymbol{f}(\boldsymbol{x},t),\\ \boldsymbol{u}'&=\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t),\quad p'=p(\boldsymbol{x}+\boldsymbol{r},t),\quad \boldsymbol{f}'=\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{r},t),\\ \delta\boldsymbol{u}&=\boldsymbol{u}'-\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t),\end{aligned}\right\} \tag{3.25}$$

and I have permitted a body force $\boldsymbol{f}$ in the Navier–Stokes equation, for greater generality. Similar exact equations were introduced somewhat later by Hill (2001, 2002) as deterministic versions of the Kármán–Howarth–Monin relations. However, the balance equation (3.24) does not yet eliminate UV divergences by point-splitting alone, which requires further multiplying through by a filter kernel $G_\ell(\boldsymbol{r})$ and integrating over $\boldsymbol{r}$. This yields a corresponding balance equation for the regularised energy density $\frac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell$:

$$\begin{aligned}&\partial_t\left(\frac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell\right)+\boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell\right)\boldsymbol{u}+\frac{1}{2}(p\bar{\boldsymbol{u}}_\ell+\bar{p}_\ell\boldsymbol{u})+\frac{1}{4}\overline{(|\boldsymbol{u}|^2\boldsymbol{u})_\ell}\right.\\ &\qquad\left.-\frac{1}{4}\overline{(|\boldsymbol{u}|^2)_\ell}\boldsymbol{u}-\nu\boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell\right)\right]\\ &=-\frac{1}{4\ell}\int\mathrm{d}^d\boldsymbol{r}(\boldsymbol{\nabla}G)_\ell(\boldsymbol{r})\boldsymbol{\cdot}\delta\boldsymbol{u}(\boldsymbol{r})\,|\delta\boldsymbol{u}(\boldsymbol{r})|^2-\nu\boldsymbol{\nabla}\boldsymbol{u}:\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell+\frac{1}{2}(\boldsymbol{f}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell+\bar{\boldsymbol{f}}_\ell\boldsymbol{\cdot}\boldsymbol{u}).\end{aligned} \tag{3.26}$$

It is now easy to check that all terms remain bounded in the limit as $\nu\to 0$.

The next step of Duchon & Robert (2000) was to study the convergence as $\nu\to 0$ of the various terms in the regularised energy balance (3.26), under the assumption of strong L^3-convergence of Navier–Stokes solutions $\boldsymbol{u}^\nu$ to some $\boldsymbol{u}$, i.e. $\lim_{\nu\to 0}\|\boldsymbol{u}^\nu-\boldsymbol{u}\|_3=0$. Note that $\boldsymbol{u}$ is then necessarily a weak Euler solution. As with the prior assumption of strong L^2-convergence, this even stronger convergence assumption is not guaranteed *a priori*, but we shall see that it is implied by other empirically observed properties. Taking strong L^3-convergence for granted, it is easy to check that all of the terms proportional to ν in (3.26) in fact disappear in the limit. Furthermore, all of the remaining terms are found to converge in the sense of distributions to the same expression but with Navier–Stokes velocity $\boldsymbol{u}^\nu$ replaced with the limiting field $\boldsymbol{u}$, yielding a modified energy balance for the inviscid Euler solution:

$$\begin{aligned}&\partial_t\left(\frac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell\right)+\boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}\boldsymbol{u}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell\right)\boldsymbol{u}+\frac{1}{2}(p\bar{\boldsymbol{u}}_\ell+\bar{p}_\ell\boldsymbol{u})+\frac{1}{4}\overline{(|\boldsymbol{u}|^2\boldsymbol{u})_\ell}-\frac{1}{4}\overline{(|\boldsymbol{u}|^2)_\ell}\boldsymbol{u}\right]\\ &\qquad=-\frac{1}{4\ell}\int\mathrm{d}^d\boldsymbol{r}(\boldsymbol{\nabla}G)_\ell(\boldsymbol{r})\boldsymbol{\cdot}\delta\boldsymbol{u}(\boldsymbol{r})|\delta\boldsymbol{u}(\boldsymbol{r})|^2+\frac{1}{2}(\boldsymbol{f}\boldsymbol{\cdot}\bar{\boldsymbol{u}}_\ell+\bar{\boldsymbol{f}}_\ell\boldsymbol{\cdot}\boldsymbol{u}).\end{aligned} \tag{3.27}$$

Note, in particular, that validity in the sense of distributions means that the above balance equation is implicitly smeared with a smooth test function $\varphi(\boldsymbol{x},t)$ in space–time and that all derivatives acting on $\boldsymbol{u}$ or related solution fields can be moved over to the test function.

From these remarks, it is then easy to check further that the limit as $\ell \to 0$ of all terms exist directly, except for the term containing $(\nabla G)_\ell(\boldsymbol{r})$. However, because all other terms in the equation converge, this term must also converge. Duchon & Robert (2000) thus concluded that a local kinetic energy balance holds in the sense of distributions for the limiting Euler solution

$$\partial_t \left(\tfrac{1}{2}|\boldsymbol{u}|^2\right) + \nabla \boldsymbol{\cdot} \left[\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + p\right)\boldsymbol{u}\right] \equiv -D(\boldsymbol{u}) + \boldsymbol{f} \boldsymbol{\cdot} \boldsymbol{u}, \tag{3.28}$$

where

$$D(\boldsymbol{u}) = \lim_{\ell \to 0} \frac{1}{4\ell} \int \mathrm{d}^d r (\nabla G)_\ell(\boldsymbol{r}) \boldsymbol{\cdot} \delta\boldsymbol{u}(\boldsymbol{r})|\delta\boldsymbol{u}(\boldsymbol{r})|^2 := \lim_{\ell \to 0} D_\ell(\boldsymbol{u}). \tag{3.29}$$

The naïve kinetic energy balance of the Euler equations is broken by the term $D(\boldsymbol{u})$, which can be easily shown to vanish unless the velocity has Besov regularity exponent $\sigma_p \leq 1/3$ for $p \geq 3$. Note in fact that the same final equation (3.28) can be obtained also by considering the coarse-grained energy balance (3.10) with both factors of $\boldsymbol{u}$ filtered and then taking the limit $\ell \to 0$ in the same manner yields

$$D(\boldsymbol{u}) = \lim_{\ell \to 0} \varPi_\ell \tag{3.30}$$

so that the 'inertial dissipation' in fact represents nonlinear energy cascade to infinitesimally small length scales.

Physically, this loss of energy must arise from viscous dissipation. Notice, however, (3.27)–(3.30) can be derived for any weak Euler solution, under the sole requirement that $\boldsymbol{u} \in L^3$. Because the notion of weak Euler solution is time-reversible but $D(\boldsymbol{u})$ is cubic in the velocity field, it follows that any Euler solution with $D(\boldsymbol{u}) > 0$ yields by time-reversal another Euler solution with $D(\boldsymbol{u}) < 0$. It is only Euler solutions obtained in the inviscid limit, or so-called 'viscosity solutions' of Euler, which can be expected to satisfy the physical expectation of positive dissipation. Because coarse-graining is purely optional, one should be able to obtain the limiting energy balance (3.28) directly in the limit $\nu \to 0$. This is what Duchon & Robert (2000) did, starting with the familiar kinetic energy balance for Navier–Stokes solutions $\boldsymbol{u}^\nu$:

$$\partial_t \left(\tfrac{1}{2}|\boldsymbol{u}^\nu|^2\right) + \nabla \boldsymbol{\cdot} \left[\left(\tfrac{1}{2}|\boldsymbol{u}^\nu|^2 + p^\nu\right)\boldsymbol{u}^\nu - \nu\nabla\left(\tfrac{1}{2}|\boldsymbol{u}^\nu|^2\right)\right] = -\nu|\nabla\boldsymbol{u}^\nu|^2 + \boldsymbol{f} \boldsymbol{\cdot} \boldsymbol{u}^\nu. \tag{3.31}$$

(Here I note in passing that Duchon & Robert (2000) considered a more general situation where the Navier–Stokes solutions are themselves singular and must be interpreted distributionally à la Leray (1934), in which case, the '=' in (3.31) is replaced by '$\leq$'. For reasons discussed later, I am not interested physically in this mathematical generality.) Now assuming once more the strong L^3 convergence $\boldsymbol{u}^\nu \to \boldsymbol{u}$, it is easy to see that all of the terms on the left-hand side converge distributionally as $\nu \to 0$ to the expected limit on the left-hand side of (3.28) and likewise $\boldsymbol{f} \boldsymbol{\cdot} \boldsymbol{u}^\nu \to \boldsymbol{f} \boldsymbol{\cdot} \boldsymbol{u}$. The only term which cannot be shown directly to converge is the viscous dissipation $\nu|\nabla\boldsymbol{u}^\nu|^2$, but, since all other terms in the Navier–Stokes balance (3.31) converge, so also does this term. One thus derives the same local energy balance (3.28) as before for the limiting Euler solution, but one obtains furthermore the new expression

$$D(\boldsymbol{u}) = \lim_{\nu \to 0} \nu|\nabla\boldsymbol{u}^\nu|^2 \geq 0. \tag{3.32}$$

This result can be understood as a matching relation for the 'viscosity solutions' of Euler equations, which equates the energy carried by nonlinear cascade to infinitesimally small scales and the usual energy dissipation by molecular viscosity.

A final important result obtained by Duchon & Robert (2000) exploited additional freedom in the regularisation of UV divergences. In fact, not only is the length scale ℓ arbitrary, but so also is the precise choice of the kernel G. This freedom can be exploited by choosing a rotationally symmetric kernel for which

$$\nabla G(\boldsymbol{r}) = \hat{\boldsymbol{r}} G'(r). \tag{3.33}$$

In that case, the d-dimensional integral over $\boldsymbol{r}$ in (3.29) for $D(\boldsymbol{u})$ can be transformed into hyperspherical coordinates so that

$$D_\ell(\boldsymbol{u}) = \frac{1}{4\ell} \int_0^\infty r^{d-1}\,\mathrm{d}r \int_{S^{d-1}} \mathrm{d}\omega_d(\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} \boldsymbol{\cdot} \delta\boldsymbol{u}(\boldsymbol{r})|\delta\boldsymbol{u}(\boldsymbol{r})|^2 (G')_\ell(r), \tag{3.34}$$

where S^{d-1} is the unit hypersphere and $d\omega_d$ is the measure on in d-dimensional solid angles. Using the integration by parts identity $\int_0^\infty r^d G'(r)\,\mathrm{d}r = d$, one can infer a new identity

$$\lim_{r\to 0} \frac{\langle \delta u_L(\boldsymbol{r})|\delta\boldsymbol{u}(\boldsymbol{r})|^2\rangle_{ang}}{r} = -\frac{4}{d} D(\boldsymbol{u}), \tag{3.35}$$

where $\langle\boldsymbol{\cdot}\rangle_{ang}$ denotes for any function of $\boldsymbol{r}$ the average with respect to ω_d over the direction $\hat{\boldsymbol{r}}$, and $\delta u_L(\boldsymbol{r}) = \hat{\boldsymbol{r}} \boldsymbol{\cdot} \delta\boldsymbol{u}(\boldsymbol{r})$ is the longitudinal velocity increment. Combined with the previous expression (3.32) for $D(\boldsymbol{u})$ in terms of the viscous dissipation, one can see that (3.35) for $d = 3$ has the form of the Kolmogorov–Yaglom 4/3rd law (Antonia *et al.* 1997). It is well known in the standard derivation by ensemble averages that the Kolmogorov 4/5th and 4/15th laws are equivalent to the 4/3rd law under the assumption of statistical isotropy. Since the angle average in the present derivations provides isotropy without any statistical hypotheses, some further manipulation (Eyink 2003; Novack 2023) yields

$$\lim_{r\to 0} \frac{\langle \delta u_L^3(\boldsymbol{r})\rangle_{ang}}{r} = -\frac{12}{d(d+2)} D(\boldsymbol{u}), \tag{3.36}$$

$$\lim_{r\to 0} \frac{\langle \delta u_L(\boldsymbol{r})\delta u_T^2(\boldsymbol{r})\rangle_{ang}}{r} = -\frac{4}{d(d+2)} D(\boldsymbol{u}), \tag{3.37}$$

where $\delta\boldsymbol{u}_T(\boldsymbol{r}) = \delta\boldsymbol{u}(\boldsymbol{r}) - \hat{\boldsymbol{r}}\delta u_L(\boldsymbol{r})$ is the full transverse velocity increment and $\delta u_T(\boldsymbol{r})$ is the magnitude of any particular (fixed) component. One can see for $d = 3$ that (3.36) has the form of the usual 4/5th law and (3.37) has the form of the 4/15th law.

Note, however, that the relations (3.35)–(3.37) are deterministic, holding for individual flow realisations, and are furthermore space–time local in the sense of distributions. The latter statement simply means that they hold with both sides smeared by an arbitrary space–time test function $\varphi(\boldsymbol{x}, t)$, taking first $\nu \to 0$ and then $r \to 0$. Thus, these results represent a considerable strengthening of the original ensemble-average relations of Kolmogorov (1941*a*). An effort was made by Taylor, Kurien & Eyink (2003) to test these deterministic local relations in pseudospectral numerical simulations, but the 512^3 resolution available at the time was insufficient. It was possible to show that angle-averaging greatly improved the validity of the observed 4/5th law, attaining results comparable to those at 1024^3 resolution even without time-averaging. Thus, the local 4/5th law appeared in fact to hold instantaneously, without smearing in time, beyond what could be proved mathematically. Note that it is only with simulations of $16\,384^3$ resolution that the 4/5th law has recently been demonstrated convincingly, exploiting both angle-averaging and time-averaging (Iyer, Sreenivasan & Yeung 2020). This brings us close to being able to check the local 4/5th law in simulations of $32\,768^3$

resolution by dividing the computational cube into eight $16\,384^3$ subcubes, and verifying that the relation holds with the structure functions and mean dissipation calculated by space-averages individually over each subcube. Of course, the result should hold with any space–time test function φ held fixed in the limit first $\nu \to 0$ and then $r \to 0$, but current computational limitations seem to make characteristic functions of octants the only choice consistent with the stringent condition that the local Reynolds number must be high. It is worth remarking that the analogue of the deterministic, local 4/5th law can be rigorously derived for the 'viscosity solutions' of the inviscid Burgers equation, $\partial_t u + \partial_x(\frac{1}{2}u^2) = 0$. In that case, the local energy balance holds

$$\partial_t \left(\tfrac{1}{2}u^2\right) + \partial_x \left(\tfrac{1}{3}u^3\right) = -D(u) \tag{3.38}$$

with the 'inertial dissipation' given by

$$\lim_{r\to 0} \frac{\langle \delta u_L^3(r)\rangle_{ang}}{|r|} = -12D(u), \tag{3.39}$$

where $\delta u_L(r) := \text{sign}(r)\delta u(r)$ and $\langle \delta u_L^3(r)\rangle_{ang} = [\delta u^3(+|r|) - \delta u^3(-|r|)]/2$. See Eyink (2007, § III) and Novack (2023). These relations correspond to the ensemble-averaged '12th law' for Burgulence, but valid in space–time local form with 'inertial dissipation' arising entirely from shock singularities.

It should be emphasised again that the deterministic versions of the 4/5th law (3.36) and the 4/15th law (3.37), and all of the earlier results in this section, require no statistical assumptions whatsoever. Neither local homogeneity, local stationarity, isotropy nor any other hypothesis about the flow statistics was invoked in their derivation. In particular, these relations will apply in the 'non-equilibrium decay regime' highlighted in the review of Vassilicos (2015) on turbulent energy dissipation. Vassilicos (2015) reported observations in a near wake region of various grids and bluff bodies that the time-averaged quantity $C_\epsilon := \langle D\rangle = \langle \varepsilon\rangle/(u'^3/L)$ scales as Re_I^m/Re_L^n, where in his notation, $Re_I = UL_b/\nu$ is the 'global' or 'inlet Reynolds number' based on the inlet flow speed U and a length L_b giving the overall size of the grid or body, whereas $Re_L = u'L/\nu$ is the 'local Reynolds number'. In the wake flows considered by Vassilicos (2015), both the r.m.s. velocity fluctuation u' and integral length L are statistical quantities that depend upon the longitudinal distance x of the observation point from the grid/body. Since wake flows involve interactions with solid walls, I shall treat the energy cascade in such flows in § 4.2.4, but here I note that all of our previous results carry over, with possible modifications only directly at the solid surface. As I shall discuss in § 4.1, for flows with walls or solid bodies, I always consider the global Reynolds number Re_I, which appears in the non-dimensionalisation of the Navier–Stokes equation. As noted already by Vassilicos (2015, p. 108), the quantity C_ϵ at each x-location is 'more or less independent of the Reynolds number' Re_I, even in the non-equilibrium decay region because '$m \approx 1 \approx n$, and Re_L increases linearly with Re_I'. In fact, $m = n$ corresponds exactly to the 'dissipative anomaly' conjectured by Onsager (1949) and all of our previous conclusions follow in the limit $Re_I \to \infty$. In particular, the local 4/5th law (3.36) and the 4/15th law (3.37) will hold in the non-equilibrium decay region, not only for time-/ensemble-averages but even for individual flow realisations. Vassilicos (2015) has observed that 'the Richardson–Kolmogorov cascade does not seem to be the ...interscale energy transfer mechanism at work', but the dissipative anomaly for $Re_I \gg 1$ must occur by Onsager's local cascade mechanism, which generalises Kolmogorov's equilibrium picture. I agree with the observation of Vassilicos (2015) that the 4/5th law of Kolmogorov (1941*a*) need not be valid in general for $r \simeq L$. In fact, our local, deterministic version (3.36) will generally only hold for $r \ll L_\varphi$, where L_φ measures the spatial diameter of the support of the test function φ required to make (3.36) meaningful.

Amazingly enough, most of the previous developments were foreseen by Onsager in his unpublished work in the 1940s. When I was on sabbatical at Yale in 2000, I was able to examine Onsager's private research notes available there on microfiche and I was astonished to discover that Onsager's own proof of his 1/3 Hölder claim was essentially identical to that given by Duchon & Robert (2000). These notes are now all available online through the Onsager Archive hosted by the NTNU in Trondheim and the reader can peruse them at https://ntnu.tind.io/record/121183. The essential calculations are on pp. 14–19 of Onsager's notes in this folder, where he derives the identity (3.27) but in a space integrated form and with the filtering function $G_\ell(\boldsymbol{r})$ denoted $F(\boldsymbol{r})$. It is interesting that on p. 15, Onsager begins the calculation by writing an overbar, his notation for ensemble-average, but then scratches out the overbar in the second line, apparently realising that volume-averaging was sufficient to derive the result. This seems to have been the moment that the theme of my essay was born. Onsager communicated his identity to von Kármán and Lin in his letter of June 1945 (reproduced as appendix B in Eyink & Sreenivasan 2006) where he gave an argument for 1/3-scaling by taking $F(\boldsymbol{r})$ to be a spherical tophat filter of radius a and then letting $a \to 0$. It is further interesting that Onsager in this same letter also foresaw that inertial-range intermittency could lead to an energy spectrum having a steeper slope than $-5/3$, writing that, 'As far as I can make out, a more rapid decrease of $\overline{a_k^2}$ with increasing $\underline{k}$ would require a "spotty" distribution of the regions in which the velocity varies rapidly between neighboring points.' Onsager then notes that all velocity increments supported in discontinuities at vortex sheets would produce $S_2(r) \propto r$, corresponding to a k^{-2} energy spectrum as for Burgers equation. Such sheets would also be consistent with $S_3(r) \propto r$, but Onsager remarks that they would not agree well with experimental observations on velocity traces. It is worth remarking that vortex sheets cannot, in fact, be the origin of anomalous energy dissipation (De Rosa & Inversi 2024). For more discussion of Onsager's insights on inertial-range intermittency, see Eyink & Sreenivasan (2006, § IV.C).

The local kinetic energy balance (3.28) derived by Duchon & Robert (2000) for weak Euler solutions is the most succinct formulation of Onsager's proposal for turbulent dissipation 'without the final assistance by viscosity'. This result has a very familiar appearance to a modern field-theorist, because it is reminiscent of anomalies in classical conservation laws due to UV divergences in the quantum field theory. This analogy seems to have been first pointed out by Polyakov (1992, 1993), who did not know about Onsager's work, of course, but who based his discussion on the statistical theory of Kolmogorov. Polyakov pointed out the essential similarity between turbulent cascades and chiral anomalies in quantum Yang–Mills theory, as both involve a constant flux through wavenumbers which vitiates a naïve conservation law. He pointed out also that the first derivation of the chiral anomaly in quantum electrodynamics by Schwinger (1951) using a point-splitting regularisation was very similar to the derivation of the 4/5th law by Kolmogorov (1941*a*). It is interesting that Schwinger seemingly did not fully appreciate the significance of his own calculation and did not point out its implication that conservation of axial charge is violated (Adler 2005). However, we now see that Onsager had used a similar point-splitting regularisation already in the 1940s and had fully appreciated its consequence that naïve conservation of kinetic energy in the inviscid limit is broken by the 'violet catastrophes' in turbulent flows. This phenomenon is thus now commonly called a turbulent dissipative anomaly, because of the close connection with quantum field theory anomalies. For more detailed discussion of this analogy, see Eyink & Sreenivasan (2006, § IV.B).

I would like to make just a brief remark why the term 'anomaly' is apt for the turbulent phenomenon, independent of the connection with quantum field theory. It should be emphasised that no fundamental microscopic conservation laws for total energy, total linear momentum, total angular momentum, etc. are ever violated by such anomalies. However, there is obviously no microscopic law of 'conservation of kinetic energy'! This conservation law is an emergent property of inviscid hydrodynamics due to its Hamiltonian character, in which the kinetic energy of the fluid is itself the conserved Hamiltonian (Salmon 1988; Morrison, Francoise & Naber 2006). It is because of this Hamiltonian structure of the Euler equations that kinetic energy conservation is formally expected and thus the breakdown of this conservation is 'anomalous'. The same is true more generally for turbulent dissipative anomalies. For example, conservation of helicity and conservation of circulation on arbitrary loops are not sacrosanct microscopic laws but are instead emergent laws connected with 'relabelling symmetry', an infinite-dimensional symmetry group of the action for three-dimensional ideal Euler equations. It is such emergent conservation laws of the ideal fluid which may be afflicted with dissipative anomalies in turbulent flows. It is interesting that some of these symmetries are preserved in a formulation of the viscous Navier–Stokes via a stochastic least-action principle (Eyink 2010), suggesting that not all of these emergent conservation laws will be explicitly broken by dissipative anomalies but may instead be realised in an unconventional stochastic form. See further discussion in § 3.1.3.2.

Finally, I point out that the work of Duchon & Robert (2000) makes a connection between the 'ideal turbulence' theory and the multifractal model of energy dissipation. After Kolmogorov (1962) had proposed his lognormal model of turbulent energy dissipation to account for intermittency corrections, Mandelbrot (1974, 1989) introduced a more general description of the energy dissipation as a multifractal measure with a spectrum of singularities. Subsequent experimental studies of Meneveau & Sreenivasan (1991) supported these predicted scaling properties of turbulent energy dissipation. The result (3.32) of Duchon & Robert (2000) shows that the dissipative anomaly term $D(\boldsymbol{u})$ in the inviscid energy balance coincides exactly with this multifractal measure. In fact, the analysis of Duchon & Robert (2000) implies that the inviscid limit of the viscous energy dissipation $\varepsilon^{\nu} = 2\nu|\boldsymbol{S}^{\nu}|^2$ exists and coincides with $D(\boldsymbol{u})$, which is a non-negative distribution and, thus, a measure. Recent rigorous results relate the fractal dimension of the support of this measure to the inertial-range intermittency of the velocity increments (De Rosa & Isett 2024).

3.1.3. *Implications and open questions*

There are many questions raised by the original analysis of Onsager (1945*a*, 1949) and also many subsequent developments extending and elaborating his ideas. Here I shall briefly review and discuss such further implications and open issues.

3.1.3.1. Origin of singularities Onsager's result and its extensions by Constantin *et al.* (1994), Eyink (1994), Duchon & Robert (2000) and others show that singularities of the velocity field are required to explain anomalous dissipation of kinetic energy in the inviscid limit of incompressible hydrodynamic turbulence. However, this theory does not explain the origin of such singularities. The traditional view has been that these singularities arise from finite-time blow-up of smooth Euler solutions, e.g. see Frisch (1995, § 9.3). The most significant argument in favour of this view is based on a set of results in mathematical theory of PDEs which goes by the name weak-strong uniqueness. Such results state that a weak Euler solution (or an Euler solution in even a more general sense: see § 3.2) which has total kinetic energy non-increasing in time must coincide with a strong (i.e. smooth) solution with the same initial data, over the entire time-interval for which the latter exists. For example, see Lions (1996), Brenier, De Lellis & Székelyhidi (2011), Bardos & Titi (2013) and Wiedemann (2018). These results can be used to infer that a weak Euler solution obtained by a zero-viscosity limit of a Navier–Stokes solution, even in a weaker sense of limit than discussed in § 3.1.2, must coincide with any such smooth Euler solution. In particular, these weak-strong uniqueness results rule out the appearance of anomalous energy dissipation over any finite time interval, if the Navier–Stokes equation is solved with initial data that are a smooth velocity field or that even converge (say in L^2) to a smooth field, and if the Euler solution with that initial data does not blow up. These arguments seem to suggest that anomalous dissipation requires finite-time Euler singularities. There has also been recent progress in showing blow-up of strong Euler solutions, both by numerical simulations (Luo & Hou 2014; Hou 2023) and by rigorous mathematical analysis (Elgindi & Jeong 2019; Chen & Hou 2021, 2022; Elgindi 2021).

However, there are significant reasons to doubt that finite-time blow-up of smooth Euler solutions has anything at all to do with empirical observations on fluid turbulence. First, weak-strong uniqueness is known to break down in the presence of solid boundaries (Bardos, Titi & Wiedemann 2019), requiring in that case additional assumptions that may well be violated. This point will be discussed at length in § 4.2.6. Furthermore, the hypothesis that dissipative singularities in incompressible turbulence form from finite-time Euler singularities does not correlate well with observations. For example, it does not account for the dichotomy demonstrated by Cadot *et al.* (1997) between flows with hydraulically smooth and rough walls, with vanishing dissipation for hydraulically smooth walls and a dissipative anomaly for hydraulically rough walls. In fact, the best numerical and mathematical evidence for a finite-time blow-up is in a cylindrical domain with smooth boundaries (Luo & Hou 2014; Chen & Hou 2021, 2022), where empirical evidence suggests anomalous dissipation does not exist! I personally am of the opinion that solid walls must play a crucial role in the appearance of anomalous energy dissipation in incompressible fluid turbulence, as I shall discuss in more detail in § 4. I just note here that all of the laboratory experiments and natural observations that support a turbulent anomaly involve fluid–solid interactions, e.g. the interactions with the solid grid for decaying turbulence in a wind-tunnel.

The obvious objection to these remarks is the evidence for a dissipative anomaly arising from numerical simulations in a periodic box, with no walls. However, the evidence for anomaly via blow-up from simulations is weak. For example, the simulation of Kaneda *et al.* (2003) does not start with smooth initial data. As already discussed by Drivas & Eyink (2019, Remark #4), the simulation reported by Kaneda *et al.* (2003) uses an iterative initialisation procedure which builds in an initial quasi-singularity, corresponding to an increasing range of Kolmogorov-type spectrum. This common device is widely regarded as a numerical short-cut to accelerate the convergence to a steady state, but it renders the simulations irrelevant to the issue of finite-time singularity. There are simulations that do employ smooth initial data, e.g. the Taylor–Green vortex as discussed recently by Fehn *et al.* (2022). However, here the evidence for anomalous dissipation in a finite time is weaker, with no completely compelling convergence to a non-zero limit within computational limitations.

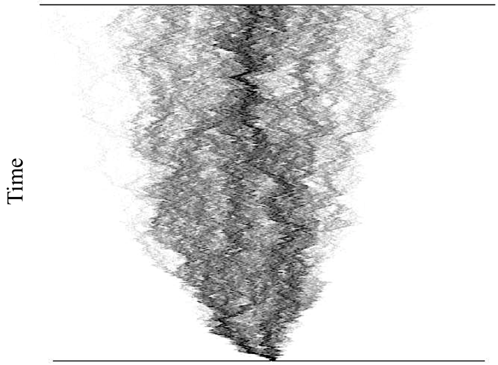

Figure 5. Sketch of the solutions of a deterministic ODE $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{u}(\boldsymbol{x}, t)$ for deterministic initial data $\boldsymbol{x}_0$ but with singular velocity $\boldsymbol{u}$. Unlike traditional unique solutions, the trajectories spread randomly, like a plume of smoke. Reproduced with permission from Falkovich, Gawędzki & Vergassola (2001). Copyright (2001) by the American Physical Society.

3.1.3.2. Lagrangian spontaneous stochasticity One of the most remarkable consequences of Onsager's Hölder singularity result was first discovered in the work of Bernard, Gawędzki & Kupiainen (1998), who studied Lagrangian fluid particle trajectories that solve the initial-value problem:

$$\dot{\boldsymbol{x}} = \boldsymbol{u}(\boldsymbol{x}, t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{3.40}$$

When $\boldsymbol{u}$ is the limiting Euler solution at infinite-Re then, as recalled by Bernard *et al.* (1998), the maximal Hölder regularity derived by Onsager (1949) is insufficient to guarantee a unique solution of (3.40) and there is generally a continuous infinity of solutions with exactly the same initial data. It was realised by Bernard *et al.* (1998) that this non-uniqueness permits stochasticity to persist in the high-Reynolds-number limit. For example, the position of a Brownian particle such as a small colloid or a dye molecule advected by a turbulent flow will satisfy instead a stochastic ODE:

$$\mathrm{d}\tilde{\boldsymbol{x}} = \boldsymbol{u}(\tilde{\boldsymbol{x}}, t)\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}\tilde{\boldsymbol{W}}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{3.41}$$

where D is the molecular diffusivity of the particle and $\tilde{\boldsymbol{W}}(t)$ is a Wiener process modelling the Brownian motion. To study the high-Re limit, one may again non-dimensionalise by defining $\hat{\boldsymbol{u}} = \boldsymbol{u}/u'$, $\hat{\boldsymbol{x}} = \boldsymbol{x}/L$, $\hat{t} = t/(L/u')$ and then D is replaced by $1/Pe$, where $Pe = u'L/D$ is the Péclet number. In the joint limit $Re \gg 1$, $Pe \gg 1$, the Lagrangian particle trajectories $\tilde{\boldsymbol{x}}(t)$ solve the limiting deterministic ODE (3.40), but Bernard *et al.* (1998) showed that they may remain random in the limit with a non-trivial transition probability density $p_{\boldsymbol{u}}(\boldsymbol{x}, t|\boldsymbol{x}_0, 0)$ to arrive at position $\boldsymbol{x}$ at time t (see figure 5). The essential physics was implicit already in the work of Richardson (1926) on two-particle turbulent dispersion, according to which initial particle separations are 'forgotten' at sufficiently long times. It is important to note however that there is no averaging over velocities in the prediction of Bernard *et al.* (1998) and that the transition probability represents stochastic particle evolution in a fixed realisation $\boldsymbol{u}$ of the turbulent velocity field. Bernard *et al.* (1998) furthermore showed that this 'spontaneous stochasticity' of Lagrangian particles provides a mechanism for anomalous dissipation of a concentration field c of such tracers, which satisfies the advection–diffusion equation

$$\partial_t c(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla c(\boldsymbol{x}, t) = D\Delta c(\boldsymbol{x}, t), \quad c(\boldsymbol{x}, 0) = c_0(\boldsymbol{x}). \tag{3.42}$$

Since the solution of this equation (in non-dimensionalised form) has the exact Feynman–Kac representation

$$c(\boldsymbol{x}, t) = \int p_{\boldsymbol{u}}^{Re,Pe}(\boldsymbol{x}, t|\boldsymbol{x}_0, 0) c_0(\boldsymbol{x}_0)\, \mathrm{d}^d x_0, \tag{3.43}$$

the existence of a non-trivial limit $\lim_{Re,Pe\to\infty} p_{\boldsymbol{u}}^{Re,Pe}(\boldsymbol{x}, t|\boldsymbol{x}_0, 0) = p_{\boldsymbol{u}}(\boldsymbol{x}, t|\boldsymbol{x}_0, 0)$ implies by convexity that $\int \frac{1}{2} c^2(\boldsymbol{x}, t)\, \mathrm{d}^d x < \int \frac{1}{2} c_0^2(\boldsymbol{x}_0)\, \mathrm{d}^d x_0$ in the limit $Re, Pe \to \infty$.

The original work of Bernard *et al.* (1998) was carried out for the synthetic turbulence model of Kraichnan (1968), who considered advection by Gaussian random velocity fields that are delta-correlated in time. Note that for suitable power-law spatial correlations to mimic inertial-range turbulent velocity fields, the realisations of the Kraichnan velocity ensemble are Hölder continuous with probability one. A very important feature of this model is that the limiting probability distributions

$$p_{\boldsymbol{u}}(\boldsymbol{x}, t|\boldsymbol{x}_0, 0) = \lim_{Re,Pe\to\infty} p_{\boldsymbol{u}}^{Re,Pe}(\boldsymbol{x}, t|\boldsymbol{x}_0, 0) \tag{3.44}$$

are independent of the particular subsequences Re_k, $Pe_k \to \infty$ and, in fact, are independent of the particular form of regularisation and noise, e.g. the Brownian motion $\tilde{\boldsymbol{W}}(t)$ in (3.41) might be replaced with a fractional Brownian motion and the limit would be the same. This strong result for the Kraichnan model follows from a rigorous study by Le Jan & Raimond (2002) who gave a direct construction of a Markov transition probability $p_{\boldsymbol{u}}(\boldsymbol{x}, t|\boldsymbol{x}_0, 0)$ for the zero-regularisation, zero-noise problem by a Weiner chaos expansion in the white-noise advecting velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. It is a consequence of this construction that any reasonable spatial regularisation and noisy perturbation will converge to this same transition probability in the zero-regularisation, zero-noise limit. It was also shown by Le Jan & Raimond (2002) that the realisations $\tilde{\boldsymbol{x}}(t)$ of the limiting Markov process with transition densities $p_{\boldsymbol{u}}(\boldsymbol{x}, t|\boldsymbol{x}_0, 0)$ are solutions (in a generalised sense) of the deterministic initial-value problem (3.40) for each fixed realisation $\boldsymbol{u}$ of the Kraichnan velocity ensemble. Thus, in this case, the Markov process $\tilde{\boldsymbol{x}}(t)$ should be regarded as the proper solution of the above initial-value problem (3.40), which is then well-posed in the sense of Hadamard. Spontaneous stochasticity in the strong sense should be understood to include such universality of the limiting random process, that is, the robustness of the stochastic solution to different regularisations and noisy perturbations.

Extensions of the work of Bernard *et al.* (1998) and further developments on the Kraichnan model are described by Falkovich *et al.* (2001). This comprehensive review makes clear that many of the results have validity extending beyond the Kraichnan model and, in fact, Drivas & Eyink (2017*a*) showed that spontaneous stochasticity is the only possible mechanism of scalar anomalous dissipation away from solid walls, for both passive and active scalars advected by a general incompressible velocity field. Note that universality of the limiting spontaneous statistics is not required in this case, so that different limiting random processes yielding anomalous dissipation may result from different subsequences and/or different regularisations. There are, however, only a handful of model velocity fields for which Lagrangian spontaneous stochasticity can be demonstrated from first principles (Drivas & Mailybaev 2021; Drivas, Mailybaev & Raibekas 2023). Renormalisation group methods from statistical physics apply to velocity fields with suitable scaling properties and yield universality of the limit (Eyink & Bandak 2020), but the underlying ergodic properties of the dynamical flows that are required for this method are difficult to establish in general. One interesting example of a velocity field solving a PDE is the solution of the inviscid Burgers equation, where Lagrangian spontaneous stochasticity holds backwards in time at shock locations and explains anomalous energy dissipation (Eyink & Drivas 2015). There is no such spontaneous stochasticity for particles advected forwards in time by the Burgers velocity field and it has been speculated that similar time asymmetry may be a more general feature of Lagrangian spontaneous stochasticity in dissipative weak solutions of PDEs obtained as inviscid limits.

As far as Navier–Stokes turbulence is concerned, there is so far no compelling evidence of Lagrangian spontaneous stochasticity from laboratory experiments (Bourgoin *et al.* 2006; Tan & Ni 2022). However, numerical simulations of homogeneous, isotropic turbulence in a periodic domain provide reasonable evidence for the 'forgetting' of initial separations of deterministic Lagrangian particles (Bitane, Homann & Bec 2013; Buaria, Sawford & Yeung 2015) and 'forgetting' of both initial separations and molecular diffusivity of stochastic Lagrangian trajectories (Eyink 2011; Buaria, Yeung & Sawford 2016). Setting aside the important issue of empirical evidence, another crucial question is what implications Lagrangian spontaneous stochasticity might have for incompressible Navier–Stokes turbulence. A Feynman–Kac representation of Navier–Stokes solutions was derived by Constantin & Iyer (2008) which is similar to (3.43) for the advection–diffusion equation but more non-trivial, because of the nonlinearity of the equations. It was pointed out by Eyink (2010) that the Constantin–Iyer representation corresponds to a stochastic principle of least-action, thus making connection with Hamiltonian fluid mechanics. In particular, the stochastic action functional for incompressible Navier–Stokes is invariant under the infinite-dimensional symmetry group of particle-relabelling just as is the action for deterministic Euler. The remarkable properties of vorticity under Euler dynamics that follow from that symmetry therefore carry over to Navier–Stokes in a stochastic fashion. For example, the Kelvin theorem of conservation of circulation on an arbitrary Lagrangian loop for Euler holds also for Navier–Stokes in the sense that the circulation on stochastically advected loops is a backwards-in-time martingale (Constantin & Iyer 2008; Eyink 2010). This property prescribes the 'arrow-of-time' of the irreversible Navier–Stokes dynamics and it is plausible that a similar property carries over to dissipative weak Euler solutions obtained in the inviscid limit (Eyink 2006). The dissipative anomaly for Navier–Stokes turbulence may be characterised also in terms of the time asymmetry of separation of stochastic Lagrangian particle (Drivas 2019; Cheminet *et al.* 2022), which in three dimensions separate faster backwards in time than they do forwards in time (Eyink 2011; Buaria *et al.* 2016).

3.1.3.3. Finite Reynolds numbers A frequently made criticism of Onsager's 'ideal turbulence' theory is that it applies only to the unrealistic limit $Re = \infty$ and is thus inapplicable to real-world turbulence which is, of course, always at a finite Reynolds number. As an example, I may quote from the monograph of Tsinober (2009, § 10.3.2): '...it is not clear why results for finite Re (i.e. for NSE having no singularities or extremely "intermittent" ones) are relevant for the limit (if such exists) $Re \to \infty$ (e.g. for Euler equation with space-filling singularities)'. In some respects, such criticisms are a simple misunderstanding of the elementary concept of limit. Indeed, the predictions of a theory for $Re \to \infty$ are experimentally falsifiable, as they must be valid to arbitrary accuracy by taking Reynolds numbers larger (but finite). There is, however, a legitimate question regarding any such predictions of an 'ultimate regime' concerning how large Re must be chosen to observe the predictions of the theory. This is especially important for Onsager's theory since, as has often been observed, the strict requirement for its validity is that the number of cascade steps must be large, that is, $\log_2 Re \gg 1$ and this condition is hard to satisfy in even the highest Reynolds number terrestrial turbulent flows. There are two responses to this very important question.

First, there are explicit error estimates in the mathematics of the Onsager theory which provide bounds on the correction terms at large but finite Re. For example, in the coarse-grained energy balance equation (3.10), there is at finite Reynolds number a viscous dissipation term $\nu|\nabla\bar{\boldsymbol{u}}_\ell|^2$ in addition to the inertial flux term $\varPi_\ell$. It is not difficult to derive bounds of the form

$$\nu|\nabla\bar{\boldsymbol{u}}_\ell|^2 = O(\nu\delta u^2(\ell)/\ell^2) \tag{3.45}$$

locally in space–time with $\delta u(\ell) = \sup_{|\boldsymbol{r}|<\ell}|\delta\boldsymbol{u}(\boldsymbol{r})|$, or similar bounds in terms of Besov norms for space-integrated dissipation. These estimates provide concrete upper estimates on the finite-Re corrections, which can explain departures from the $Re = \infty$ theory.

Second, Onsager's RG-type arguments can also be applied directly to the Navier–Stokes equations at large but finite Re and do not require any assumptions about existence of weak Euler solutions in the limit $Re \to \infty$. For example, Drivas & Eyink (2019) study the balance of the subscale kinetic energy $k_\ell := (1/2)\mathrm{Tr}\,\boldsymbol{\tau}_\ell$ for the forced Navier–Stokes equation, which takes the form

$$\partial_t k_\ell + \nabla\boldsymbol{\cdot}\boldsymbol{J}_\ell = -\overline{(\nu|\nabla\boldsymbol{u}|^2)}_\ell + \nu|\nabla\bar{\boldsymbol{u}}_\ell|^2 + \varPi_\ell + \tau_\ell(\boldsymbol{f};\boldsymbol{u}), \tag{3.46}$$

where $\boldsymbol{J}_\ell$ is a suitable spatial flux of the subscale kinetic energy and $\tau_\ell(\boldsymbol{f};\boldsymbol{u}) := \overline{(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{f})}_\ell - \bar{\boldsymbol{u}}_\ell\boldsymbol{\cdot}\bar{\boldsymbol{f}}_\ell$ is the direct power input by the force into unresolved scales. If one assumes suitable Besov regularity of the Navier–Stokes solutions, $\boldsymbol{u}^\nu \in B_p^{\sigma,\infty}$, $p \geq 3$, uniform in the Reynold number, then one may derive bounds of the form

$$D = (1/Re)|\hat{\nabla}\hat{\boldsymbol{u}}|^2 = O(Re^{(1-3\sigma)/(1+\sigma)}). \tag{3.47}$$

These bounds follow by the same basic principle of independence of the physics on the arbitrary coarse-graining scale ℓ, which allows one to choose $\ell/L \propto Re^{-1/(1+\sigma)}$ to optimise the estimates. Following the same ideas of Onsager's argument, one can then deduce the existence of 'quasi-singularities' in the Navier–Stokes solutions from the observation of energy dissipation vanishing slowly with Re:

$$D = (1/Re)|\hat{\nabla}\hat{\boldsymbol{u}}|^2 \propto Re^{-\alpha}, \tag{3.48}$$

which will require $\sigma \leq \sigma_\alpha := (1+\alpha)/(3-\alpha)$. Note that when $\alpha = 0$, one recovers Onsager's critical value $\sigma = \frac{1}{3}$, but when $\alpha > 0$ instead, $\sigma_\alpha > \frac{1}{3}$. This is a significant strengthening of the Onsager's original result, because empirical observations alone could never allow one to distinguish $\alpha = 0$ from a very tiny value of α. Nevertheless, Onsager's conclusions about 'quasi-singularities' are robust, remaining valid even under the weaker condition (3.48).

The condition (3.48) on the dimensionless dissipation rate for $\alpha > 0$ has been evocatively termed a 'weak dissipative anomaly' by Bedrossian *et al.* (2019). Note that whenever $\alpha < 1$ in (3.48), then $|\hat{\nabla}\hat{\boldsymbol{u}}|^2 \propto Re^{1-\alpha} \to \infty$ as $Re \to \infty$ and thus classical solutions of the PDEs can no longer exist in the limit of infinite Re. Although no assumption about the existence of weak solutions is necessary to derive the bound

(3.47), such weak Euler solutions nevertheless emerge under the natural assumption of some Besov regularity with $\sigma < \sigma_\alpha$ that is uniform in *Re* (Drivas & Eyink 2019). Note furthermore that Bedrossian *et al.* (2019) have proved that the Kolmogorov 4/5th law remains valid under the assumption of a weak dissipative anomaly only, unlike the original derivation of Kolmogorov (1941*a*) which assumed a strong dissipative anomaly ($\alpha = 0$). It is easy to check that the Taylor microscale defined by $\lambda^2 = 15\nu u'^2/\varepsilon$ satisfies $\lim_{Re\to\infty} \lambda/L = 0$ precisely when $\alpha < 1$ in (3.48) and then it is proved in Bedrossian *et al.* (2019) that the maximum difference between $\langle \delta u_L^3(r)\rangle/r$ and $-\frac{4}{5}\varepsilon^\nu$ becomes vanishingly small over an increasing range of length scales $\lambda \ll r \ll L$. This important result shows that validity of the 4/5th law cannot be taken as evidence for a strong anomaly. The proof by Bedrossian *et al.* (2019) considers the statistical steady-state of turbulence in a periodic domain driven by a body force which is a Gaussian random field, delta-correlated in time and takes advantage of simplifications of that stochastic forcing. More recently, analogous results have been derived by Novack (2023) for incompressible Navier–Stokes equations with a deterministic forcing and are valid for individual flow realisations.

These results have gained some added significance by a recent surprising observation that the scaling exponent ζ_3 of absolute third-order structure functions defined by (3.19) or (3.23) for $p = 3$ in fact satisfies $\zeta_3 > 1$ in some high-*Re* numerical simulations of forced turbulence in a periodic domain (Iyer *et al.* 2024). This observation is, of course, inconsistent with the bound $\zeta_3 \leq 1$ derived by Constantin *et al.* (1994) under the assumption of a strong dissipative anomaly. Iyer *et al.* (2024) appear to have found instead that the normalised dissipation rate *D* very slowly decays at high *Re* according to the bound (3.47) derived by Drivas & Eyink (2019). Therefore, only a weak dissipative anomaly appears to occur in this particular simulation of forced homogeneous and isotropic turbulence in a periodic box. Consistent with the results of Bedrossian *et al.* (2019) and Novack (2023), however, the Kolmogorov 4/5th law is still observed to hold in this simulation, in agreement with earlier observations of Iyer *et al.* (2020). These observations need to be confirmed with other forcing schemes and extended to even higher Reynolds numbers, but they raise additional doubts about the simplification of forced turbulence in a periodic box as a valid paradigm for incompressible fluid turbulence more generally.

3.1.3.4. Extensions One of the great virtues of Onsager's 'ideal turbulence' theory is that it extends readily to turbulent flows in other physical systems than incompressible fluids. For example, consequences have been worked out for turbulence in compressible fluids, both barotropic (Feireisl *et al.* 2017) and with a general thermodynamic equation of state (Drivas & Eyink 2018; Eyink 2018*a*), in quantum superfluids (Tanogami 2021), in relativistic fluids (Eyink & Drivas 2018), in magnetohydrodynamics, both incompressible (Caflisch, Klapper & Steele 1997; Galtier 2018; Faraco, Lindberg & Székelyhidi 2022) and compressible (Lazarian *et al.* 2020, § IV.B), and in kinetic plasmas at low collisionality (Eyink 2015, 2018*a*; Bardos, Besse & Nguyen 2020). In many of these examples, the dimensional and statistical reasoning of Kolmogorov (1941*a*,*b*) does not obviously apply and the Onsager theory provides the only first-principles formulation. Almost all of the just-listed examples have applications in astrophysics and space science and, in particular, most astrophysical fluids and plasmas are compressible. Astrophysics provides, in many ways, the optimal arena for Onsager's theory. For one thing, the Reynolds numbers in astrophysics are often much larger than the highest Reynolds numbers attainable in terrestrial turbulence. In addition, problems such as the physical origin of fluid singularities are much easier to address in compressible fluids. A well-advanced mathematical theory already exists for generic development of shocks in smooth solutions of compressible Euler equations and the consequent production of vorticity (Buckmaster, Shkoller & Vicol 2023). Shocks and baroclinic generation of vorticity are expected to be a common source of astrophysical turbulence, e.g. supernovae-driven shocks in the interstellar medium (Gatto *et al.* 2015).

Other recent results on the $Re \to \infty$ limit involve incompressible fluids, but fully incorporating the usually neglected effect of thermal fluctuations. Starting with the fluctuating hydrodynamic equations (2.5) of Landau & Lifshitz (1959) for a low-Mach incompressible fluid (see also Forster, Nelson & Stephen 1977; Donev *et al.* 2014), Eyink & Peng (2024) have studied the limiting dynamics in the inertial range. The earlier work of Bandak *et al.* (2024), which shall be discussed more below, already considered the standard large-scale non-dimensionalisation of the equations via $\hat{\boldsymbol{u}} = \boldsymbol{u}/u'$, $\hat{\boldsymbol{x}} = \boldsymbol{x}/L$, $\hat{t} = u't/L$, which takes the form

$$\partial_{\hat{t}}\hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}}\boldsymbol{\cdot}\hat{\boldsymbol{\nabla}})\hat{\boldsymbol{u}} = -\hat{\boldsymbol{\nabla}}\hat{p} + \frac{1}{Re}\hat{\Delta}\hat{\boldsymbol{u}} + \sqrt{\frac{2\theta_\eta}{Re^{15/4}}}\hat{\boldsymbol{\nabla}}\boldsymbol{\cdot}\hat{\boldsymbol{\xi}} + \mathcal{F}\hat{\boldsymbol{f}}, \tag{3.49}$$

where the dimensionless quantity $\theta_\eta = k_BT/\rho u_\eta^2\eta^3$ is the thermal energy relative to the energy of an eddy of Kolmogorov scale η and with Kolmogorov velocity $u_\eta = (\nu\varepsilon)^{1/4}$, with θ_η generally of order 10^{-6} or even smaller. In deriving (3.49), Bandak *et al.* (2024) assumed the Taylor relation $\varepsilon \sim u'^3/L$ and an external body force with dimensionless magnitude $\mathcal{F} = f_{rms}L/u'^2$ has also been included to generate turbulence. It is naïvely clear that, on inertial-range scales, the direct effect of the thermal noise term must be negligible, even smaller than the viscous diffusion. In fact, the noise term leads to a mathematically ill-defined dynamics, unless the stochastic noise field $\tilde{\boldsymbol{\xi}}$ and velocity field $\tilde{\boldsymbol{u}}$ are both cut off at some high wavenumber $\varLambda$, or $\hat{\varLambda} = \varLambda L$ after non-dimensionalisation. In the statistical physics literature, such a cutoff is standard and represents physically a coarse-graining length $\ell = \varLambda^{-1}$ of the measured velocity field, which is chosen somewhere between η and the molecular mean-free path length ℓ_{mfp}. In the limit $Re \to \infty$ and $\hat{\varLambda} \to \infty$, Eyink & Peng (2024) prove under natural conditions that are observed in experiment (see § 3.2.3 below) that the limiting velocity fields are weak solutions of incompressible Euler, as assumed in the Onsager (1949) theory.

It is important to stress that Onsager's 'ideal turbulence' description does not apply in the dissipation range and, indeed, Onsager never discussed dissipation-range turbulence in any of his published or unpublished writings, to my knowledge. This is perhaps one reason why he never discussed the possible interactions of thermal fluctuations and turbulence. This issue was raised approximately a decade later, however, by Betchov (1957, 1961), who pointed out that the kinetic energy spectrum of thermal fluid fluctuations

$$E(k) \sim \frac{k_BT}{\rho}\frac{4\pi k^2}{(2\pi)^3} \tag{3.50}$$

must surpass the spectrum of turbulent fluctuations at a wavenumber of order $k_\eta \sim 1/\eta$. This same idea was rediscovered much later by Bandak *et al.* (2022) who provided numerical evidence for the conjecture in a shell model, before full verification by Bell *et al.* (2022) in a numerical simulation of the Landau–Lifschitz fluctuating hydrodynamic equations (2.5). Confirmation by laboratory experiment is crucial but appears much more difficult at this time. In principle, all turbulent processes at sub-Kolmogorov scales should be strongly affected by thermal noise, including droplet formation, combustion, biolocomotion, etc. but it is unclear whether such probes of the small-scale physics can show the clear signature of thermal noise at large scales. A prototypical example is high-Schmidt-number turbulent advection, where the classical theories of Batchelor (1959) and Kraichnan (1968) that ignore thermal fluctuations miss power-law scaling of the concentration spectrum below the Batchelor dissipation scale (Eyink & Jafari 2022). However, the classical prediction of a k^{-1} spectrum in the viscous-convective range remains intact, because the strong effect of thermal hydrodynamic fluctuations in renormalising the molecular diffusivity in that range is exactly the same as in laminar flows (Onsager 1945*b*) and thus hidden phenomenologically. There are effects of thermal noise even in the turbulent inertial range (Bandak *et al.* 2024), but these are much more subtle and will be discussed further below.

3.1.3.5. Physicality of weak solutions A frequent objection made by physicists and fluid mechanicians to the ideal turbulence theory is that the mathematical concept of a 'weak solution' is unphysical. In fact, such criticisms are directed not only at Onsager's proposed Euler solutions, but also against the weak solutions of the incompressible Navier–Stokes equation constructed by Leray (1934). To quote again from Tsinober (2009, § 10.3.1), regarding Leray's work: 'An important point is that if one looks at real turbulence at finite Reynolds numbers (however large) there seems to be no need for weak solutions at all.' Here the presumption is that smooth, strong solutions of Navier–Stokes must exist, as there is no obvious empirical evidence for the severe singularities required to overcome the viscous regularisation. This issue of the physical meaning of weak solutions is quite important, since it is at the centre of the empirical testability of Onsager's theory. Thus, I must discuss the matter briefly here. Furthermore, my own views may not coincide with those of most mathematicians who work on Onsager's theory, and these differences must be carefully explained. The issues are subtle and intimately related with the famous Sixth Problem of Hilbert (1900), which was 'the problem of developing mathematically the limiting processes ...which lead from the atomistic view to the laws of motion of continua' and which remains still largely unresolved.

As I have already emphasised, 'weak solutions' are mathematically equivalent to 'coarse-grained solutions' such as (3.4) for Navier–Stokes and (3.5) for Euler, when those equations are imposed for all $\ell > 0$ (Drivas & Eyink 2018). In reality, the 'coarse-grained solution' of Euler equation (3.5) is an accurate physical description of a turbulent flow only in the inertial range of scales for $\ell \gg \eta$. The most important fact about the formulation in (3.5) of 'weak Euler solutions', or its equivalent (3.8), is that it involves only the coarse-grained fields $\bar{\boldsymbol{u}}_\ell$ and $\bar{p}_\ell$ for $\ell \gg \eta$, since the subscale stress $\boldsymbol{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u})$ depends as well only upon $\bar{\boldsymbol{u}}_{\ell'}$ with $\ell' \lesssim \ell$ by the fundamental property of scale locality. To state my view succinctly, it is the coarse-grained fields such as $\bar{\boldsymbol{u}}_\ell$ and $\bar{p}_\ell$ which are physical, because they correspond to what is experimentally measurable. In fact, every experiment has some spatial resolution ℓ, such that only averaged properties for length scales $> \ell$ are obtained. It is instead the fine-grained/bare fields such as $\boldsymbol{u}(\boldsymbol{x})$ which are unphysical, because they are unobservable objects corresponding to a mathematical idealisation $\bar{\boldsymbol{u}}_\ell \to \boldsymbol{u}$ as $\ell \to 0$, which goes beyond the validity of a hydrodynamic description and is physically unachievable.

The above views are probably not the same as those held currently by the majority of mathematicians and fluid mechanicians who work on turbulence, who mostly have followed von Neumann (1949) in believing that

> 'Turbulent' and 'molecular' disorder are in general distinct. A macroscopic and yet turbulent $\boldsymbol{u}$ (and, more generally, a complete system of system of fluid dynamics with such characteristics) can be defined. (von Neumann 1949, § 3.2, p. 446)

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According to this common view, the deterministic Navier–Stokes equations are assumed valid for scales $\ell \gtrsim \lambda_{mfp}$, the molecular mean free path length, with $\bar{\boldsymbol{u}}_\ell \approx \boldsymbol{u}$, the Navier–Stokes solution, for all ℓ in the range $\eta \gtrsim \ell \gtrsim \lambda_{mfp}$. There is, however, no rigorous mathematical theory that justifies this assumption and there are no experimental measurements available at scales $\ell < \eta$ to confirm it. A major problem with this view is that it omits the physical effects of thermal fluctuations, which leads to deviations from the predictions of deterministic Navier–Stokes at scales $\ell \gg \lambda_{mfp}$ that are experimentally well observed in laminar flows (de Zarate & Sengers 2006). In fact, deterministic Navier–Stokes is a physically inconsistent set of equations, because it incorporates molecular dissipation without including the corresponding molecular fluctuations. It thus violates the fundamental fluctuation-dissipation theorem of statistical physics (Onsager & Machlup 1953). I therefore do not regard solutions of deterministic Navier–Stokes, either weak or strong, as having fundamental physical significance. Deterministic Navier–Stokes is instead just a reasonably accurate 'large-eddy simulation' model of turbulence for resolved scales down to $\ell \sim \eta$.

A more fundamental model of molecular fluids are the fluctuating hydrodynamic equations of Landau & Lifshitz (1959) which incorporate the fluctuation-dissipation relation and which explain the experiments on thermal fluctuations in laminar flows (de Zarate & Sengers 2006). It was first pointed out by Betchov (1961) that fluctuating hydrodynamics is an appropriate model of the turbulent dissipation range and, in fact, the incompressible, low-Mach-number equations (2.5) were proposed in his work independently of Landau & Lifshitz (1959). However, as observed in the previous subsection, fluctuating hydrodynamics is not a continuum theory but instead involves an explicit cutoff $\varLambda = \ell^{-1}$ and it is believed to describe the evolution of the coarse-grained fields $\bar{\boldsymbol{u}}_\ell$ for all scales $\ell \gtrsim \lambda_{mfp}$ (or, in fact, for the incompressible version (2.5), only at scales where $|\bar{\boldsymbol{u}}_\ell| \lesssim c_s$, the sound speed Donev *et al.* 2014). Furthermore, the viscosity ν_ℓ in the model depends upon the scale ℓ, as shown, for example, by the renormalisation group analysis of Forster *et al.* (1977). Thus, even in laminar flows, the observed viscosity ν_ℓ at length scale ℓ is an 'eddy-viscosity', although the fluctuating eddies arise there from thermal fluctuations rather than turbulent fluctuations. In the language of modern physics, the Landau–Lifschitz fluctuating hydrodynamics equations are low-wavenumber 'effective field theories' (Schwenk & Polonyi 2012; Liu & Glorioso 2018).

An important consequence of these considerations is that velocity-gradients are ℓ-dependent at all length scales ℓ in a turbulent flow and not only for $\ell > \eta$. However, it should be true to a good approximation that, for typical thermal fluctuations,

$$\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell \sim \boldsymbol{\nabla}\boldsymbol{u} + O\left(\left(\frac{k_B T}{\rho \ell^5}\right)^{1/2}\right), \quad \ell \lesssim \eta, \tag{3.51}$$

where $\boldsymbol{u}$ is again the solution of deterministic Navier–Stokes. It then easily follows that

$$\nu_\ell \langle |\boldsymbol{\nabla}\bar{\boldsymbol{u}}_\ell|^2\rangle \simeq \nu \langle |\boldsymbol{\nabla}\boldsymbol{u}|^2\rangle, \quad \nu = \nu_\eta, \tag{3.52}$$

where $\langle \cdot \rangle$ denotes a space–time average and ℓ/η spans approximately 2–3 decades of scales. For more details, see Eyink (2007, § II.E). In this sense, typical thermal fluctuations should matter little for mean energy dissipation. However, this statement is presumably not true for large, rare thermal fluctuations. Indeed, in a simple particle model of hydrodynamics, the 'totally asymmetric exclusion process', dissipative weak solutions of inviscid Burgers equation arise with overwhelming probability (law of large numbers) but rare fluctuations (large deviations) can lead to weak solutions with local energy production in some space–time regions rather than dissipation everywhere (Jensen 2000; Varadhan

2004). Furthermore, higher-order gradients $\nabla^k \bar{\boldsymbol{u}}_\ell$ with $k > 1$ will be much more strongly affected by thermal noise and, indeed, the numerical simulation of Bell *et al.* (2022) found that such higher-order gradients are completely dominated by thermal noise already for $\ell \lesssim \eta$.

I therefore fundamentally disagree with a common view, clearly expressed by Tsinober (2009), that hypothetical fine-grained fields $\boldsymbol{u}$ and their gradients are physically more 'objective' than coarse-grained fields $\bar{\boldsymbol{u}}_\ell$:

> There is a generic ambiguity in defining the meaning of the term *small scales* (or more generally scales) and consequently the meaning of the term *cascade* in turbulence research. As mentioned in chapter 5, the specific meaning of this term and associated inter-scale energy exchange/cascade (e.g. spectral energy transfer) is essentially decomposition/representation dependent. Perhaps, the only common element in all decompositions/ representations (D/R) is that the small scales are associated with the field of velocity derivatives. Therefore, it is natural to look at this field as the one *objectively* (i.e. D/R independent) representing the small scales. Indeed, the dissipation is associated precisely with the strain field, s_{ij}, both in Newtonian and non-Newtonian fluids. (Tsinober 2009, § 6.2, p. 127)

This point of view is based on the belief, without evidence, that there is some hypothetical fine-grained field $\boldsymbol{u} = \lim_{\ell \to 0} \bar{\boldsymbol{u}}_\ell$ and that objectivity is achieved by 'convergence' as $\ell \to 0$. The existence of such a fine-grained velocity in turbulent flow was probably first questioned by the visionary scientist, Lewis Fry Richardson, whose famous paper on turbulent pair-dispersion contained a section entitled 'Does the Wind possess a Velocity?' He wrote there:

> This question, at first sight foolish, improves on acquaintance. A velocity is defined, for example, in Lamb's "Dynamics" to this effect: Let Δx be the distance in the x direction passed over in a time Δt, then the x-component of velocity is the limit of $\Delta x/\Delta t$ as $\Delta t \to 0$. But for an air particle it is not obvious that $\Delta x/\Delta t$ attains a limit as $\Delta t \to 0$. (Richardson 1926, § 1.2, p. 709)

Indeed, each individual air molecule is moving at approximately 346 m s^{-1}, the speed of sound in that fluid, with the mean wind velocity a small correction, and the local measured velocity will depend entirely upon the resolution scale ℓ which is adopted.

I certainly agree that science should be concerned with objective facts. However, in contrast to the 19th-century continuum mechanics perspective that objectivity is achieved by convergence, the modern tool to achieve objectivity is the renormalisation group. I may cite here the physics Nobel laureate Kenneth G. Wilson:

> A procedure is now being developed to understand the statistical continuum limit. The procedure is called the renormalisation group. It is the tool that one uses to study the statistical continuum limit in the same way that the derivative is the basic procedure for studying the ordinary continuum limit. (Wilson 1975, p. 774)

The fundamental understanding that arose in physics was that for systems with strong fluctuations at all length scales – a class which includes both quantum field theories and turbulent flows – the effective description varies with the length scale ℓ of resolution. In that case, the proper goal is not to seek for some idealised and unobservable 'truth' at $\ell \to 0$, but instead to attain objectivity by understanding how the description changes as ℓ is varied. The application of this approach to hydrodynamics has mostly been restricted to flows much simpler than turbulence, cf. Forster *et al.* (1977), where weak-coupling perturbation expansions are applicable, and only recently has some progress been made on non-perturbative application to turbulence by functional renormalisation group methods (Canet 2022). It is conceptually important, however, to understand that LES models resolved at length scale ℓ are not really continuum models. Indeed, it is a truism in the LES community that the true model is not the 'continuum' LES model but instead the discretisation of the model used to solve it numerically. It therefore makes sense to combine the steps of coarse-graining and numerical discretisation, as in some approaches to modelling thermal fluctuations (Español, Anero & Zúñiga 2009). I shall return to this theme later in discussing the relation of Onsager's 'ideal turbulence' theory with LES modelling.

3.2. *Onsager's conjecture*

The quote from Onsager (1949) on which I base this essay began with a provocative suggestion that 'turbulent dissipation as described could take place just as readily without the final assistance by viscosity'. It is important to emphasise that none of the mathematical results that I have reviewed so far, including those of Onsager himself, establish this hypothesis. Such arguments give only necessary conditions for anomalous dissipation, but not sufficient conditions. An example of an instantaneous velocity field in C^h was found by Eyink (1994) showing that the proofs of conservation for $h > 1/3$ cannot be extended to $h \leq 1/3$, but such examples also fall far short of proving Onsager's conjecture, as already stated there:

> It must not, of course, be concluded that, simply because our argument fails when $h \leq 1/3$, that non-conservation is actually possible for $h \leq 1/3$. We emphasize that to demonstrate this it is necessary to construct an appropriate solution $\boldsymbol{v}(.,.)$ with $\boldsymbol{v}(.,t) \in C^h$, $0 < h < 1/3$ for $t \in [0, T]$, for which the energy indeed decreases or increases in the interval. (Eyink 1994)

There are several elements involved in Onsager's conjecture which are highly non-trivial, e.g. that weak Euler solutions exist in the limit $Re \to \infty$ and that dissipation remains positive in that limit. As to the first matter, I note that inviscid limits of Navier–Stokes solutions can be proved to exist without any further assumptions, at least for flows in infinite Euclidean space or in a periodic domain. However, the limiting Euler solutions so obtained are much weaker even than those postulated by Onsager, consisting of 'measure-valued solutions' with a distribution $p_{\boldsymbol{x},t}(d\boldsymbol{u})$ of possible velocity values at each space–time point $(\boldsymbol{x}, t)$ (DiPerna & Majda 1987; Wiedemann 2018) or satisfying cumbersome energy inequalities (Lions 1996). In addition, such generalised weak Euler solutions are proved to exist only for subsequences of viscosity $\nu_k \to 0$ and may differ from subsequence to subsequence.

Recently, however, considerable progress has been made on these problems, in particular, a rigorous proof has been found that dissipative Euler solutions as conjectured by Onsager (1949) do exist with $\boldsymbol{u} \in C^{1/3-\epsilon}$ for any $\epsilon > 0$ (Buckmaster *et al.* 2018; Isett 2018). Furthermore, advances have been made on deriving such results from the incompressible Navier–Stokes equations in the limit $Re \to \infty$ (Bruè & De Lellis 2023) and even more progress has been made on related problems in passive scalar turbulence (Bedrossian, Blumenthal & Punshon-Smith 2022). In this section, I shall attempt to give a broad overview of these developments as appropriate to this essay, but certainly no comprehensive review, which would be difficult in any case for such a rapidly moving and growing literature. See De Lellis & Székelyhidi (2019) and Buckmaster & Vicol (2020) for some recent reviews by lead researchers in this area.

It must be emphasised at the outset that, as spectacular as these recent mathematical developments might be, they cannot in principle justify the correctness of Onsager's 'ideal turbulence' as a physical theory. In fact, the final judge of the truth of any physical theory is experiment, not mathematics. I wholeheartedly agree with the famous dictum of Einstein:

> Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit. [Insofar as the propositions

of mathematics refer to reality, they are not certain, and insofar as they are certain, they do not refer to reality.] (Einstein 1921, p. 3)

Even if Onsager's conjectures could be rigorously derived from a fundamental model such as the Landau–Lifschitz fluctuating hydrodynamic equations (2.5), serious questions arise whether those equations themselves might break down locally in space–time in the vicinity of extreme intermittent events (Bandak *et al.* 2022). Instead, the novel predictions of Onsager's theory must be carefully compared with experiments to test their validity. In fact, we know already from the brief review of experimental observations in § 2 that Onsager's $1/3$ Hölder result cannot be the whole story, because it makes no reference to solid walls and, in particular, it does not elucidate the difference in the wall geometries where anomalous dissipation occurs and where it does not. I discuss this crucial issue more in § 4.

3.2.1. *Existence of dissipative Euler solutions*

The program which has successfully led to a mathematical construction of the weak Euler solutions conjectured by Onsager (1949) in breakthrough work by Isett (2018) was initiated by the works of De Lellis & Székelyhidi (2009) and De Lellis & Székelyhidi (2010), which both appeared as preprints already in 2007. A very good review of this early work is given by De Lellis & Székelyhidi (2013) and an updated review is to be found in De Lellis & Székelyhidi (2019). These authors pointed out a remarkable connection of Onsager's conjecture with famous work of the mathematician John Nash (1954) on C^1 isometric embeddings. I give here a very succinct review, following closely the discussions in the previous references.

Nash (1954) addressed a classical problem of differential geometry, whether a smooth manifold M of dimension $n \geq 2$ with Riemannian metric g may be isometrically imbedded in m-dimensional Euclidean space $\mathbb{R}^m$, i.e. whether a C^1 embedding map $\boldsymbol{u} : M \to \mathbb{R}^m$ exists so that the Riemannian metric induced by the embedding agrees with g, or

$$\partial_i \boldsymbol{u} \boldsymbol{\cdot} \partial_j \boldsymbol{u} = g_{ij}. \tag{3.53}$$

To answer this question, Nash considered a more general problem of short embeddings which do not preserve lengths of curves on M but can only decrease lengths, so that

$$\partial_i \bar{\boldsymbol{u}} \boldsymbol{\cdot} \partial_j \bar{\boldsymbol{u}} \leq g_{ij} \tag{3.54}$$

in the matrix sense. The startling result obtained by Nash, with some improvement due to Kuiper (1955), is the following:

NASH–KUIPER THEOREM. *Let* (M, g) *be a smooth closed n-dimensional Riemannian manifold, and let* $\bar{\boldsymbol{u}} : M \to \mathbb{R}^m$ *be a* C^∞ *strictly short embedding with* $m \geq n + 1$. *For any* $\epsilon > 0$, *there exists a* C^1 *isometric embedding* $\boldsymbol{u} : M \to \mathbb{R}^m$ *with* $|\boldsymbol{u} - \bar{\boldsymbol{u}}|_{C^0} < \epsilon$.

This result is surprising for two reasons. First, the condition (3.53) is a set of $n(n+1)/2$ equations in m unknowns. A reasonable guess would be that the system is solvable, at least locally, when $m \geq n(n+1)/2$ and this indeed was a classical conjecture of Schläfli. However, for $n \geq 3$ and $m = n + 1$, the system (3.53) is hugely overdetermined! It is not obvious that there should be any solutions at all, but the Nash–Kuiper theorem shows that there exists an enormous set of C^1-solutions that are C^0-dense in the set of short embeddings. I shall not discuss here the details of Nash's proof of his astonishing result, but just remark that his construction of the isometry $\boldsymbol{u}$ was in a series of stages, by adding at

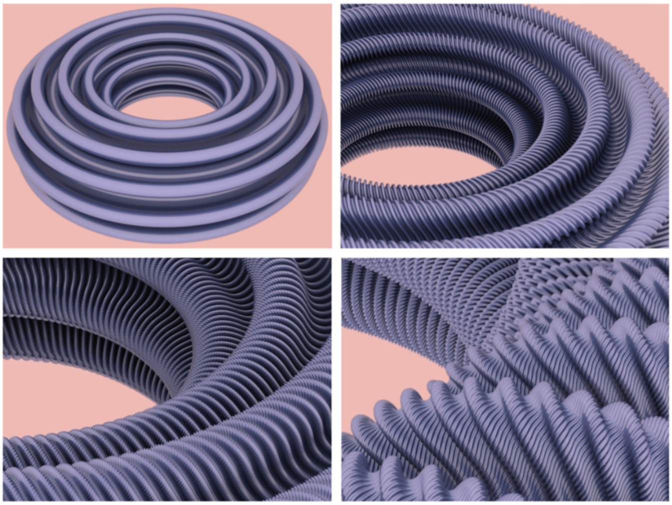

Figure 6. First four stages in the iterative construction of a C^1-isometric embedding of the flat 2-torus $\mathbb{T}^2$ into $\mathbb{R}^3$. The initial map is corrugated along the meridians to increase their length. Corrugations are then applied repeatedly in various directions to produce a sequence of maps. Each successive map is strictly short, with reduced isometric default. Reproduced from Borrelli *et al.* (2012), with permission of PNAS.

each stage a new small, high-frequency perturbation. The construction is not an 'abstract nonsense' argument that requires the Axiom of Choice or some other non-constructive element, and, in fact, the entire procedure can be implemented to any finite stage of iteration, in principle, by a computer algorithm. For example, see the work of Borrelli *et al.* (2012) which calculates numerically a C^1-isometric embedding of the flat 2-torus $\mathbb{T}^2$ into $\mathbb{R}^3$ and figure 6 for the first four stages of iteration. It is interesting to note that Nash himself regarded his 1954 paper as a 'sidetrack' on the problem of isometric embeddings (Raussen & Skau 2016) and attached greater importance to his later work that constructed C^∞ embeddings in higher dimensions (Nash 1956). The latter paper is notable for its introduction of the Nash–Moser implicit function theorem and also for its exploitation of a smooth mollification scheme with a continuous parameter ℓ, involving a careful study of variations with ℓ similar to an RG approach (L. Székelyhidi, Jr., private communication). A very intriguing recent stochastic formulation of Menon (2021) and Inauen & Menon (2023) makes this analogy more clear.

The fundamental contribution of De Lellis and Székelyhidi Jr. was to realise that there is a very close mathematical analogy between the problem of isometrically embedding a smooth manifold by a map of low regularity and the problem of solving the Cauchy initial-value problem for incompressible Euler equations by a velocity field of low regularity, and that Nash's method of construction can be carried over to the latter. The analogue of a 'short mapping' for the Euler system is what De Lellis and Székelyhidi Jr. call a smooth subsolution, i.e. a smooth triple $(\bar{\boldsymbol{u}}, \bar{p}, \bar{\boldsymbol{\tau}})$ with $\bar{\boldsymbol{\tau}}$ a symmetric, positive-definite tensor such that

$$\partial_t \bar{\boldsymbol{u}} + \boldsymbol{\nabla} \boldsymbol{\cdot} (\bar{\boldsymbol{u}}\, \bar{\boldsymbol{u}} + \bar{\boldsymbol{\tau}}) = -\boldsymbol{\nabla} \bar{p}, \quad \boldsymbol{\nabla} \boldsymbol{\cdot} \bar{\boldsymbol{u}} = 0. \tag{3.55}$$

Everyone from the turbulence modelling community will recognise at once that this has the form of an LES model equation incorporating a positive-definite 'turbulent stress' tensor $\bar{\boldsymbol{\tau}}$. Approximation results can be obtained for subsolutions entirely analogous to the

Nash–Kuiper theorem on approximation of short embeddings. As an example, a theorem of Buckmaster *et al.* (2018) states the following result.

THEOREM (Buckmaster *et al.* 2018). *Let* $(\bar{\boldsymbol{u}}, \bar{p}, \bar{\boldsymbol{\tau}})$ *be any smooth, strict subsolution of the Euler equations on* $\mathbb{T}^3 \times [0, T]$ *and let* $h < 1/3$. *Then there exists a sequence* $(\boldsymbol{u}_k, p_k)$ *of weak Euler solutions such that* $\boldsymbol{u}_k \in C^h(\mathbb{T}^3 \times [0, T])$ *satisfy, as* $k \to \infty$,

$$\int_{\mathbb{T}^3} \mathrm{d}^3x f\, \boldsymbol{u}_k \to \int_{\mathbb{T}^3} \mathrm{d}^3x f \bar{\boldsymbol{u}}, \quad \int_{\mathbb{T}^3} \mathrm{d}^3x f\, \boldsymbol{u}_k\boldsymbol{u}_k \to \int_{\mathbb{T}^3} \mathrm{d}^3x f (\bar{\boldsymbol{u}}\, \bar{\boldsymbol{u}} + \bar{\boldsymbol{\tau}}) \tag{3.56}$$

for all $f \in L^1(\mathbb{T}^3)$ *uniformly in time, and furthermore for all* $t \in [0, T]$ *and all* k,

$$\int_{\mathbb{T}^3} \mathrm{d}^3x \tfrac{1}{2}|\boldsymbol{u}_k|^2 = \int_{\mathbb{T}^3} \mathrm{d}^3x\, \tfrac{1}{2}(|\bar{\boldsymbol{u}}|^2 + \mathrm{Tr}\, \bar{\boldsymbol{\tau}}). \tag{3.57}$$

The notion of convergence in this theorem can be made more physically transparent by taking $f(\boldsymbol{r}) = \tilde{G}_\delta(\boldsymbol{x} + \boldsymbol{r})$ for $\delta > 0$, in which case the approximation property implies that pointwise,

$$\tilde{\boldsymbol{u}}_{k,\delta} \to \tilde{\bar{\boldsymbol{u}}}_\delta, \quad \tilde{\tau}_\delta(\boldsymbol{u}_k, \boldsymbol{u}_k) \to \tilde{\tau}_\delta(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}) + \tilde{\bar{\boldsymbol{\tau}}}_\delta. \tag{3.58}$$

Thus, if the subsolution $(\bar{\boldsymbol{u}}, \bar{p}, \bar{\boldsymbol{\tau}})$ arose from a coarse-grained Euler solution in the sense of Drivas & Eyink (2018), so that $\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}_\ell, \bar{p} = \bar{p}_\ell, \bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_\ell(\boldsymbol{u}, \boldsymbol{u})$, then the simple identity of Germano (1992) for the convolution filter $\tilde{\bar{G}}_{\ell,\delta} = \bar{G}_\ell * \tilde{G}_\delta$,

$$\tilde{\bar{\boldsymbol{\tau}}}_{\ell,\delta}(\boldsymbol{u}, \boldsymbol{u}) = \tilde{\tau}_\delta(\bar{\boldsymbol{u}}_\ell, \bar{\boldsymbol{u}}_\ell) + (\widetilde{\bar{\boldsymbol{\tau}}_\ell(\boldsymbol{u}, \boldsymbol{u})})_\delta, \tag{3.59}$$

implies that for any $\delta > 0$,

$$\tilde{\boldsymbol{u}}_{k,\delta} \to \tilde{\bar{\boldsymbol{u}}}_{\ell,\delta}, \quad \tilde{\tau}_\delta(\boldsymbol{u}_k, \boldsymbol{u}_k) \to \tilde{\bar{\tau}}_{\ell,\delta}(\boldsymbol{u}, \boldsymbol{u}) \tag{3.60}$$

pointwise in space. When $\delta \ll \ell$, this yields the remarkable statement that a coarse-grained Euler solution $(\tilde{\bar{\boldsymbol{u}}}_{\ell,\delta}, \tilde{\bar{p}}_{\ell,\delta}) \simeq (\bar{\boldsymbol{u}}_\ell, \bar{p}_\ell)$ at scale ℓ can be well approximated pointwise in space by a sequence $(\tilde{\boldsymbol{u}}_{k,\delta}, \tilde{p}_{k,\delta})$ of coarse-grained Euler solutions at the much smaller scale δ.

The constructions of such weak Euler solutions follow a strategy similar to that of Nash, by a sequence of stages $\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots$. At stage n, one has, after coarse-graining, a subsolution

$$\partial_t \boldsymbol{u}_n + \boldsymbol{\nabla} \boldsymbol{\cdot} (\boldsymbol{u}_n\boldsymbol{u}_n + \boldsymbol{\tau}_n) = -\boldsymbol{\nabla} p_n, \tag{3.61}$$

which is supported on wavenumbers $< \Lambda_n$. By adding a small-scale carefully chosen perturbation, one can succeed to cancel a large part of the stress $\boldsymbol{\tau}_n$ so that, in the limit, $\boldsymbol{\tau}_n \to \boldsymbol{0}$ weakly and one obtains a weak limit $\boldsymbol{u}$ which is a distributional Euler solution. A number of different models of the small scales have been employed, such as Beltrami flows (De Lellis & Székelyhidi 2013) and Mikado flows (Buckmaster *et al.* 2018; Isett 2018), together with other operations, such as evolving under smooth Euler dynamics locally in time and 'gluing' the different time-segments. In the language of LES modelling, the constructions can be regarded as a sort of iterative 'defiltering', in which unresolved scales are successively restored. A physicist might prefer to call this an 'inverse renormalisation group', the reverse of the successive coarse-graining employed by Wilson (1975). Clearly, there is a huge number of subsolutions, since one may adopt any positive definite tensor $\bar{\boldsymbol{\tau}}$ whatsoever. A consequence is therefore results such as the following theorem:

THEOREM (Buckmaster *et al.* 2018). *Let* $e : [0, T] \to \mathbb{R}^+$ *be any strictly positive, smooth function. Then for any* $0 < h < 1/3$*, there exists a weak Euler solution* $\boldsymbol{u} \in C^h(\mathbb{T}^3 \times [0, T])$ *such that*

$$\int_{\mathbb{T}^3} \mathrm{d}^3x \, \tfrac{1}{2}|\boldsymbol{u}|^2 = e(t). \tag{3.62}$$

In particular, one may take $e(t)$ to be any function strictly decreasing in time and then the Euler solution of the above theorem (globally) dissipates kinetic energy.

The details of the constructions are complex and rapidly evolving, so that they may be best gleaned from the journal articles themselves. I therefore make here just a few remarks. First, in addition to solutions that satisfy physical constraints, one can construct by the same methods solutions with obviously unphysical behaviours (e.g. kinetic energy increasing in time for unforced 3-D incompressible flows!) Thus, it is clear that the methods suffice to answer some very fundamental PDE questions, but they still miss much essential physics. Another important point is that the existence proofs of the weak Euler solutions are constructive, exactly as are the Nash constructions of isometric embeddings. Frisch, Székelyhidi & Matsumoto (2018) have attempted to compute numerically the weak Euler solutions constructed by Mikado flows (Buckmaster *et al.* 2018; Isett 2018). Visualisation, however, appears difficult with current constructions because of the very tiny scales of the added eddies. These weak Euler solutions can be regarded as a sophisticated sort of 'synthetic turbulence' (Juneja *et al.* 1994), manufactured space–time fields with many of the same properties as a physical turbulent flow velocity, but in addition solving exactly the inviscid equations of motion. Such solutions have been constructed which satisfy additional observed properties such as spatial intermittency (Novack & Vicol 2023; De Rosa & Isett 2024). We might hope in coming years to see weak Euler solutions with features in ever closer agreement with physical turbulent flows. As in the field of artificial intelligence and the Turing test, it would then be crucial to design tests of increasing sensitivity that could distinguish a true turbulent velocity field and such artificially constructed flows. This exercise could help to clarify better the essential properties that characterise physical turbulence.

3.2.2. *Non-uniqueness for the initial-value problem*

One very fundamental feature of the result of Nash (1954) which I have not yet duly emphasised is the profligate non-uniqueness of the C^1 isometric embeddings produced by his construction. This stands in stark contrast to embeddings with higher smoothness C^n, $n \geq 2$ for which the the Gauss–Codazzi equations relate the second fundamental form of the embedded surface to the Riemann curvature tensor. In fact, it is known that a compact Riemannian surface with positive Gauss curvature may be isometrically embedded into $\mathbb{R}^3$ by a C^2 map in just one way, up to a rigid-body motion. Thus it is clear that isometric embeddings have very different qualitative behaviour at low and high regularity, i.e. C^1 versus C^2, often referred to as a dichotomy between rigidity at high regularity versus flexibility at low regularity. This type of wild non-uniqueness at low regularity is a central aspect of the h-principle introduced by mathematician Mikhail Gromov (1971, 1986), with the isometric embedding problem as a primary example. The theorem of Buckmaster *et al.* (2018) that I cited in the previous subsection, as emphasised by those authors, is an extension of the h-principle to low-regularity weak solutions of the Euler equations, which exhibit a similar 'flexibility'. The name convex integration for the discussed techniques to construct these Euler solutions, by the way, goes back also to the work of Gromov (1971, 1986), who introduced it as one of several methods to prove the h-principle.

The origin of the name has to do with differential relations of the type $Df \in A$, which satisfy the h-principle when the convex hull of A contains a small neighbourhood of the origin.

Convex integration methods have fundamental implications for non-uniqueness of weak solutions to the Cauchy initial-value problem for the Euler equations, as discussed already by De Lellis & Székelyhidi (2010). Theorem 1 from that paper shook many previous expectations:

THEOREM (De Lellis & Székelyhidi 2010). *Let $d \geq 2$. There exist compactly supported divergence-free vector fields $\boldsymbol{u}_0 \in L^\infty$ for which there are infinitely many weak Euler solutions with that initial data, satisfying both the global energy equality*

$$\int_{\mathbb{R}^d} |\boldsymbol{u}(\boldsymbol{x}, t)|^2 \, \mathrm{d}^n x = \int_{\mathbb{R}^d} |\boldsymbol{u}(\boldsymbol{x}, s)|^2 \, \mathrm{d}^d x \quad \textit{for all } t > s \tag{3.63}$$

and the local energy equality

$$\partial_t \left(\tfrac{1}{2}|\boldsymbol{u}|^2\right) + \boldsymbol{\nabla} \boldsymbol{\cdot} \left[\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + p\right) \boldsymbol{u}\right] = 0. \tag{3.64}$$

Furthermore, there are infinitely many weak Euler solutions with that initial data satisfying the global energy inequality:

$$\int_{\mathbb{R}^d} |\boldsymbol{u}(\boldsymbol{x}, t)|^2 \, \mathrm{d}^d x < \int_{\mathbb{R}^n} |\boldsymbol{u}(\boldsymbol{x}, s)|^2 \, \mathrm{d}^d x \quad \textit{for all } t > s. \tag{3.65}$$

This result showed that one cannot add a local energy inequality

$$\partial_t \left(\tfrac{1}{2}|\boldsymbol{u}|^2\right) + \boldsymbol{\nabla} \boldsymbol{\cdot} \left[\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + p\right) \boldsymbol{u}\right] \leq 0 \tag{3.66}$$

(or even an equality) and obtain a unique weak solution to the Euler equations for certain L^∞ initial data. Furthermore, non-uniqueness occurs even if total energy is strictly decreasing. This is in stark contrast to simpler equations such as the inviscid Burgers model (or general hyperbolic scalar conservation laws) where it is known that insisting on such a local energy inequality selects a unique weak solution, which also coincides with the 'viscosity solution' obtained by incorporating viscosity and passing to the inviscid limit. The above theorem shows that such a simple selection principle for 'physical weak solutions' cannot succeed for the incompressible Euler equations.

Note that such initial data with non-unique solutions cannot have smoothness such as $C^{1+\epsilon}$ (velocity-gradients existing, with C^ϵ Hölder regularity) or higher smoothness, because this would violate the following important type of result.

THEOREM. *Let $\boldsymbol{u} \in L^\infty((0, T); L^2(\mathbb{T}^d))$ be a weak Euler solution, $\boldsymbol{U} \in C^1(\mathbb{T}^d \times [0, T])$ a strong solution, and assume that $\boldsymbol{u}$ and $\boldsymbol{U}$ share the same initial datum $\boldsymbol{u}_0$. Assume moreover that*

$$\int_{\mathbb{T}^d} |\boldsymbol{u}(\boldsymbol{x}, t)|^2 \, \mathrm{d}^d x \leq \int_{\mathbb{T}^d} |\boldsymbol{u}_0(\boldsymbol{x})|^2 \, \mathrm{d}^d x \tag{3.67}$$

for almost every $t \in (0, T)$. Then $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{U}(\boldsymbol{x}, t)$ for almost every $(\boldsymbol{x}, t)$.

Results of this type go by the name of strong-weak uniqueness. For a clear and lucid review, see Wiedemann (2018). The conclusion of such results is that any 'admissible' weak Euler solution, i.e. satisfying the global kinetic energy inequality (3.67), must coincide with a classical Euler solution, as long as that exists. Since local-in-time existence of a classical solution is guaranteed for $\boldsymbol{u}_0 \in C^{1+\epsilon}$, then the results in the cited theorem of De Lellis & Székelyhidi (2010) cannot hold for such initial data, at least for some finite time interval. Note, by the way, that strong-weak uniqueness applies even to 'measure-valued weak Euler solutions' such as those constructed by DiPerna & Majda (1987). See Wiedemann (2018).

These non-uniqueness results have since been considerably extended and the question is still currently under active investigation. Some very important results are contained in the more recent paper of Daneri, Runa & Szekelyhidi (2021), who prove the following theorem:

THEOREM (Daneri *et al.* 2021). *For any* $h \in (0, 1/3)$*, there is a set of divergence-free vector fields* $\boldsymbol{u}_0 \in C^h(\mathbb{T}^3)$ *which is a dense subset of the divergence-free vector fields in* $L^2(\mathbb{T}^3)$ *such that infinitely many Euler solutions exist with that initial data for which* $\boldsymbol{u}(t) \in C^{h'}(\mathbb{T}^3)$ *for all* $h' < h$, $t \in (0, T)$ *and for which the global energy inequality (3.67) holds.*

This theorem shows that the non-uniqueness holds right up to the critical Onsager 1/3 exponent and for a dense set of initial data. Thus, non-uniqueness is in some sense 'typical'. In a recent remarkable work of Giri, Kwon & Novack (2023), it has been shown by an L^3-generalisation of convex integration methods that weak Euler solutions exist, with $\boldsymbol{u} \in C([0, T], B_3^\sigma(\mathbb{T}^3))$ for any $\sigma < 1/3$, which are even locally dissipative, satisfying the local energy inequality (3.66) and not merely the global inequality (3.67). This result suggests that non-uniqueness may hold also for locally dissipative Euler solutions up to Onsager's $1/3$ exponent. To my knowledge, such results have been proved in the C^h-setting so far only for $h < 1/15$ by Isett (2022) and subsequently for $h < 1/7$ by De Lellis & Kwon (2022). Very intriguingly, De Lellis & Székelyhidi (2022) have argued heuristically that $h = 1/3$ should be the critical Hölder exponent not only for conservation of kinetic energy, but also for convex integration constructions. For a physicist, this remarkable coincidence smacks of a turbulent 'fluctuation-dissipation relation'.

These startling results on the 'flexibility' or non-uniqueness of the dissipative weak Euler solutions that were conjectured by Onsager pose a fascinating problem for their physical interpretation. One possible attitude is that additional conditions should be added to select the 'physically correct' weak solution obtained in the inviscid limit, presumed also to be unique. It is certainly true that the Euler solutions constructed by current convex integration methods have unphysical features and thus solutions obtained in the inviscid limit should plausibly have additional properties, e.g. the 'martingale property' of fluid circulations conjectured by Eyink (2006). However, I believe myself that the non-uniqueness or 'flexibility' uncovered by the convex integration theory is probably physically real and that, even if sufficient conditions are identified to select physical solutions, infinitely many, non-unique solutions for the same initial-data will remain. One motivation for this conjecture is the 'spontaneous stochasticity' phenomenon discovered by Bernard *et al.* (1998), which was proposed by Eyink (2014) to manifest not only in Lagrangian particle trajectories but also in the Eulerian evolution of the turbulent velocity field itself. Independently and much more concretely, Mailybaev (2016) presented strong evidence from numerical simulations that such Eulerian spontaneous stochasticity appears in the Sabra shell model in the high-*Re* limit. Mailybaev (2016) used another evocative term 'stochastic anomaly' for this phenomenon, stressing the close formal relation with the 'dissipative anomaly', and pointed out also connections with the classic paper of Lorenz (1969) on turbulence predictability. Indeed, the pioneering work of Lorenz (1969) nearly anticipated the modern concept of Eulerian spontaneous stochasticity, as is clear from the first sentences in the abstract of that paper:

> It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error. (Lorenz 1969, p. 289)

Stated differently, the claim of Lorenz (1969) was that deterministic fluid systems at high Reynolds numbers with 'many scales of motion' may exhibit stochastic solutions, even as random errors in the initial data are taken to zero. This is the essence of spontaneous stochasticity. The only thing missing from the concept of Bernard *et al.* (1998) is the clear understanding that such stochastic behaviour is made possible by non-uniqueness of the solutions of the initial-value problem for the limiting ideal evolution equation as $Re \rightarrow \infty$.

Mailybaev and his collaborators have gone on to provide numerical evidence for Eulerian spontaneous stochasticity in some prototypical fluid instabilities, such as the Rayleigh–Taylor mixing layer (Biferale *et al.* 2018) and a Kelvin–Helmholtz unstable vortex sheet (Thalabard, Bec & Mailybaev 2020). It is noteworthy that for both of these problems, Rayleigh–Taylor (Gebhard, Kolumbán & Székelyhidi 2021) and Kelvin–Helmholtz (Székelyhidi 2011; Mengual & Székelyhidi 2023), non-unique solutions of the Cauchy problem for the limiting inviscid equations have been demonstrated by convex integration methods. These examples suggest that spontaneous stochasticity must be a fairly widespread phenomenon in geophysics and astrophysics. Mailybaev & Raibekas (2023*b*) discuss also some simple multiscale models with known ergodic mixing properties, based on the Arnold cat map or irrational rotations of the circle, where Eulerian spontaneous stochasticity can be rigorously proved *a priori*. Furthermore, Mailybaev & Raibekas (2023*a*) have set up a general renormalisation group approach which can demonstrate Eulerian spontaneous stochasticity by identifying a suitable RG fixed point and establish also universality properties of the limiting spontaneous statistics. Finally, Bandak *et al.* (2024) provide evidence that the Landau–Lifschitz fluctuating Navier–Stokes equations (2.5) should exhibit spontaneously stochastic solutions in the limit $Re \rightarrow \infty$ for turbulent initial data, triggered just by thermal noise, by means of shell-model simulations.

I therefore believe that the non-uniqueness or 'flexibility' phenomenon uncovered by convex integration theory is manifested by the essential unpredictability of individual turbulent flow realisations, but whose statistics are universal and predictable. The ground-breaking work of Lorenz (1969) identified a physical mechanism for such turbulent unpredictability through an 'inverse error cascade', in which dissipation-scale errors propagate up to the largest scales of the flow in a time of the order of the large-eddy turnover time (Bandak *et al.* 2024). The ideas of Lorenz (1969) are paradigm-altering, fundamentally changing the methods and aims of scientific prediction. I believe that the revolutionary nature of the work by Lorenz (1969) was obscured by his use of a quasi-normal closure to enable numerical calculations, but which hid the underlying mathematical foundations of his ideas. The paper of Palmer, Döring & Seregin (2014) has recalled attention to the 'real butterfly effect' of Lorenz (1969) but, in my opinion, mis-identified the origin of the phenomenon in the non-uniqueness of solutions of the viscous Navier–Stokes dynamics. However, their proposal does not explain the observations of Thalabard *et al.* (2020) on the 2-D vortex sheet, for example, since the 2-D Navier–Stokes equation has unique solutions. I believe that the fundamental unpredictability identified by Lorenz (1969) has its origin instead in the non-uniqueness of solutions of the ideal Euler equations obtained in the limit $Re \to 0$ and the Eulerian spontaneous stochasticity associated with statistical distributions over those non-unique solutions (Bandak *et al.* 2024). I hope that proper understanding of the mathematical foundations will help to accelerate progress on this important problem.

We see furthermore that probability re-enters Onsager's 'ideal turbulence' theory, as an essential aspect. Although your stirred morning cup of coffee, or similar turbulent flows at higher Reynolds numbers, are described by a single velocity realisation, the same is not true if one considers the ensemble of flows that arise from all of the cups prepared over a long sequence of mornings. No matter how carefully one arranges to stir the coffee in the same manner each day, the individual flows in the cup and the patterns and whorls of cream will differ on each individual day. In fact, the coarse-grained Euler equation (3.8) becomes a stochastic equation when considered as an initial-value problem for $\bar{\boldsymbol{u}}_\ell$ alone, since the initial data for the unresolved eddies $\boldsymbol{u}'_\ell := \boldsymbol{u} - \bar{\boldsymbol{u}}_\ell$ are unknown. In that case, the turbulent stress term $\boldsymbol{\tau}_\ell$ becomes a random quantity (Eyink 1996), rationalising the introduction of 'eddy noise' (Rose 1977) or 'stochastic backscatter' (Leith 1990) in large-eddy simulation modelling. What spontaneous stochasticity tells us is that such randomness persists even in the idealised limit $\ell \to 0$. It is in the prediction of the future from present initial data that probability enters turbulence theory intrinsically.

3.2.3. *The infinite-Reynolds-number limit*

None of the weak Euler solutions discussed so far are obtained by taking infinite-Reynolds-number limits of the solutions of fluid equations valid at dissipation-range scales, such as the incompressible Navier–Stokes equation (2.4) (valid down to approximately the Kolmogorov scale η) or the incompressible Landau–Lifschitz equations (2.5) (valid down to approximately the mean-free-path length ℓ_{mfp}). The Euler solutions of physical relevance are just approximations to these more basic equations in the inertial range of length scales ($\ell \gg \eta$) and such solutions must possess very special properties not shared by general weak Euler solutions, including those manufactured by convex integration methods. Several very challenging problems remain in the investigation of the infinite-*Re* limits, which include the existence and uniqueness of the limits themselves. Another important issue is the selection problem, which is the question whether suitable constraints or conditions can be formulated which fully characterise *a priori* the Euler solutions obtained in the inviscid limit, without actually passing to the limit. The latter problem is important for potential practical applications, since one of the great hopes of Onsager's 'ideal turbulence' theory is that it might provide a shortcut to describe and compute the infinite-*Re* limit directly, avoiding the great expense to resolve the small dissipation-range scales. I provide here a very brief review of a few significant mathematical results that bear on these questions.

A basic problem in the theory of the inviscid limit is that the principal *a priori* bounds on the velocity field follow from the decay of kinetic energy and such 'L^2-bounds' alone provide insufficient compactness to guarantee that inviscid limits are standard weak Euler solutions, even along suitable subsequences. Without additional conditions, the inviscid limits are more general objects such as 'measure-valued Euler solutions' (DiPerna & Majda 1987; Wiedemann 2018) or clumsy 'dissipative Euler solutions' in the sense of Lions (1996). There are some researchers who take measure-valued solutions as the proper physical description and who investigate numerical schemes converging to such solutions (Fjordholm *et al.* 2017). However, an important observation was made by Chen & Glimm (2012) and Isett (2022) (see also Drivas & Eyink 2019; Drivas & Nguyen 2019) who observed that empirical observations on scaling laws of energy spectra and velocity structure functions guarantee strong inviscid limits that are standard weak Euler solutions. The criterion can be stated in terms of instantaneous structure functions of absolute velocity increments, where, as in (3.19), strict power-law scaling is not required but only a power-law upper bound of the form

$$S_p(\boldsymbol{r}, t) \leq C_p(t) U^p |\boldsymbol{r}/L|^{\zeta_p}, \quad \eta_p \leq |\boldsymbol{r}| \leq L, \tag{3.68}$$

where $\eta_p/L \to 0$ as $Re \to \infty$. If the pth-order moment condition (3.20) holds and if also

$$\bar{C}_p := \frac{1}{T}\int_0^T C_p(t)\,\mathrm{d}t < \infty \tag{3.69}$$

for some $p \geq 2$ and for some Re-independent values $\zeta_p > 0$ and $\bar{C}_p > 0$, then it follows that strong limits of Navier–Stokes solutions exist in $L^p([0, T] \times \mathbb{T}^d)$ along subsequences $Re_k \to \infty$ and the limiting velocities are standard weak Euler solutions. Of course, these solutions may not be unique for the same initial data $\boldsymbol{u}_0$ and different subsequences $Re_k \to \infty$ may yield different limiting Euler solutions. These results have recently been extended to the Landau–Lifschitz equations (2.5) in a probabilistic setting (Eyink & Peng 2024), where the argument now yields a probability distribution $\mathbb{P}$ on space–time velocity fields in the limit $Re_k \to \infty$ whose realisations, with probability one, are standard weak Euler solutions with the specified initial data $\boldsymbol{u}_0$ (either deterministic or random).

Perhaps the most impressive results that have been achieved to date are on the problem of scalar turbulence, in particular, for the viscous-convective regime described by the theories of Batchelor (1959) and Kraichnan (1968) for the limit of high Schmidt number. In the breakthrough work of Bedrossian *et al.* (2022), the Onsager 'ideal turbulence' predictions have been derived from first principles for the vanishing diffusivity limit of a passive scalar advected by a 2-D Navier–Stokes flow (or a 3-D hyperviscous Navier–Stokes flow) at fixed Reynolds number, when both the scalar advection–diffusion equation (3.42) and the Navier–Stokes equation (2.4) are forced by low-wavenumber random sources, Gaussian and white-in-time. Bedrossian *et al.* (2022) study the unique stationary measures $\mu^{Re,Sc}$ of the joint velocity-scalar fields $(\boldsymbol{u}, c)$ in the limit as $Sc = \nu/D \to \infty$ and establish the existence of limiting measures $\mu^{Re,\infty}$ for which the scalar concentration field c has the Batchelor k^{-1} spectrum and a constant scalar flux χ to infinite wavenumbers. Furthermore, Bedrossian *et al.* (2022) show that the scalar advection equation obtained from (3.42) in the formal limit $D \to 0$ still has unique weak solutions in a natural integral form and the limiting measures $\mu^{Re,\infty}$ are invariant under this limiting joint dynamics of $(\boldsymbol{u}, c)$ for $Sc \to \infty$. The analogue of Onsager's 1/3 criterion for dissipative weak solutions of the scalar in the Batchelor range advected by a smooth velocity field $\boldsymbol{u}$ is the statement that $c \notin B_2^{\sigma,\infty}$ for any $\sigma > 0$ and Bedrossian *et al.* (2022) prove that the scalar fields c for the limiting measures $\mu^{Re,\infty}$ indeed cannot possess such regularity. This is the only example I know where, in a quite physical setting, the analogue of Onsager's conjectures can be rigorously derived from first principles. Although probabilistic methods are not 'intrinsic' to this problem in the same sense as for spontaneous stochasticity, this is a beautiful example where a probabilistic approach is able to achieve physical results where purely deterministic techniques are still inadequate. Extending such results appears quite difficult. The enstrophy cascade of two-dimensional turbulence (Kraichnan 1967; Batchelor 1969) is physically very similar to the Batchelor regime, but here the vorticity ω is an active scalar with $Sc = 1$. The analogue of the Onsager theory for the 2-D enstrophy cascade (Eyink 2001; Lopes Filho, Mazzucato & Nussenzveig Lopes 2006) closely resembles the results derived by Bedrossian *et al.* (2022) for the Batchelor regime, but now the limit $Re \to \infty$ must be tackled directly to prove anything rigorously.

It seems very difficult to extend the results of Bedrossian *et al.* (2022) even to the passive scalar in the inertial-convective range, where likewise Sc is held fixed and $Re \to \infty$ or, equivalently, Péclet number $Pe = ScRe \to \infty$. However, there has been interesting progress on this problem by Colombo, Crippa & Sorella (2023), and especially by Armstrong & Vicol (2023) and Burczak, Székelyhidi & Wu (2023). None of these authors consider an advecting velocity field which satisfies the Navier–Stokes equations, but instead they construct 'synthetic turbulence' velocity fields which are divergence-free and such that the passive scalar satisfying the advection–diffusion equation (3.42) exhibits anomalous dissipation. The velocity fields in these constructions all have (in a suitable sense) the spatial Hölder regularity C^h with $0 < h < 1$ that is expected for an inertial-range velocity field and the advected scalar exhibits anomalous dissipation in the sense that, with dimensionless diffusivity $\hat{D} = 1/Pe$,

$$\limsup_{\hat{D}\to 0} \int_0^{\hat{T}} \mathrm{d}\hat{t} \int_{\mathbb{T}^2} \mathrm{d}^2\hat{x}\, \hat{D}|\hat{\nabla}\hat{c}|^2 > 0, \tag{3.70}$$

where $\widehat{(\cdot)}$ denotes outer-scale non-dimensionalisation. A striking result of these constructions is that there are distinct limit points $\hat{c}_*$ for the scalar field obtained along different subsequences $\hat{D}_k \to 0$, which give different weak solutions of the ideal advection equation for the same initial data $\hat{c}_0$. Similar non-uniqueness was found also by Huysmans & Titi (2023) for a passive scalar advected by a rougher divergence-free velocity field that is only L^∞ in space, and here the non-unique solutions of the ideal advection equation conserve the scalar energy. Such non-uniqueness is somewhat surprising for passive scalar advection. Recall that the scalar advection equation for the Batchelor regime at $Sc \to \infty$ was shown by Bedrossian *et al.* (2022) to have unique solutions and likewise the white-noise advection model of Kraichnan (1968) in the inertial-convective regime is known to have unique weak solutions of the scalar advection equations (Lototskii & Rozovskii 2004; note that the latter solutions are 'weak' in the PDE sense but strong in the stochastic sense, with exactly one solution for each realisation of the advecting random velocity field). The recent results raise the intriguing possibility of Eulerian spontaneous stochasticity also for passive scalar advection in the inertial-convective range.

There are important differences between the methods in the above papers which may be appreciated by reading the originals, but which deserve to be emphasised briefly here. See also the extensive review by Armstrong & Vicol (2023, § 1.1). The construction of Colombo *et al.* (2023) combines alternating shear flows that focus in a singular manner at the final time step. A drawback of this approach is that the anomalous scalar dissipation occurs only at the final time and the velocity field has only one active scale at each time, both features very uncharacteristic of physical turbulence. Armstrong & Vicol (2023) overcome these problems, synthesising a true multiscale velocity field for $h < 1/3$ that produces anomalous scalar dissipation at all times. Their approach is an iterative homogenisation technique which has much in common with renormalisation group methods in physics, generating an eddy-diffusivity at each scale by integrating over the smaller scales. Furthermore, the synthetic turbulent field which is constructed by Armstrong & Vicol (2023) has a very fundamental feature of true turbulent flows that small eddies are advected by larger eddies, which turns out to be essential in obtaining anomalous dissipation for the scalar at all times. To my knowledge, this is the first example of a synthetic turbulent velocity which captures the sweeping properties of a physical Navier–Stokes or Euler solution. Burczak *et al.* (2023) have extended this approach by combining it with convex integration techniques to ensure that the advecting velocity field is in fact a weak solution of the Euler equations.

The methods of Colombo *et al.* (2023) have recently been applied to prove the existence of kinetic energy dissipation anomaly for the forced 3-D Navier–Stokes equation (Bruè *et al.* 2023; Bruè & De Lellis 2023). For a sequence of body forces $\boldsymbol{f}^{Re}$ that satisfy some modest uniform smoothness properties, such as

$$\sup_{Re>0} \|\boldsymbol{f}^{Re}\|_{C([0,T],C^h(\mathbb{T}^3))} < \infty, \tag{3.71}$$

for any $h \in (0, 1)$, it has been proved that the viscous dissipation by the Navier–Stokes solution is anomalous for $\hat{\nu} = 1/Re \to 0$ in the sense that

$$\limsup_{\hat{\nu}\to 0} \int_0^{\hat{T}} \mathrm{d}\hat{t} \int_{\mathbb{T}^3} \mathrm{d}^3\hat{x}\, \hat{\nu}|\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}|^2 > 0. \tag{3.72}$$

Although the body force is not a smooth, large-scale forcing as usually assumed in turbulence theory, it is regular enough that the previous lim-sup vanishes for the linear Stokes equation, which lacks nonlinear energy cascade. Even more interestingly, the inviscid limits are found to be non-unique exactly as in the passive scalar case, with different subsequences of Reynolds numbers $Re_k \to \infty$ yielding distinct weak solutions of the Euler equations with the same initial data $\boldsymbol{u}_0$ and the same body force $\boldsymbol{f} = \lim_{Re\to\infty} \boldsymbol{f}^{Re}$ (Bruè *et al.* 2023).

4. Turbulence interactions with solid walls

4.1. *Overture on turbulence and solid surfaces*

All of the previous theoretical results that I have discussed involve turbulence away from walls and solid boundaries, although terrestrial turbulence arises most frequently at fluid–solid interfaces (and somewhat less often at gas–liquid interfaces). Surprisingly, it is only recently that the methods of Onsager (1949) have been applied to such problems, in what may be called the 'Onsager theory of wall-bounded turbulence'. Here I shall briefly review this recent work following Eyink (2023), where more details can be found. It is not entirely ahistorical to refer to an 'Onsager theory of wall-bounded turbulence', since Onsager (1949) mentioned turbulent pipe flow as an example of this type:

> Such a familiar type of turbulence as exists in a liquid flowing through a cylindrical tube is neither homogeneous nor isotropic. The mean fluctuations of the velocity vary over the cross-section of the tube, the local macroscale is generally comparable to the distance from the wall, and fluctuations as well as correlations are more or less anisotropic. (Onsager 1949, p. 283)

The observations reported in this quotation were attributed by Onsager to a paper by Montgomery (1943) on smooth- and rough-walled pipes. Onsager was also a friend of Theodore Theodorsen from their time in Trondheim together, the originator of the concept of 'hairpin vortices' in wall-bounded turbulence (Theodorsen 1952), and they reportedly discussed turbulence, among other scientific topics. I shall unfortunately not have space in this essay to explore how Onsager's ideas relate to the modern descendent of Theodorsen's ideas, the 'Attached Eddy Model' (Marusic & Monty 2019), although both attempt to describe the high-Reynolds-number limit of wall-bounded flows. Connections can be glimpsed from the results presented in §§ 4.2.3–4.2.5. In fact, we shall see that Onsager pursued his interest in wall-bounded turbulence further in his unpublished notes and that his calculations anticipate some modern research directions.

However, it is clear already from the brief summary of the experimental evidence presented in § 2 that Onsager's 1/3 Hölder result and the various analyses that follow it, ignoring the crucial role of the solid walls, cannot be a complete theory of the turbulent dissipative anomaly for incompressible fluid flows. Onsager's result applies indeed to wall-bounded turbulence, as we shall see, but it gives only a necessary condition for anomalies, without explaining the sufficient conditions that may generate them. In particular, the literature reports a sharp dichotomy in certain flows which have been characterised as 'internal' or 'closed' (Cadot *et al.* 1997), but which I believe are better described as 'wall-parallel', with the mean flow everywhere tangent to the solid boundary. In these flows – which include such common examples as flows through straight pipes and flat-walled channels, flat-plate boundary layers, and Taylor–Couette flows – there is no empirical evidence for energy dissipative anomaly with walls that are hydraulically smooth, but only with walls that are hydraulically rough. Furthermore, this dichotomy does not apply to turbulent wakes behind solid bodies, either bluff or streamlined, where there is solid evidence for a dissipative anomaly in the wake independent of the wall roughness. No theory of the turbulent dissipative anomaly can be complete which does not account for these crucial dependencies on the nature of the wall. It is these conundrums, among others, which makes the problem of high-*Re* wall-bounded turbulence such a fascinating science problem.

It should be emphasised at the outset that the very meaning of 'infinite-Reynolds-number limit' becomes ambiguous for flows with wall roughness, and such roughness is always present in any physical surface to some degree (Jiménez 2004). The difficulty is that the typical roughness height k introduces a new length scale in addition to the overall dimension L of the flow and thus Reynolds' law of hydrodynamic similarity breaks down. Dimensional analysis yields that, in addition to the usual Reynolds number $Re = UL/\nu$, there is another dimensionless number group, which may be taken to be either the length ratio $\hat{k} = k/L$ or else the 'roughness Reynolds number' $Re_k = Uk/\nu$. It now matters greatly how the parameter regime $Re \gg 1$ is reached. In a fixed laboratory apparatus of dimension L, it is most common to vary Re by varying the flow speed U, but this has the effect also of varying Re_k. When $Re_k \simeq 1$, the transitionally rough regime, then roughness effects begin to be felt and these effects become dominant in the fully rough regime for $Re_k \gg 1$. The high-*Re* limit with smooth walls that is most popular with mathematicians instead corresponds to flow around bodies of large size L so that $Re = UL/\nu \gg 1$ while simultaneously $Re_k \lesssim 1$ and $\hat{k} \ll 1$.

Another crucial difference with turbulence away from solid walls is that kinetic energy is not the only inviscid invariant of smooth Euler solutions that is subject to possible dissipative anomalies. For example, fluid linear momentum is strictly conserved in smooth wall-parallel Euler flows, as pressure stress cannot transfer the wall-parallel momentum component normal to the wall. Instead, in viscous flow, the skin friction vector

$$\boldsymbol{\tau}_w = 2\nu \boldsymbol{S} \boldsymbol{\cdot} \boldsymbol{n}, \tag{4.1}$$

with $\boldsymbol{n}$ the surface unit normal pointing into the fluid, can transfer parallel momentum, possibly even in the limit as $Re \to \infty$. This possibility seems to have been first raised by Taylor (1915) in a paper on eddy motion in the atmosphere:

> ...a very large amount of momentum is communicated by means of eddies from the atmosphere to the ground. This momentum must ultimately pass from the eddies to the ground by means of the almost infinitesimal viscosity of the air. The actual value of the viscosity of the air does not affect the rate at which momentum is communicated to the ground, although it is the agent by means of which the transference is effected.
>
> ...
>
> The finite loss of momentum at the walls due to an infinitesimal viscosity may be compared with the finite loss of energy due to infinitesimal viscosity at a surface of discontinuity in a gas. (Taylor 1915, pp. 25–26)

It is remarkable that Taylor in this early paper not only recognised that there could be a finite loss of momentum due to an 'infinitesimal viscosity', but also compared this phenomenon with discontinuous shock solutions which we now understand to be described, in modern language, by weak solutions of inviscid fluid equations. This proposal goes well beyond the laminar boundary-layer theory of Prandtl (1905). In addition to momentum, another conserved quantity is the total volume-integrated vorticity, which vanishes identically according to the theorem of Föppl (1913) and is thus, trivially, conserved. However, it has been emphasised by Lighthill (1963) and Morton (1984) that this 'trivial' conservation law imposes very important constraints on the flow and, in particular, vorticity has a local source at solid surfaces of the form

$$\boldsymbol{\sigma} = -\boldsymbol{n} \times \nabla p = \boldsymbol{n} \times \nu(\nabla \times \boldsymbol{\omega}). \tag{4.2}$$

The middle expression in (4.2) describes the creation of vorticity by tangential pressure gradients (which occurs even in smooth Euler flows) and the final expression describes the viscous diffusion of vorticity from the surface, with both terms instantaneously in balance for stick boundary conditions (b.c.) $\boldsymbol{u} = \mathbf{0}$ on the velocity field. In a turbulent flow, the source (4.2) drives an 'inverse cascade of vorticity' away from the solid surface (Eyink 2008; Kumar, Meneveau & Eyink 2023) and one naturally wonders if this cascade may persist in the infinite-*Re* limit.

Before undertaking any mathematical analysis, it is crucial to review briefly some representative results from experiments and numerical simulations, providing a few more details than in § 2, so that we have some clear idea of the observations that require explanation. McKeon *et al.* (2004) report on experimental measurements of the friction factor (2.3) in a smooth-wall pipe with $Re_k \lesssim 1$, finding good agreement with the classical Prandtl–Kármán law for appropriate constants. Extrapolated to very high Reynolds numbers, this result gives $\lambda \sim C/(\ln Re)^2$ for $Re \gg 1$. Although there is considerable ongoing debate about the precise asymptotics, all observations are consistent with λ tending to zero as $Re \to \infty$, although much more slowly than the rate $\lambda \sim 64/Re$ for laminar pipe flow. Noting the simple relation $\partial \bar{p}/\partial x = 2\bar{\tau}_{w,x}/R$ from mean momentum balance, with R the pipe radius, we see that $\bar{\tau}_{w,x}/\bar{U}^2 \sim C'/(\ln Re)^2$ according to the results of McKeon *et al.* (2004). There is thus no 'strong momentum anomaly' in this flow of the type conjectured by Taylor (1915), but only a 'weak anomaly' in the terminology of Bedrossian *et al.* (2019). However, the experiment of Nikuradse (1933) with 'sand-grain' roughness, which varied *Re* by changing $\bar{U}$ for fixed values of $\hat{k} = k/R$, found that $\lambda \sim C\hat{k}^{1/3}$ for $Re \gg 1$. Although this experiment indicates a 'strong anomaly', it does not mean that $\bar{\tau}_{w,x}/\bar{U}^2$ tends to a non-vanishing value. In fact, various numerical simulations of rough-walled pipe and channel flow such as that of Busse, Thakkar & Sandham (2017) indicate that indeed $\bar{\tau}_{w,x}/\bar{U}^2 \to 0$ as $Re \to \infty$ and the non-vanishing drag is due to the net pressure force acting on the roughness elements, or the 'form drag'. There is therefore a close correspondence between the wall-parallel flows with hydraulically rough walls and the wake flows past bodies, where likewise the time-averaged skin friction tends to zero with increasing *Re* and non-vanishing drag is due to the asymmetric pressure forces acting on the body. A good example of the latter type is the flow past the sphere studied at high-*Re* by Achenbach (1972), who measured both skin friction and pressure profiles on the surface of the sphere. Experimental visualisations of flow around individual 'sand-grain' roughness elements in turbulent duct flow (Gao, Agarwal & Katz 2021) show remarkable similarity to flow around bluff bodies such as a sphere, with separating boundary layers and recirculation zones in the wake. The apparently non-vanishing surface pressure gradients in these flows suggest also a 'strong vorticity anomaly', due to persistent vorticity flux into the flow.

Most of the attempts to understand these observations have been based on the incompressible Navier–Stokes equation (2.4) in the flow domain Ω, with stick b.c. $\boldsymbol{u}=\boldsymbol{0}$ of the velocity on the boundary $\partial\Omega$ of the domain and kinematic pressure obtained from the Poisson problem with Neumann boundary conditions:

$$-\Delta p=\mathrm{tr}(\boldsymbol{\nabla}\boldsymbol{u})^2, \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n}=\boldsymbol{n}\boldsymbol{\cdot}\nu\Delta\boldsymbol{u}, \quad \text{on } \partial\Omega. \tag{4.3}$$

Of course, this mathematical model is just an approximation to reality for molecular fluids in nature and various refinements have been considered. For example, many analyses have considered also the slip conditions of Navier that determine a velocity slip $\delta\boldsymbol{u}_s$ at the wall

$$\delta\boldsymbol{u}_s=\ell_s\frac{\partial\boldsymbol{u}}{\partial n}, \quad \text{on } \partial\Omega \tag{4.4}$$

rather than stick b.c. for the velocity field, with slip length ℓ_s estimated by Maxwell to be of the order of the molecular mean free path length ℓ_{mfp}. One also cannot rule out effects of thermal fluctuations, since these generally appear at lengths much larger than ℓ_{mfp}. Here it should be noted that the correct formulation of the Landau–Lifschitz fluctuating hydrodynamics equations in the vicinity of a fluid–solid interface is still a hotly debated issue; see Reichelsdorfer (2016) for a very readable summary. I leave such issues mostly to the side in my review below, where I shall discuss how far the 'standard model' of incompressible Navier–Stokes with stick b.c. can be used to explain the empirical observations. I just note finally that the various conditions for convergence of Navier–Stokes solutions to weak Euler solutions in the limit $Re\rightarrow\infty$, which were discussed in § 3.2.3, carry over also to wall-bounded flows. See Drivas & Nguyen (2019) for an excellent overview of such results.

4.2. *Onsager RG analysis*

4.2.1. *Regularisation of ultraviolet divergences*

A key conclusion that may be be drawn from the brief summary of empirical observations in the previous section is that new UV divergences appear at the solid interfaces in wall-bounded turbulence for $Re\gg 1$. In fact, whenever the non-dimensionalised wall shear stress $\hat{\boldsymbol{\tau}}_w=\boldsymbol{\tau}_w/U^2=(1/Re)(\partial\hat{\boldsymbol{u}}/\partial\hat{n})$ vanishes more slowly than the laminar rate $\propto 1/Re$, then the non-dimensionalised velocity-gradient at the wall diverges:

$$\lim_{Re\rightarrow\infty}\frac{\partial\hat{\boldsymbol{u}}}{\partial\hat{n}}=\infty. \tag{4.5}$$

Note that such divergences appear at the wall even if there is only a 'weak anomaly' in the turbulent momentum balance. To obtain a dynamical description that remains valid in the limit $Re\rightarrow\infty$, these new divergences at the wall must be regularised along with the divergences of the non-dimensionalised gradients $\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}$ in the bulk of the flow.

A recent paper of Bardos & Titi (2018) has initiated the mathematical investigation of Onsager's 'ideal turbulence' theory for wall-bounded flows, followed already by several works with improvements (Drivas & Nguyen 2018; Bardos *et al.* 2019; Chen, Liang & Wang 2022). These works employ as regulariser a modification of the spatial coarse-graining of the velocity defined in (3.2). It is convenient to assume that the filter kernel G is supported in a ball of radius 1, so that the definition of $\bar{\boldsymbol{u}}_\ell(\boldsymbol{x},t)$ makes

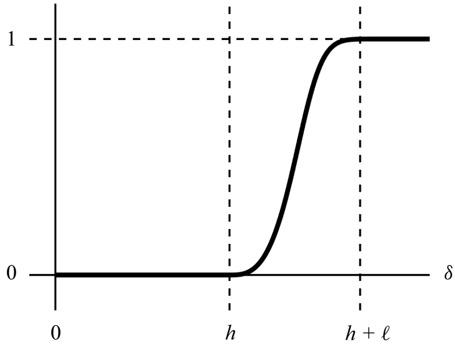

Figure 7. Window function $\theta_{h,\ell}$ used to screen from observation fluid eddies near the wall.

sense for points $\boldsymbol{x} \in \Omega$ with distance at least ℓ from the boundary $\partial\Omega$. To eliminate also the divergences of the velocity-gradients at the wall and to obtain a well-defined coarse-grained velocity, one may also smoothly 'window out' eddies at distances $< h$ to the wall, with $h > \ell$. This can be accomplished by taking a smooth windowing function $\theta_{h,\ell}(\delta)$ with the properties that

$$\theta_{h,\ell}(\delta) = \begin{cases} 0, & \delta < h, \\ 1, & \delta > h + \ell \end{cases} \tag{4.6}$$

and $\theta_{h,\ell}(\delta)$ monotone increasing on the interval $[h, h+\ell]$ (see figure 7). Finally, one defines $\eta_{\ell,h}(\boldsymbol{x}) := \theta_{\ell,h}(d(\boldsymbol{x}))$ and for all $\boldsymbol{x} \in \Omega$,

$$\tilde{\boldsymbol{u}}_{\ell,h}(\boldsymbol{x}, t) = \eta_{\ell,h}(\boldsymbol{x})\bar{\boldsymbol{u}}_\ell(\boldsymbol{x}, t), \tag{4.7}$$

where $d(\boldsymbol{x})$ measures the distance of $\boldsymbol{x} \in \Omega$ to the boundary:

$$\mathrm{d}(\boldsymbol{x}) = \inf_{\boldsymbol{y}\in\partial\Omega} |\boldsymbol{x} - \boldsymbol{y}|. \tag{4.8}$$

With this definition, $\nabla\, \mathrm{d}(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{y}_{\boldsymbol{x}}) := \boldsymbol{n}(\boldsymbol{x})$, where $\boldsymbol{n}(\boldsymbol{y})$ is the inward-pointing unit normal vector at a point $\boldsymbol{y} \in \partial\Omega$ and $\boldsymbol{y}_{\boldsymbol{x}} \in \partial\Omega$ is the point at which the infimum in (4.8) is achieved for each $\boldsymbol{x} \in \Omega$. See Bardos & Titi (2018). The coarse-grained velocity defined by (4.7) may be described picturesquely as the fluid velocity seen by an observer who is myopic and who also has tunnel vision, with parameter ℓ characterising the blurriness of their eyesight and h their loss of peripheral vision.

It is important to emphasise here that this specific choice of regularisation is not essential to the theory. Just as in quantum field theory, there are many possible regularisers to eliminate UV divergences (Gross 1976), and it is ultimately a matter of convenience and ease of application which to choose for a particular problem. Onsager in his unpublished research notes in fact explored a different regularisation method for plane-parallel channel flow based on expansion of the velocity field in eigenmodes of the linear Stokes operator. See the folder of notes in the online Onsager Archive at https://ntnu.tind.io/record/121186, pp. 4–8, where Onsager solves the eigenproblem for the Stokes operator $A_D\boldsymbol{w} := P(-\Delta)\boldsymbol{w}$ with Dirichlet b.c. $\boldsymbol{w} = \boldsymbol{0}$, where P is the Leray projection onto divergence-free fields:

$$A_D\boldsymbol{w}_n = \lambda_n\boldsymbol{w}_n, \quad n = 0, 1, 2, \ldots. \tag{4.9}$$

Identical results were published later by Rummler (1997). Onsager clearly had the idea to expand the turbulent velocity field in the channel into such eigenmodes, thus regularising all UV divergences. With a smooth exponential cutoff, this would correspond to taking

$$\bar{\boldsymbol{u}}_\ell = \sum_{n=0}^{\infty} e^{-\lambda_n \ell^2} c_n \boldsymbol{w}_n \tag{4.10}$$

for some regularisation scale ℓ, where $\boldsymbol{u} = \sum_{n=0}^{\infty} c_n \boldsymbol{w}_n$. With this approach, $\nabla \boldsymbol{\cdot} \bar{\boldsymbol{u}}_\ell = 0$ in Ω exactly and $\bar{\boldsymbol{u}}_\ell = \boldsymbol{0}$ on $\partial\Omega$. Note that the above expansion is equivalent to defining the regularised velocity field by the auxiliary initial-value problem for the Stokes equation:

$$\frac{\partial \bar{\boldsymbol{u}}_\ell}{\partial \ell^2} = -A_D \bar{\boldsymbol{u}}_\ell, \quad \bar{\boldsymbol{u}}_\ell|_{\ell=0} = \boldsymbol{u}. \tag{4.11}$$

Exactly this PDE approach to spatial filtering in wall-bounded domains was proposed recently by Johnson (2022), which can be regarded as a version of the heat-kernel regularisation of Isett & Oh (2016), adapted to manifolds with boundaries. This PDE approach is more flexible than the original eigenmode-expansion of Onsager, since it is difficult for general domains to calculate the eigenfunctions explicitly. Note that one may apply this Stokes regularisation also with the operator A_N for Neumann b.c. $\partial \boldsymbol{w}/\partial n = \boldsymbol{0}$ on $\partial\Omega$, so that $\int_\Omega \bar{\boldsymbol{u}}_\ell \, \mathrm{d}V = \int_\Omega \boldsymbol{u} \, \mathrm{d}V$.

In general, the necessity of some spatial filtering or coarse-graining closely connects the Onsager theory with the turbulence modelling method of LES. Note that wall-bounded turbulence is currently one of the most challenging areas of LES research and the subject of intense effort (Bose & Park 2018). Many other types of spatial filtering or regularisation have been explored in that context, including regularisations specified by elliptic PDE problems such as

$$\bar{\boldsymbol{u}}_\ell - \frac{\partial}{\partial x_k}\left(\ell^2 \frac{\partial \bar{\boldsymbol{u}}_\ell}{\partial x_k}\right) = \boldsymbol{u}, \quad \text{in } \Omega, \tag{4.12}$$

for some chosen length scale $\ell(\boldsymbol{x})$, which may now be chosen to depend on position. See Germano (1986*a*,*b*), Bose & Moin (2014) and Bae *et al.* (2019). As stated in the quote of Onsager (1949), the integral length in wall-bounded turbulence is generally proportional to the distance from the wall, or $L(\boldsymbol{x}) \sim \kappa \, \mathrm{d}(\boldsymbol{x})$ with κ the so-called Kármán constant. Thus, one might be tempted to take $\ell(\boldsymbol{x}) \propto L(\boldsymbol{x})$, which would correspond to integrating out all eddies smaller than the local integral length. This so-called 'wall-resolved LES' is, however, almost as computationally expensive as DNS of the full Navier–Stokes equation (Yang & Griffin 2021), so that one is generally interested in taking $\ell(\boldsymbol{x}) \gg L(\boldsymbol{x})$ for practical applications. This problem of coarse-graining and LES modelling in the presence of solid walls is an important application area where mathematicians and engineers can very productively interact.

4.2.2. *Coarse-grained equations*

The filtering procedures employed in LES lead to effective coarse-grained equations which contain quantities unclosed in terms of the coarse-grained velocity itself, which may then be modelled. The same is true also of the regularisation procedures used by mathematicians to study the infinite-Reynolds limit. For example, the combined spatial filtering and windowing introduced by Bardos & Titi (2018) when applied to the Navier–Stokes equation (2.4) leads to an exact regularised equation for the coarse-grained field $\tilde{\boldsymbol{u}}_{\ell,h}$ of the form

$$\frac{\partial \tilde{\boldsymbol{u}}_{\ell,h}}{\partial t} + \nabla \boldsymbol{\cdot} \left[\tilde{\tau}_{\ell,h}(\boldsymbol{u},\boldsymbol{u}) + \tilde{\boldsymbol{u}}_{\ell,h}\tilde{\boldsymbol{u}}_{\ell,h} + \tilde{p}_{\ell,h}\boldsymbol{I} - \nu\widetilde{\nabla\boldsymbol{u}}_{\ell,h}\right] = -\boldsymbol{f}_{\ell,h}. \tag{4.13}$$

Here, the turbulent (or subgrid) stress due to eddies of size $< \ell$ may be defined as usual by

$$\tilde{\tau}_{\ell,h}(\boldsymbol{u},\boldsymbol{u}) = (\widetilde{\boldsymbol{u}\boldsymbol{u}})_{\ell,h} - \tilde{\boldsymbol{u}}_{\ell,h}\tilde{\boldsymbol{u}}_{\ell,h}. \tag{4.14}$$

In addition, a new inertial drag force appears associated with the eliminated near-wall eddies

$$\boldsymbol{f}_{\ell,h} = -\theta'_{\ell,h}(d(\boldsymbol{x}))\,\boldsymbol{n}(\boldsymbol{x}) \boldsymbol{\cdot} \left[(\overline{\boldsymbol{u}\boldsymbol{u}})_\ell + \bar{p}_\ell\boldsymbol{I} - \nu\nabla\bar{\boldsymbol{u}}_\ell\right], \tag{4.15}$$

and represents momentum transfer to the unresolved near-wall eddies. It is non-vanishing only for $h < d(\boldsymbol{x}) < h+\ell$. This force includes resolved viscous momentum diffusion, but nonlinear transfer dominates for $Re \gg 1$ at fixed ℓ, h. Mathematically, the regularised equation (4.13), when considered for all possible choices of $h > \ell$, is equivalent to the standard weak formulation of the incompressible Navier–Stokes equation in a domain with boundary (see Boyer & Fabrie 2012, Ch. V.1.2).

If (4.13) were considered for the purpose of LES modelling, both the subgrid stress $\tilde{\boldsymbol{\tau}}_{\ell,h}$ and the inertial drag force $\boldsymbol{f}_{\ell,h}$ would need to be somehow closed or parametrised. Indeed, one can expect that these quantities have universal statistical properties independent of the small-scale dissipation and of the detailed properties of the wall, if the distance h is chosen in the inertial sublayer and length scale ℓ is chosen also sufficiently small. In addition, however, another quantity must be modelled which appears in the coarse-grained mass balance:

$$\nabla \boldsymbol{\cdot} \tilde{\boldsymbol{u}}_{\ell,h} = \tilde{\sigma}_{\ell,h}, \tag{4.16}$$

which I call the inertial mass source

$$\tilde{\sigma}_{\ell,h} := \theta'_{h,\ell}(d(\boldsymbol{x}))\boldsymbol{n}(\boldsymbol{x}) \boldsymbol{\cdot} \bar{\boldsymbol{u}}_\ell. \tag{4.17}$$

This quantity measures the mass-exchange with the unresolved near-wall eddies and it is also non-vanishing only for $h < d(\boldsymbol{x}) < h+\ell$, with furthermore a vanishing space integral $\int_{h<d<h+\ell} \tilde{\sigma}_{\ell,h}\,\mathrm{d}V = 0$. The windowing operation has thus introduced effective 'compressibility' so that the Poisson equation for coarse-grained pressure becomes

$$-\triangle\tilde{p}_{\ell,h} = \partial_t\tilde{\sigma}_{\ell,h} + \nabla\nabla : \left[\tilde{\boldsymbol{u}}_{\ell,h}\tilde{\boldsymbol{u}}_{\ell,h} + \tilde{\boldsymbol{\tau}}_{\ell,h} - \nu\widetilde{\nabla\boldsymbol{u}}_{\ell,h}\right] - \nabla \boldsymbol{\cdot} \boldsymbol{f}_{h,\ell}, \tag{4.18}$$

which involves derivatives of all three modelled quantities but which can yield the resolved pressure by applying standard Poisson solvers. Similar effects appear with other coarse-graining methods, e.g. the velocity field from the elliptic filtering (4.12) satisfies an equation $\nabla \boldsymbol{\cdot} \bar{\boldsymbol{u}}_\ell = \bar{\sigma}_\ell$ analogous to (4.16) with

$$\bar{\sigma}_\ell = \overline{\left(\frac{\partial}{\partial x_k}\left[\nabla\ell^2 \boldsymbol{\cdot} \frac{\partial \bar{\boldsymbol{u}}_\ell}{\partial x_k}\right]\right)}_\ell. \tag{4.19}$$

In this case, however, it is not true that $\int_\Omega \bar{\sigma}_\ell\,\mathrm{d}V = 0$ unless $\partial\bar{\boldsymbol{u}}_\ell/\partial n = 0$ at $\partial\Omega$.

The regularised equation (4.13) involves two arbitrary lengths ℓ and h and, as well, three arbitrary functions G, η and d. Other choices of distance function might be more useful for some purposes, e.g. $d(\boldsymbol{x}) = \min\{|y-H|, |y+H|\}$ could be useful in an iterative RG analysis of a rough-wall channel flow, with H the channel half-width, to establish universal statistics in the 'inertial sublayer'. A similar type of 'time windowing' was employed in the recent RG analysis of Lagrangian spontaneous stochasticity (Eyink & Bandak 2020), where it corresponds to ignoring non-universal initial times of the particle position histories. The 'principle of renormalisation group invariance' is that no objective physics can depend upon these arbitrary quantities introduced for the purpose of regularisation (Gross 1976; Eyink 2018*b*). The present example is a case of a several-parameter renormalisation group involving changes of the entire regularisation scheme, which was encountered already in quantum field theory (Stückelberg & Petermann 1953; Stevenson 1981; Peterman 1982) and which has been applied since to PDEs, including boundary-value problems (Chen, Goldenfeld & Oono 1996; Kovalev & Shirkov 1999). A key idea in RG methods is that arbitrariness in regularisation parameters may be exploited by choosing them in some optimal way to deduce non-trivial consequences. I describe some applications of that principle in the following section.

4.2.3. *Momentum cascade in space*

Most of the prior work on the Onsager theory for wall-bounded flows has considered energy cascade and energy dissipation (Bardos & Titi 2018; Drivas & Nguyen 2018; Bardos *et al.* 2019; Chen *et al.* 2022), but I prefer to begin first with the momentum anomaly suggested by Taylor (1915). I shall employ the regularisation by spatial filtering and windowing introduced by Bardos & Titi (2018), which leads to the important concept of spatial momentum cascade. To avoid any confusion, I emphasise at the outset that the notion of momentum cascade arising in the Onsager theory does not coincide with the one which is now standard in the literature on wall-bounded turbulence (Tennekes & Lumley 1972; Jiménez 2012). I shall discuss the essential differences below. My analysis shall follow closely the work of Quan & Eyink (2022*a*), where momentum balance in the limit $Re \to \infty$ is treated. It is also possible to extend this analysis to finite Reynolds numbers and some beginning steps in this direction have been taken by Eyink, Kumar & Quan (2022), following the ideas of Drivas & Eyink (2019) for energy cascade, but I consider here only the regime $Re \gg 1$.

The work of Quan & Eyink (2022*a*) followed the same RG strategy as in prior works on the Onsager theory such as Duchon & Robert (2000), by considering the limit $Re \to \infty$ for both the fine-grained description and the coarse-grained description with regularisation scales ℓ, h. Non-trivial consequences can be deduced simply by requiring that both descriptions lead to the same observable conclusions and exploiting the freedom to vary h, ℓ by taking $h, \ell \to 0$. The specific example analysed by Quan & Eyink (2022*a*) was external flow around a finite, smooth body B, as considered in the famous paradox of d'Alembert (1749, 1768). As seen from the empirical results reviewed in § 4.1, this is probably the simplest flow which provides evidence for a strong dissipation anomaly. However, the analysis of Quan & Eyink (2022*a*) carries over straightforwardly to related more complex flows, such as turbulent flows through pipes with hydraulically rough (but mathematically smooth) walls. Since only mathematically smooth surfaces are considered, we are investigating the limit $Re = UL/\nu \gg 1$ as achieved by taking L very large (where L would be a length such as the diameter of the body or the radius of the pipe). In the case of a hydraulically rough wall, we keep the ratio $\hat{k} = k/L$ fixed, so that the body remains geometrically similar as Re increases.

The analysis of Quan & Eyink (2022*a*) (with some pedagogical simplifications introduced by Eyink (2023)) begins with the fine-grained momentum balance/

Navier–Stokes equation smeared against a smooth space–time test function $\boldsymbol{\varphi}(\boldsymbol{x}, t)$:

$$\int_0^T \int_\Omega \left[(\partial_t\boldsymbol{\varphi} + \nu\Delta\boldsymbol{\varphi})\cdot\boldsymbol{u}^\nu + \nabla\boldsymbol{\varphi} : \left(\boldsymbol{u}^\nu\otimes\boldsymbol{u}^\nu + p^\nu\boldsymbol{I}\right)\right] \mathrm{d}V\,\mathrm{d}t = \int_0^T \int_{\partial\Omega} \left[\boldsymbol{\tau}_w^\nu\cdot\boldsymbol{\varphi} - p_w^\nu(\boldsymbol{n}\cdot\boldsymbol{\varphi})\right] \mathrm{d}S\,\mathrm{d}t. \tag{4.20}$$

Note that I am using the standard non-dimensionalisation with $\hat{\nu} = 1/Re$, but I drop hats $\widehat{(\cdot)}$ to keep notation simple. In contrast to usual discussions of weak solutions in domains Ω with boundaries, however, the test functions adopted in this study need not vanish near the surface $\partial\Omega$, and thus the smeared balance equation gets surface contributions both from the skin friction $\boldsymbol{\tau}_w$ and also from p_w, the pressure at the wall. Under the assumption of strong L^2-convergence in the space–time domain $[0, T]\times\Omega$ as $\nu\to 0$ (see § 3.2.3), we see that the left-hand side converges and thus also the limits exist, $\boldsymbol{\tau}_w = \lim_{\nu\to 0}\boldsymbol{\tau}_w^\nu$, $p_w = \lim_{\nu\to 0} p_w^\nu$, as distributions on the surface $\partial\Omega$, so that

$$\int_0^T \int_\Omega \left[\partial_t\boldsymbol{\varphi}\cdot\boldsymbol{u} + \nabla\boldsymbol{\varphi} : (\boldsymbol{u}\otimes\boldsymbol{u} + p\boldsymbol{I})\right] \mathrm{d}V\,\mathrm{d}t = \int_0^T \int_{\partial\Omega} \left[\boldsymbol{\tau}_w\cdot\boldsymbol{\varphi} - p_w(\boldsymbol{n}\cdot\boldsymbol{\varphi})\right] \mathrm{d}S\,\mathrm{d}t. \tag{4.21}$$

The limiting fields $(\boldsymbol{u}, p)$ in $[0, T]\times\Omega$ thus constitute a weak Euler solution, as seen by restricting (4.21) to test functions $\boldsymbol{\varphi}$ whose supports vanish on $\partial\Omega$, but (4.21) gives additional information at the surface inherited from the Navier–Stokes solutions.

The key step is now to consider the corresponding smeared version of the coarse-grained momentum balance (4.13):

$$\int_0^T \int_\Omega \partial_t\boldsymbol{\varphi}\cdot(\eta_{h,\ell}\bar{\boldsymbol{u}}_\ell^\nu)\,\mathrm{d}V\,\mathrm{d}t + \int_0^T \int_\Omega \nabla\boldsymbol{\varphi} : \eta_{h,\ell}(\bar{\boldsymbol{T}}_\ell^\nu + \bar{p}_\ell^\nu\boldsymbol{I} - \nabla\bar{\boldsymbol{u}}_\ell^\nu)\,\mathrm{d}V\,\mathrm{d}t = -\int_0^T \int_\Omega \boldsymbol{\varphi}\cdot(\bar{\boldsymbol{T}}_\ell^\nu\cdot\nabla\eta_{h,\ell} + \bar{p}_\ell^\nu\nabla\eta_{h,\ell})\,\mathrm{d}V\,\mathrm{d}t, \tag{4.22}$$

where I have defined $\bar{\boldsymbol{T}}_\ell^\nu := \overline{(\boldsymbol{u}^\nu\boldsymbol{u}^\nu)}_\ell$ and also I have recalled the definition $\tilde{a}_{h,\ell} = \eta_{h,\ell}\bar{a}_\ell$ for any quantity a. Taking the double limits, first $\nu\to 0$ and subsequently $h, \ell\to 0$, then it is easy to see that the left-hand side of (4.22) converges exactly to the left-hand side of (4.21). In that case, however, the right-hand side of (4.22) must converge also to the right-hand side of (4.21). We thereby get results on spatial cascade of wall-parallel momentum:

$$\lim_{h,\ell\to 0} -\boldsymbol{n}\times\bar{\boldsymbol{T}}_\ell\cdot\nabla\eta_{h,\ell} = \boldsymbol{n}\times\boldsymbol{\tau}_w \tag{4.23}$$

with $\bar{\boldsymbol{T}}_\ell := \overline{(\boldsymbol{u}\boldsymbol{u})}_\ell$, and on spatial cascade of wall-normal momentum:

$$\lim_{h,\ell\to 0} \boldsymbol{n}\cdot\left[\bar{\boldsymbol{T}}_\ell\cdot\nabla\eta_{h,\ell} + \bar{p}_\ell\nabla\eta_{h,\ell}\right] = p_w, \tag{4.24}$$

both convergence statements interpreted in the sense of distributions on $[0, T]\times\partial\Omega$. To obtain these results, I had to make suitable choices for test functions $\boldsymbol{\varphi}$ and, in particular, that allowed us to neglect $\lim_{h,\ell\to 0} -\boldsymbol{n}\times\bar{p}_\ell\nabla\eta_{h,\ell} = 0$ in the first statement. The physical meaning of these results are straightforward. The first result (4.23) simply states that the spatial cascade of transverse momentum to the wall via turbulent nonlinear advection in the limiting Euler solution must match onto the direct infinite-*Re* limit of the viscous skin friction at the wall. Recall that $\boldsymbol{\nabla}\eta_{h,\ell}(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x})\theta'_{h,\ell}(d(\boldsymbol{x})) \simeq \boldsymbol{n}(\boldsymbol{x})\delta(d(\boldsymbol{x})-h)$, so that the quantity on the left-hand side of (4.23) indeed represents advective momentum flux towards the wall at distance h from the wall. Likewise, the second result (4.24) states that the spatial cascade of wall-normal momentum away from the wall (note the change of sign) in the limiting Euler solution must match onto the direct infinite-*Re* limit of the kinematic pressure at the wall. These two results are exact analogues of that derived for scale-cascade by Duchon & Robert (2000), i.e. the matching of the expressions (3.29), (3.30) for nonlinear energy cascade in the limiting Euler solution and the direct infinite-*Re* limit of viscous energy dissipation (3.32).

Unlike the scale-cascade of kinetic energy studied by Duchon & Robert (2000), which can be understood very roughly as a deterministic version of the cascade of Kolmogorov (1941*a*,*b*), in our case, the meaning of 'momentum cascade' is distinctly different from the traditional one in the literature on wall-bounded turbulence. I may recall that Tennekes & Lumley (1972) and Jiménez (2012), and many others, posit the existence in canonical wall-bounded flows (pipe, channel, boundary-layer) of an 'inertial layer' where the Reynolds stress or mean momentum flux $-\langle u_x u_y\rangle$ (x wall-parallel, y wall-normal) is approximately constant and equal to $u_\tau^2 = \langle\tau_{w,xy}\rangle$, the mean skin friction. This looks superficially similar to (4.23), except that the latter is a deterministic result for the weak Euler solutions obtained in the infinite-*Re* limit. However, the result (4.23) apparently becomes trivial in all of these canonical flows where there is only a weak anomaly and $\boldsymbol{\tau}_w = \boldsymbol{0}$! Conversely, both the traditional and the deterministic cascades occur in a turbulent pipe flow with a hydraulically rough wall, but at completely different wall normal distances. Indeed, the standard mean momentum cascade then holds in the traditional inertial layer $k \ll y \ll H$, whereas result (4.23) holds in the range $\nu/u_\tau \ll h \ll k$, i.e. within the roughness sublayer! Recall that our limit is one in which $Re_k \to \infty$ so that our flows are in the fully rough regime and completely inertial-dominated turbulence exists between the roughness elements.

So far, I have not established a result equivalent to Onsager's $1/3$ Hölder exponent result. As glimpsed already by Taylor (1915), a corresponding condition has to do with continuity of the velocity at the wall. In fact, Bardos & Titi (2018); Bardos *et al.* (2019) understood that the essential condition has to do with the vanishing of the normal velocity component approaching the wall, since it is the normal velocity which transports momentum and other conserved quantities to and from the wall. The specific condition that I consider is that of Drivas & Nguyen (2018):

$$\lim_{\delta\to 0} \|\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{u}\|_{L^2([0,T],L^\infty(\varOmega_\delta))} = 0, \tag{4.25}$$

which means that the maximum wall-normal velocity in the δ-neighbourhood $\varOmega_\delta = \{\boldsymbol{x}\in\varOmega : d(\boldsymbol{x}) < \delta\}$ of the boundary $\partial\varOmega$ must vanish as $\delta\to 0$. With this assumption, the advective transport of momentum to the wall can be shown without much difficulty to vanish:

$$\lim_{h,\ell\to 0} \bar{\boldsymbol{T}}_\ell \boldsymbol{\cdot}\boldsymbol{\nabla}\eta_{h,\ell} = \boldsymbol{0}, \tag{4.26}$$

see Quan & Eyink (2022*a*) and Eyink (2023) for details. In that case, the previous results simplify drastically. The first result (4.23) reduces to

$$\boldsymbol{\tau}_w = \boldsymbol{0}, \tag{4.27}$$

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which is the statement that condition (4.25) rules out a strong momentum anomaly. The second result (4.24) furthermore becomes

$$\lim_{h,\ell\to 0} \bar{p}_\ell \theta'_{h,l}(d(\boldsymbol{x})) = p_w. \tag{4.28}$$

This is essentially a statement about commutation of two limits: the same pressure field at the wall is obtained either by taking the $Re \to \infty$ limit p_w of the pressure p_w^{Re} directly at the wall or instead by taking $Re \to \infty$ first and then taking the limit of the pressure p of the Euler solution in the flow interior at points approaching to the wall. This commutation of limits may be loosely termed 'continuity of pressure at the wall'.

The last two results seem very satisfactory. As I have already discussed in § 4.1, the mean skin friction $\langle \boldsymbol{\tau} \rangle$ (e.g. a long-time average) seems to vanish quite generally. Thus, the absence of a strong momentum anomaly (4.27) looks consistent with experiment. The second result (4.28) on the continuity of pressure at the wall is very good news for LES modelling. As I have discussed in § 4.1, the limiting drag force on a moving body at high-*Re* seems to arise entirely from pressure forces (form drag) with negligible drag arising from viscous skin friction. Thus, the result (4.28) means that the asymptotic drag on a body can be successfully calculated from the weak Euler solution, with the limiting pressure on the body correctly calculated from the Euler pressure in the flow interior after taking the limit approaching the body. However, it is a bit premature to take this as good news! First, there is currently no consensus how to determine the pressure uniquely for the weak Euler solution and in particular what boundary condition should be imposed to replace the Neumann condition in (4.3) for the Navier–Stokes pressure (Bardos & Titi 2022; Bardos, Boutros & Titi 2023; De Rosa, Latocca & Stefani 2023, 2024). Second, and more seriously, we shall see in § 4.2.6 that the assumption that $\boldsymbol{\tau}_w \equiv \boldsymbol{0}$ identically leads to the paradoxical conclusion that all drag vanishes in the limit $Re \to 0$ and that the Navier–Stokes solution converges to the smooth potential Euler solution of d'Alembert (1749, 1768) with no drag.

4.2.4. *Energy cascade in space and in scale*

Kinetic energy balance can be analysed in a very similar manner. Quan & Eyink (2022*b*) considered external flow past a body B with fluid velocity constant at infinity, so that the total kinetic energy was infinite. The flow was thus divided into a potential part $\boldsymbol{u}_\phi$ and a rotational part $\boldsymbol{u}_\omega = \boldsymbol{u} - \boldsymbol{u}_\phi$, so that the 'relative energy' $E_{rel}(t) = \int_\Omega [\boldsymbol{u}_\phi \boldsymbol{\cdot} \boldsymbol{u}_\omega + \frac{1}{2}|\boldsymbol{u}_\omega|^2]\,\mathrm{d}V$ (heuristically, total energy minus energy of the potential flow) remains finite. To keep notation simple, however, I consider here internal flow in a bounded domain Ω as did Bardos & Titi (2018), Bardos *et al.* (2019), Drivas & Nguyen (2018) and Chen *et al.* (2022).

The fine-grained kinetic energy balance for incompressible Navier–Stokes when smeared with a scalar test function φ is easily calculated to be

$$\int_0^T \int_\Omega (\partial_t + \nu\Delta)\varphi \times \tfrac{1}{2}|\boldsymbol{u}^\nu|^2\,\mathrm{d}V\,\mathrm{d}t + \int_0^T \int_\Omega \boldsymbol{\nabla}\varphi \boldsymbol{\cdot} \left(\tfrac{1}{2}|\boldsymbol{u}^\nu|^2 + p^\nu\right)\boldsymbol{u}^\nu\,\mathrm{d}V\,\mathrm{d}t \\ + \int_0^T \int_\Omega \nu(\boldsymbol{\nabla\nabla}\varphi) \boldsymbol{:} \boldsymbol{u}^\nu\boldsymbol{u}^\nu\,\mathrm{d}V\,\mathrm{d}t = \int_0^T \int_\Omega \varphi\,\varepsilon^\nu\,\mathrm{d}V\,\mathrm{d}t, \tag{4.29}$$

with $\varepsilon^\nu = 2\nu|\boldsymbol{S}^\nu|^2$ the viscous energy dissipation. As in the previous section, I allow the smooth test function φ to be non-vanishing on the boundary $\partial\Omega$ but no boundary terms arise after integration by parts because of the stick b.c. Assuming strong convergence of

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$\boldsymbol{u}^\nu, p^\nu$ in the bulk to $\boldsymbol{u}, p$ as $\nu \to 0$, it is easy just as in a periodic domain to obtain the inviscid limit

$$\int_0^T \int_\Omega \partial_t \varphi \times \tfrac{1}{2}|\boldsymbol{u}|^2 \, \mathrm{d}V \, \mathrm{d}t + \int_0^T \int_\Omega \nabla\varphi \boldsymbol{\cdot} \left(\tfrac{1}{2}|\boldsymbol{u}|^2 + p\right) \boldsymbol{u} \, \mathrm{d}V \, \mathrm{d}t \\ = \int_0^T \int_\Omega \varphi \, D \, \mathrm{d}V \, \mathrm{d}t, \tag{4.30}$$

where

$$\lim_{\nu\to 0} \int_0^T \int_\Omega \varphi \varepsilon^\nu \, \mathrm{d}V \, \mathrm{d}t = \int_0^T \int_\Omega \varphi \, D \, \mathrm{d}V \, \mathrm{d}t. \tag{4.31}$$

One can also write the inviscid coarse-grained kinetic energy balance (3.10), window out the near-wall region and then smear with the test function φ to obtain

$$\int_0^T \int_\Omega \partial_t \varphi \times \eta_{h,\ell} \tfrac{1}{2}|\bar{\boldsymbol{u}}_\ell|^2 \, \mathrm{d}V \, \mathrm{d}t + \int_0^T \int_\Omega \nabla\varphi \boldsymbol{\cdot} \eta_{h,\ell} \bar{\boldsymbol{J}}_e \, \mathrm{d}V \, \mathrm{d}t \\ = \int_0^T \int_\Omega \varphi \eta_{h,\ell} \varPi_\ell \, \mathrm{d}V \, \mathrm{d}t - \int_0^T \int_\Omega \varphi \bar{\boldsymbol{J}}_\ell \boldsymbol{\cdot} \nabla \eta_{h,\ell} \, \mathrm{d}V \, \mathrm{d}t, \tag{4.32}$$

where I have introduced the spatial flux of coarse-grained kinetic energy:

$$\bar{\boldsymbol{J}}_e = \left(\tfrac{1}{2}|\bar{\boldsymbol{u}}_\ell|^2 + \bar{p}_\ell\right) \bar{\boldsymbol{u}}_\ell + \boldsymbol{\tau}_\ell \boldsymbol{\cdot} \bar{\boldsymbol{u}}_\ell. \tag{4.33}$$

Taking the limit $h, \ell \to 0$, the left-hand side of the coarse-grained balance (4.32) coincides with the left-hand side of the fine-grained balance (4.30). Thus, the right-hand sides must also coincide and one obtains

$$\lim_{h,\ell\to 0} \left(\eta_{h,\ell} \varPi_\ell - \bar{\boldsymbol{J}}_\ell \boldsymbol{\cdot} \nabla \eta_{h,\ell}\right) = D. \tag{4.34}$$

In contrast to the result (3.30) obtained in periodic domains where anomalous dissipation D matches to inertial energy flux $\varPi_\ell$ alone, in wall-bounded domains, there is an additional contribution from $-\bar{\boldsymbol{J}}_\ell \boldsymbol{\cdot} \nabla \eta_{h,\ell}$, which represents spatial energy flux to the wall. I mention in passing that at any finite distance from the wall, the expression (3.29) of Duchon & Robert (2000) can also be derived here for D, as well as the 4/5th law (3.36) and 4/15th law (3.37).

A condition on vanishing of the wall-normal velocity analogous to (4.25) (but L^3 in time) suffices to show that the additional spatial flux contribution must vanish. However, the potential presence of this additional term raises the question whether the anomalous dissipation measure might have some contribution from the boundary, so that $D(\partial\Omega) > 0$. This question recalls the famous theorem of Kato (1984) (see also Sueur 2012; Bardos & Titi 2013, among others), according to which vanishing of the viscous dissipation in the boundary strip $\Omega_{c\nu} = \{\boldsymbol{x} \in \Omega : d(\boldsymbol{x}) < c\nu\}$ for any constant $c > 0$, or

$$\lim_{\nu\to 0} \int_0^T \int_{\Omega_{c\nu}} \varepsilon^\nu \, \mathrm{d}V \, \mathrm{d}t = 0 \tag{4.35}$$

is necessary and sufficient that Navier–Stokes solution $\boldsymbol{u}^\nu$ converges strongly in $L^2([0, T] \times \Omega)$ to the smooth Euler solution $\boldsymbol{u}$ with the same initial data $\boldsymbol{u}_0$, over any time period $[0, T]$ for which this smooth Euler solution exists. This type of result is closely allied to the strong-weak uniqueness theory which I shall discuss in § 4.2.6. A consequence is that vanishing of the energy dissipation in the 'Kato layer' $\Omega_{c\nu}$ implies vanishing energy dissipation everywhere in Ω, unless the smooth Euler solution $\boldsymbol{u}$ develops a singularity before time T. Bardos & Titi (2013) have shown furthermore that the vanishing of energy dissipation in $\Omega_{c\nu}$ implies also that $\boldsymbol{\tau}_w = \mathbf{0}$. Here I may recall the numerical study by Nguyen van yen *et al.* (2018) of a compact quadrupole vortex in two dimensions impacting on a flat, solid wall. At very high Reynolds numbers, these authors find evidence for anomalous energy dissipation arising in the Kato layer from maximum vorticity values scaling as $\omega^\nu \sim 1/\nu$. These high values at the wall imply that for some points on the boundary, at least, skin friction $\boldsymbol{\tau}_w = \nu\boldsymbol{\omega}^\nu \times \boldsymbol{n} \not\to \mathbf{0}$ as $\nu \to 0$. I return to these issues again in § 4.2.6.

4.2.5. *Vorticity cascade in space*

The same methods as discussed previously can be applied also to vorticity and, in some respects, the analysis is much easier, because the vorticity balance is just the curl of the momentum balance already treated. I describe here only a few key results, following the more extended treatment of Eyink (2023).

The fine-grained balance of vorticity for the incompressible Navier–Stokes equation is, of course, the viscous Helmholtz equation:

$$\partial_t\boldsymbol{\omega}^\nu = \boldsymbol{\nabla} \times (\boldsymbol{u}^\nu \times \boldsymbol{\omega}^\nu - \nu\boldsymbol{\nabla} \times \boldsymbol{\omega}^\nu) := -\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}^\nu, \tag{4.36}$$

where $\boldsymbol{\Sigma}^\nu$ is a suitable anti-symmetric tensor representing vorticity flux through the flow volume (Huggins 1971, 1994). We might adopt this as the starting point to derive the fine-grained vorticity balance in the limit $\nu \to 0$, but it is easier just to take $\boldsymbol{\varphi} \to \boldsymbol{\nabla} \times \boldsymbol{\varphi}$ in the ideal momentum balance equation (4.21). This yields directly

$$\begin{aligned}\int_0^T \int_\Omega \left[\partial_t(\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \cdot \boldsymbol{u} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \times \boldsymbol{\varphi}) : \boldsymbol{u} \otimes \boldsymbol{u}\right] \mathrm{d}V\,\mathrm{d}t \\ = \int_0^T \int_{\partial\Omega} \left[\boldsymbol{\tau}_w \cdot (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) - \boldsymbol{\sigma}_w \cdot \boldsymbol{\varphi}\right] \mathrm{d}A\,\mathrm{d}t,\end{aligned} \tag{4.37}$$

where surface integration by parts $\int_{\partial\Omega} p_w(\boldsymbol{n} \times \boldsymbol{\nabla}) \cdot \boldsymbol{\varphi}\,\mathrm{d}A = -\int_{\partial\Omega}(\boldsymbol{n} \times \boldsymbol{\nabla})p_w \cdot \boldsymbol{\varphi}\,\mathrm{d}A$ was used in the last line to introduce the vorticity source $\boldsymbol{\sigma}_w = -\boldsymbol{n} \times \boldsymbol{\nabla}p_w$ of Lighthill (1963). The terms in this equation have mostly transparent physical meaning. Note in particular that the $\boldsymbol{\tau}_w$-term arises from viscous diffusion of vorticity away from the surface in the limit $\nu \to 0$, through the integration-by-parts identity

$$\int_\Omega (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \cdot \nu\boldsymbol{\nabla} \times \boldsymbol{\omega}^\nu = \nu \int_\Omega \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \cdot \boldsymbol{\omega}^\nu + \int_{\partial\Omega} (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \cdot \boldsymbol{\tau}_w^\nu\,\mathrm{d}A. \tag{4.38}$$

We thus see that anomalous viscous diffusion of vorticity may remain as $\nu \to 0$ if $\boldsymbol{\tau}_w \neq \mathbf{0}$.

The corresponding coarse-grained vorticity balance is obtained by taking the curl of the coarse-grained momentum balance equation (4.13) and can be written most simply as

$$\partial_t(\eta_{h,\ell}\bar{\boldsymbol{\omega}}_\ell + \boldsymbol{\nabla}\eta_{h,\ell} \times \bar{\boldsymbol{u}}_\ell) + \boldsymbol{\nabla} \times \left[\boldsymbol{\nabla} \cdot (\eta_{h,\ell}\bar{\boldsymbol{T}}_\ell)\right] = -\boldsymbol{\nabla} \times \boldsymbol{f}_{\ell,h}. \tag{4.39}$$

In the time-derivative term, I used the identity $\boldsymbol{\nabla} \times \tilde{\boldsymbol{u}}_{h,\ell} = \eta_{h,\ell}\bar{\boldsymbol{\omega}}_\ell + \boldsymbol{\nabla}\eta_{h,\ell} \times \bar{\boldsymbol{u}}_\ell$, which makes clear that the balance equation is for the sum of a vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ in the flow interior and a vortex sheet $\boldsymbol{n} \times \boldsymbol{u}$ on the solid surface. Note that $\boldsymbol{f}_{\ell,h}$ is just the inviscid limit of the inertial-drag force defined in (4.15), which may also be written as

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$\boldsymbol{f}_{\ell,h} = -(\bar{\boldsymbol{T}}_\ell + \bar{p}_\ell \boldsymbol{I}) \boldsymbol{\cdot} \boldsymbol{\nabla}\eta_{h,\ell}$. The term $-\boldsymbol{\nabla} \times \boldsymbol{f}_{\ell,h}$ in the coarse-grained vorticity balance rationalises the force field method used in aerodynamics and wake flows behind complex bodies, where the solid surface is replaced by a suitable body force (van Kuik 2022). Smearing the coarse-grained vorticity balance (4.39) with a vector test function $\boldsymbol{\varphi}$ and taking the limit $h, \ell \to 0$ recovers the same (4.37) obtained from the fine-grained balance. Note in particular that

$$-\lim_{h,\ell\to 0} \int_0^T \int_\Omega \boldsymbol{\varphi} \boldsymbol{\cdot} \boldsymbol{\nabla} \times \boldsymbol{f}_{\ell,h} \, \mathrm{d}V \, \mathrm{d}t = \int_0^T \int_{\partial\Omega} \left[-\boldsymbol{\tau}_w \boldsymbol{\cdot} (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) + \boldsymbol{\sigma}_w \boldsymbol{\cdot} \boldsymbol{\varphi} \right] \mathrm{d}A \, \mathrm{d}t, \quad (4.40)$$

which relates $-\boldsymbol{\nabla} \times \boldsymbol{f}_{\ell,h}$ to inviscid generation of vorticity at the body surface and possible anomalous viscous diffusion out of the surface. Equation (4.40) follows directly from (4.23) and (4.24).

If the strong continuity condition for wall-normal velocity (4.25) holds so that $\boldsymbol{\tau}_w \equiv \boldsymbol{0}$, then one may derive another simple relation for the surface vorticity source of Lighthill (1963). In that case (Quan & Eyink 2022*a*), the pressure-continuity result (4.28) implies that

$$\begin{aligned} -\int_0^T \mathrm{d}t \int_{\partial\Omega} \mathrm{d}A \, \boldsymbol{\varphi} \boldsymbol{\cdot} (\boldsymbol{n} \times \boldsymbol{\nabla}) p_w &= \int_0^T \mathrm{d}t \int_{\partial\Omega} \mathrm{d}A \, (\boldsymbol{n} \times \boldsymbol{\nabla}) \boldsymbol{\cdot} \boldsymbol{\varphi} p_w \\ &= \lim_{h,\ell\to 0} \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}V \, \boldsymbol{\nabla}\eta_{h,\ell} \times \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\varphi} \bar{p}_\ell \\ &= \lim_{h,\ell\to 0} \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}V \, -\boldsymbol{\varphi} \boldsymbol{\cdot} \boldsymbol{\nabla}\eta_{h,\ell} \times \boldsymbol{\nabla}\bar{p}_\ell \\ &= \lim_{h,\ell\to 0} \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}V \, \boldsymbol{\varphi} \boldsymbol{\cdot} \left(\boldsymbol{\nabla}\eta_{h,\ell} \times \partial_t \bar{\boldsymbol{u}}_\ell + \boldsymbol{\nabla}\eta_{h,\ell} \boldsymbol{\cdot} \boldsymbol{\Sigma}_\ell \right), \end{aligned} \quad (4.41)$$

where in the last line, I used the coarse-grained momentum balance in the flow interior, written as

$$\partial_t \bar{\boldsymbol{u}}_\ell = -\boldsymbol{\nabla} \boldsymbol{\cdot} \bar{\boldsymbol{T}}_\ell - \boldsymbol{\nabla}\bar{p}_\ell = \boldsymbol{\Sigma}^*_\ell - \boldsymbol{\nabla}\bar{p}_\ell, \quad (4.42)$$

where $\boldsymbol{\Sigma}^*_\ell$ is the Hodge dual to anti-symmetric vorticity flux tensor $\boldsymbol{\Sigma}_\ell$, or in components $\boldsymbol{\Sigma}^*_{\ell,i} = \frac{1}{2}\epsilon_{ijk}\boldsymbol{\Sigma}_{\ell,jk}$. Equation (4.41) states that under the condition (4.25), tangential pressure gradients at the body surface act in the inviscid limit as a local source of vorticity, which either accumulates in the surface vortex sheet or else flows away via turbulent advection.

Vorticity flux from the body surface can be directly related to drag. Relations of this type have been derived by many authors, including Burgers, Lighthill and many others, as discussed in the very comprehensive review by Biesheuvel & Hagmeijer (2006). Eyink (2021) derives such a relation for the power consumed by the instantaneous drag force $\boldsymbol{F}_B(t)$ on a body B held fixed in a flow with velocity $\boldsymbol{V}$, making connection with ideas in quantum superfluids. This Josephson–Anderson relation,

$$\begin{aligned} \boldsymbol{F}^\nu_B(t) \boldsymbol{\cdot} \boldsymbol{V} &= -\rho \int_\Omega \boldsymbol{u}_\phi \boldsymbol{\cdot} (\boldsymbol{u}^\nu \times \boldsymbol{\omega}^\nu - \nu \boldsymbol{\nabla} \times \boldsymbol{\omega}^\nu) \, \mathrm{d}V \\ &= -\rho \int_\Omega \boldsymbol{\nabla}\boldsymbol{u}_\phi : \boldsymbol{u}^\nu_\omega \boldsymbol{u}^\nu_\omega \, \mathrm{d}V + \rho \int_{\partial\Omega} \boldsymbol{u}_\phi \boldsymbol{\cdot} \boldsymbol{\tau}^\nu_w \, \mathrm{d}A \end{aligned} \quad (4.43)$$

is in fact a special case of a relation derived by Howe (1995). In the last line of (4.43), integration by parts has been used to rewrite the relation in a form which should remain valid in the limit $\nu \to 0$, as pointed out by Eyink (2021). This inviscid limit has been rigorously derived by Quan & Eyink (2022*b*) when the body surface ∂B is mathematically smooth, i.e. when the body is very large. The Josephson–Anderson relation and the drag on the body are related to Onsager's dissipation anomaly by the balance equation of the kinetic energy $(1/2)|\boldsymbol{u}_\omega|^2$ in the rotational flow $\boldsymbol{u}_\omega = \boldsymbol{u} - \boldsymbol{u}_\phi$, where $\boldsymbol{u}_\phi$ is the stationary potential flow of d'Alembert (1749, 1768) with no drag. This balance takes the form

$$
\begin{aligned}
&-\int_0^T \int_\Omega \tfrac{1}{2}\partial_t\varphi|\boldsymbol{u}_\omega|^2\,\mathrm{d}V\,\mathrm{d}t - \int_0^T \int_\Omega \nabla\varphi \boldsymbol{\cdot} \left[\tfrac{1}{2}|\boldsymbol{u}_\omega|^2\boldsymbol{u} + p_\omega\boldsymbol{u}_\omega\right]\mathrm{d}V\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \varphi\boldsymbol{u}_\phi \boldsymbol{\cdot} \boldsymbol{\tau}_w\,\mathrm{d}A\,\mathrm{d}t - \int_0^T \int_\Omega \varphi\nabla\boldsymbol{u}_\phi : \boldsymbol{u}_\omega \otimes \boldsymbol{u}_\omega\,\mathrm{d}V\,\mathrm{d}t \\
&\qquad - \int_0^T \int_\Omega \varphi\, D(\boldsymbol{u})\,\mathrm{d}V\,\mathrm{d}t,
\end{aligned} \tag{4.44}
$$

with φ a smooth test function that may not vanish at the boundary (Quan & Eyink 2022*b*). The Josephson–Anderson relation represents a transfer of kinetic energy into rotational fluid motions, which is then disposed by the dissipation D into heat energy.

4.2.6. *Weak-strong uniqueness and extreme near-wall events*

A fact of fundamental importance for the theory of incompressible turbulence is that weak-strong uniqueness, as reviewed in § 3.1.3, is not valid unconditionally in flows with solid walls, as it is for incompressible flows in periodic domains or in infinite space. Bardos, Székelyhidi & Wiedemann (2014) (and see also Bardos & Titi 2013; Wiedemann 2018) have noted that additional conditions are required in wall-bounded flows to obtain weak-strong uniqueness and the first work uses convex integration methods to construct a counterexample showing that uniqueness indeed fails without such added hypotheses. One sufficient condition for weak-strong uniqueness is that the inviscid limit of the skin friction, $\boldsymbol{\tau}_w = \lim_{\nu\to 0} \boldsymbol{\tau}_w^\nu$, should be vanishing as a distribution at all times on the solid surface, i.e. $\boldsymbol{\tau}_w \equiv \boldsymbol{0}$ when smeared with any smooth space–time test function (Bardos & Titi 2013). Another condition which guarantees the later is the uniform vanishing of wall-normal velocity (4.25), which I have already noted in (4.27) implies vanishing skin friction (Quan & Eyink 2022*b*).

The gist of the weak-strong uniqueness result can be easily understood from the following sketch of a proof taken from Quan & Eyink (2024), which shows in the context of flow around a smooth body B that even the weaker condition $\boldsymbol{\tau}_w \boldsymbol{\cdot} \boldsymbol{u}_\phi \equiv 0$ on $[0, T] \times \partial B$ implies weak-strong uniqueness. The argument is based on the inviscid balance of global kinetic energy in the rotational flow, $E_\omega(\tau) = \frac{1}{2}\int_\Omega |\boldsymbol{u}_\omega(\cdot, \tau)|^2\,\mathrm{d}V$, which is

$$
E_\omega(\tau) = E_\omega(0) + \int_0^\tau \int_{\partial\Omega} \boldsymbol{u}_\phi \boldsymbol{\cdot} \boldsymbol{\tau}_w\,\mathrm{d}A\,\mathrm{d}t - \int_0^\tau \int_\Omega \nabla\boldsymbol{u}_\phi : \boldsymbol{u}_\omega\boldsymbol{u}_\omega\,\mathrm{d}V\,\mathrm{d}t - \int_0^\tau \int_\Omega D\,\mathrm{d}V\,\mathrm{d}t. \tag{4.45}
$$

This result can be obtained formally by taking $\varphi \equiv \chi_{[0,\tau]\times\Omega}$, the characteristic function of the set $[0, \tau] \times \Omega$, in the local balance (4.44) and can be obtained rigorously by a careful limiting argument. Using the condition $\boldsymbol{\tau}_w \boldsymbol{\cdot} \boldsymbol{u}_\phi \equiv 0$ and the fact that $D \geq 0$, one obtains

$$
E_\omega(\tau) \leq E_\omega(0) - \int_0^\tau \int_\Omega \nabla\boldsymbol{u}_\phi : \boldsymbol{u}_\omega\boldsymbol{u}_\omega\,\mathrm{d}V\,\mathrm{d}t. \tag{4.46}
$$

Since $\boldsymbol{u}_\phi$ is smooth and globally bounded, application of Cauchy–Schwartz inequality gives

$$E_\omega(\tau) \le E_\omega(0) + C\|\nabla \boldsymbol{u}_\phi\|_\infty \int_0^\tau \mathrm{d}t\, E_\omega(t) \tag{4.47}$$

and then the Gronwall inequality finally yields

$$E_\omega(\tau) \le E_\omega(0) \exp\left(C\|\nabla \boldsymbol{u}_\phi\|_\infty \times \tau\right). \tag{4.48}$$

If $E_\omega(0) = 0$ or equivalently $\boldsymbol{u}(0) \equiv \boldsymbol{u}_\phi$, then $E_\omega(\tau) \equiv 0$ for all $\tau > 0$ and thus $\boldsymbol{u} \equiv \boldsymbol{u}_\phi$ identically for all time. Careful examination of the proof shows that this result remains valid even if $\|\boldsymbol{u}^\nu(0) - \boldsymbol{u}_\phi\|_{L^2(\Omega)} \to 0$ as $\nu \to 0$, which allows for a viscous boundary layer in the Navier–Stokes initial data, as long as the energy $E^\nu_\omega(0)$ in that layer shrinks to zero as $\nu \to 0$. Note that the estimate (4.48) allows for instability of $\boldsymbol{u}_\phi$ in the dynamical systems sense, so that a small perturbation may grow exponentially. However, the inequality (4.48) shows that if $\boldsymbol{u}_\phi \boldsymbol{\cdot} \boldsymbol{\tau}_w \equiv 0$, then the smooth potential Euler solution $\boldsymbol{u}_\phi$ is stable in the sense of Hadamard well-posedness and that any Navier–Stokes solution whose initial data $\boldsymbol{u}^\nu(0)$ converges to $\boldsymbol{u}_\phi$ in L^2 as $\nu \to 0$ will converge globally to $\boldsymbol{u}_\phi$ in the inviscid limit.

In my opinion, a plausible conclusion is that weak-strong uniqueness is an inappropriate condition to select physically correct weak solutions of the Euler equations, unlike what has been often proposed in the mathematics literature. The clearest example is the constant flow past a smooth body, the subject of the famous paradox of d'Alembert (1749, 1768), where the potential Euler solution is smooth and stationary in time, thus existing globally without blow-up. Nevertheless, drag does not seem to vanish in the $\nu \to 0$ limit, as shown by laboratory experiments like that of Achenbach (1972) for flow past a sphere and by numerical simulations such as that of Chatzimanolakis, Weber & Koumoutsakos (2022) for 2-D flow past a disk of diameter D impulsively accelerated to velocity U. In the latter case, the initial conditions are exactly $\boldsymbol{u}(0) = \boldsymbol{u}_\phi$ at $t = 0$ and, after a short period of a few mean-free-times when a kinetic description is required, the dynamics is well described by incompressible Navier–Stokes. In principle, empirical observations like those above might be explained by instability of the potential Euler solution in a dynamical systems sense, so that small perturbations, e.g. due to external flow disturbances, molecular viscosity or even thermal noise, grow exponentially in time. However, excluding for the moment external and thermal noise, there is an asymptotically exact solution of the 2-D incompressible Navier–Stokes equation for the case of the impulsively accelerated disk as $\nu \to 0$ (Bar-Lev & Yang 1975) and although its time of validity is expected to be only $\sim D/U$, this nevertheless suffices to show that $E^\nu_\omega(0+) \to 0$, where $t = 0+$ is the initial time of validity of the Navier–Stokes description. In that case, the bound (4.48) expressing weak-strong uniqueness, if valid, would imply that $\boldsymbol{u} = \lim_{\nu\to 0} \boldsymbol{u}^\nu$ coincides with $\boldsymbol{u}_\phi$ identically. Although I have explained this result in the case of the impulsively accelerated body via the energy balance (4.45), similar arguments apply when the body is accelerated over a finite time-interval. However, the empirical observations suggest instead that the limit $\boldsymbol{u}$ is a dissipative weak Euler solution with initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_\phi$ but distinct from $\boldsymbol{u}_\phi$ and co-existing with that smooth, stationary Euler solution.

The latter conclusion can be correct only if all of the sufficient conditions guaranteeing the weak-strong uniqueness result (4.48) are in fact invalid. The possibility that $\boldsymbol{\tau}_w \not\equiv \boldsymbol{0}$ is not in contradiction with the general empirical observation that $\langle \boldsymbol{\tau}_w \rangle = \boldsymbol{0}$, since the long-time average could be zero if the condition $\boldsymbol{\tau}_w \neq \boldsymbol{0}$ existed only in some early period of the acceleration. The condition (4.25) on uniform vanishing of wall-normal velocity must also be violated if weak-strong uniqueness in fact fails. It is worth noting that experimental and numerical investigations of wall-bounded turbulence at high-*Re* have already observed very extreme events both of skin friction and of wall-normal velocity (Lenaers *et al.* 2012; Örlü & Vinuesa 2020). Numerical simulations of turbulent channel flow show that, while $u_\tau^2 = \langle \boldsymbol{\tau}_w \rangle$ decreases slowly with *Re*, the probability of large, rare fluctuations of the non-dimensionalised variable $\boldsymbol{\tau}_w^+ := \boldsymbol{\tau}_w / u_\tau^2$ increase with *Re* (Lenaers *et al.* 2012). The same study shows also very large fluctuations of the wall-normal velocity $v = \boldsymbol{u} \boldsymbol{\cdot} /\boldsymbol{n}$ in the viscous sublayer and buffer layer, with the probability of large values of $v^+ = v/u_\tau$ increasing with *Re*. These large v^+ fluctuations seem to be associated with explosive boundary-layer separation induced by strong near-wall vortices (Doligalski, Smith & Walker 1994). The events predicted by the breakdown of the condition (4.25), however, are much more extreme and occur at very different wall distances in the flows with strong dissipative anomaly, such as flow past a sphere or flow in a hydraulically rough pipe. In fact, restated in dimensional variables, the maximum value of $\boldsymbol{n} \boldsymbol{\cdot} \boldsymbol{u}/U$ attained on the interval $\nu/u_\tau \ll h < \delta \times L$ must be independent of the choice of $\delta > 0$ and also independent of Reynolds number for *Re* sufficiently large. This is a striking prediction of the breakdown of weak-strong uniqueness.

There is some serious question, however, whether the high-*Re* ranges considered are within the regime of validity of the standard Navier–Stokes description. Using the simple kinetic theory estimate for kinematic viscosity $\nu \sim c_s \ell_{mfp}$, where c_s is the thermal velocity/sound speed and ℓ_{mfp} is the mean-free-path length, one can see that the usual viscous unit of length at the wall $\delta_\nu := \nu/u_\tau$ (Tennekes & Lumley 1972) can be rewritten as $\delta_\nu \sim \ell_{mfp}/Ma_\tau$, where $Ma_\tau = u_\tau/c_s$ is the Mach number based on u_τ. In flows with a strong dissipative anomaly where $u_\tau \sim U$, this estimate becomes $\delta_\nu \sim \ell_{mfp}/Ma$, which is only larger than ℓ_{mfp} by a factor of the inverse of the global Mach number $Ma = U/c_s$. Note that this coincides with the estimate $\delta_\nu/L \sim 1/Re$ and thus corresponds to what in mathematics is called the 'Kato layer thickness'. This distance is often just several microns in realistic flows. Furthermore, the value of $\boldsymbol{\tau}_w^\nu$ is strongly fluctuating, as I have just discussed, and the local fluctuating wall unit $\delta_\nu(\boldsymbol{x}, t) := \nu/|\boldsymbol{\tau}_w^\nu(\boldsymbol{x}, t)|^{1/2}$ may therefore take values even a couple of orders of magnitude smaller. Such tiny lengths are dangerously close to ℓ_{mfp}, where the validity of any hydrodynamic description must break down. Furthermore, thermal fluctuation effects may be expected to appear at much larger length scales than ℓ_{mfp}, just as in the bulk flow (§§ 2 and 3.1.3.5). Understanding the ultimate origin of drag on solid bodies at very high Reynolds numbers may therefore require an extension of Landau–Lifschitz fluctuating hydrodynamics to fluid–solid interfaces (Reichelsdorfer 2016).

4.3. *Dissipative Euler solutions and zero-viscosity limit*

Given the recent vintage of the work applying Onsager's 'ideal turbulence' theory to wall-bounded flows, there is relatively little mathematical research attempting to construct corresponding weak Euler solutions. I thus remark only briefly about relevant works.

I have already mentioned the application of convex integration methods by Bardos *et al.* (2014) to establish non-uniqueness of globally dissipative weak Euler solutions in a wall-bounded flow. A more recent, interesting work by Vasseur & Yang (2023) has applied the convex integration technique of Székelyhidi (2011) for vortex sheets to the problem of turbulent channel flow, constructing weak Euler solutions $\boldsymbol{u}$ exhibiting boundary-layer separation, although the initial data $\boldsymbol{u}_0$ are given by the plug flow $\boldsymbol{U} = U\hat{\boldsymbol{x}}$, which is a smooth, stationary, potential Euler solution. These weak solutions thus differ from plug flow, although the space-L^2 norm of the difference at each time is bounded as $\|\boldsymbol{u}(\cdot,t)-\boldsymbol{U}\|^2_{L^2(\Omega)} \le CHWU^3t$, where H and W are the half-height and width of the channel, respectively. Furthermore, Vasseur & Yang (2023) study also Navier–Stokes solutions $\boldsymbol{u}^\nu$ with stick b.c. in the limit as $\nu \to 0$ and with initial data $\boldsymbol{u}_0^\nu$ converging in L^2 to the plug flow $\boldsymbol{U}$, proving that all weak limits $\boldsymbol{u}^*$ as $\nu \to 0$ satisfy similar upper bounds as the Euler solutions from convex integration, or $\|\boldsymbol{u}^*(\cdot,t)-\boldsymbol{U}\|^2_{L^2(\Omega)} \le C'HWU^3t$. As noted already by Vasseur & Yang (2023), vanishing energy dissipation in the 'Kato layer' near the wall would imply that all sequences $\boldsymbol{u}^\nu$ of Navier–Stokes solutions must converge strongly in L^2 to $\boldsymbol{U}$ as $\nu \to 0$. In standard terminology of wall-bounded flows, this 'Kato layer' in turbulent channel flow corresponds to the viscous sublayer and buffer layer, over which volume the energy dissipation rate integrates to u_τ^3WL for channel length L (Tennekes & Lumley 1972). This integrated dissipation tends to zero according to the standard Prandtl–Kármán theory and then the theorem of Kato (1984) implies that the inviscid limit in this geometry is just plug flow $\boldsymbol{U}$, in agreement with expectations in the fluid mechanics community (Cantwell 2019). The methods of Vasseur & Yang (2023) should yield more realistic results for bluff-body flows, such as flow past a sphere, with observable separation and anomalous dissipation.

The existing mathematical results already suggest that Eulerian spontaneous stochasticity as discussed in § 3.2.2 should appear at very high Reynolds numbers also in wall-bounded turbulence. In particular, the non-uniqueness seen in convex integration constructions such as that of Vasseur & Yang (2023) suggests that such stochasticity should be a general feature of separating turbulent boundary layers at high Reynolds numbers. The limits on predictability of turbulent wall-bounded flows due to spontaneous stochasticity will exacerbate the difficulties in data assimilation already arising due to ordinary chaos (Mons, Du & Zaki 2021; Zaki & Wang 2021). Lagrangian spontaneous stochasticity should also appear in wall-bounded flows (Drivas & Eyink 2017*b*) and this will have implications for turbulent vorticity dynamics via stochastic Lagrangian representations (Constantin & Iyer 2011; Eyink, Gupta & Zaki 2020*a*,*b*; Wang, Eyink & Zaki 2022). In particular, Lagrangian spontaneous stochasticity should allow an alternative formulation of vorticity creation and flux from the boundary surface in the inviscid limit, alternative to the Eulerian description of turbulent vortex dynamics that was sketched in § 4.2.5.

5. Prospects

The ideas proposed by Onsager (1949) on an 'ideal turbulence' at very high *Re* have been the inspiration for extensive work by many researchers over the past three decades. I have attempted to provide an overview of the great progress on this problem, emphasising intuitive ideas rather than rigorous mathematical details or exhaustive references to the literature. I address finally here the key question: what is the promise for future advances?

5.1. *How do we check if it is true?*

The works of the past decades have confirmed the remarkable insights of Onsager (1949) on the mathematical description of Navier–Stokes solutions in the limit of high Reynolds numbers by weak Euler solutions exhibiting dissipative anomalies. As a matter of pure mathematics, several conjectures of Onsager (1949) are now established by rigorous proofs based on ideas with unexpected connections to other areas of mathematics and others are partially demonstrated or reduced to empirically testable propositions. This does not, of course, mean that Onsager's theory is a correct description of physical reality, even apart from the remaining mathematical gaps. For one thing, the basic assumptions of the theory could be invalid. Onsager (1949) began his deliberations by assuming that the incompressible Navier–Stokes model (2.4) is valid for the turbulent flows at $Re \gg 1$, but even a more realistic model such as the Landau–Lifschitz equations (2.5) might break down locally during extreme turbulent events (Bandak *et al.* 2022). Furthermore, the 'ultimate regime' posited by the theory for the limit $\log Re \gg 1$ might not be attained in many practical circumstances and the observations on anomalous dissipation and drag could possibly have more mundane explanations.

The only way to really test the physical validity of the Onsager (1949) theory is by careful laboratory experiments at high Reynolds numbers and, to a lesser extent, by numerical simulations. Some very basic predictions of the theory still remain unverified, such as the deterministic, local 4/5th law (Duchon & Robert 2000; Eyink 2003), which are probably within the range of the current highest-resolution Navier–Stokes simulations. However, numerical simulations, while very useful for their ability to access quantities that are difficult to measure by experiment, obviously incorporate too many theoretical assumptions and biases to provide definitive scientific tests. I hope that one of the consequences of this essay may be a renewed interest in the experimental study of fluid turbulence at extremely high Reynolds numbers. It goes without saying that any experimental study must involve solid walls and obstacles (such as the wire mesh or grid in a wind tunnel). In fact, in my opinion, experimental investigations of turbulence have been too strongly influenced by the bias of many leading theoreticians for 'simple' or 'building block' flows. Homogeneous and isotropic turbulence is one such flow that has obviously been in the past accorded a 'building block' status, but to a lesser extent also the 'canonical wall-bounded flows' – plane-parallel channel flow, smooth pipe flow and flat-plate boundary layer – have been accorded such a status. It is far from clear based on the considerations presented in this essay, however, that any of the previous 'simple' flows are a good starting point for understanding more realistic turbulence. For example, in all of the 'canonical wall-bounded flows', form drag from pressure forces vanishes identically and instead drag arises entirely from skin friction, whereas in most turbulent flows with walls, the drag at high Reynolds numbers gets a non-negligible contribution from form drag, even in flows past streamlined bodies designed to minimise form drag. There are thus legitimate reasons to believe that the 'canonical flows' are not the best stepping-stones to understand wall-bounded turbulence in general.

If I could be granted as a wish one new experiment, I would probably choose it to be an update of the experiment of Achenbach (1972), at much higher Reynolds numbers and taking advantage of modern capabilities for measuring surface pressure, skin friction and near-surface flow characteristics. Even better would be an experimental study of the transient regime of acceleration of the sphere from rest, at a sequence of ever higher Reynolds numbers. It could be somewhat simpler to investigate the flow past a normal disk since, as illustrated in figure 2(*b*), there is no drag crisis and the drag coefficient appears to be nearly constant over four orders of Reynolds number (Roos & Willmarth 1971). A proper such study along these lines should vary also the external turbulence level, to determine its potential effect on all observations. Such a study could possibly detect the extreme events in the skin friction and the near-surface normal velocity that are predicted by breakdown of weak-strong uniqueness, under the assumption of a non-vanishing drag at asymptotically infinite Reynolds numbers (§ 4.2.6). It is unclear what would be the exact nature of such events but they are plausibly associated with strong eruptions of fluid away from the surface. Otherwise, the experiments could verify the uniform vanishing condition (4.25) or more directly the local vanishing of skin friction. In that case, one might hope to verify the consequence (4.28) on continuity of pressure at the wall. Or perhaps we would observe that drag eventually tends to zero after all! Either way, we would have more clarity about the mechanism by which drag appears and how it persists.

However, any good experimentalist who reads this essay can doubtless think of their own tests that can be made of the predictions at very high Reynolds numbers. Such active and thriving experimental work is vital not only for Onsager's theory, but for any theory of wall-bounded turbulence in the very-high-Reynolds-number regime.

5.2. *Why does it matter?*

If the 'ideal turbulence' theory deals only with the presumed ultimate regime of near-infinite Reynolds numbers, then one might wonder how much it matters. Transition to turbulence and intermediate ranges of Reynolds numbers certainly have their own importance and not everything can be understood from a high-Reynolds-number theory. My own response to this important question has several parts.

Ultimately, my basic interest is understanding the physics of how turbulence works and making sense of the diverse array of experimental observations. If there is indeed such an asymptotic high-*Re* regime governed by weak Euler solutions, then describing this state has a fundamental importance for the understanding of turbulence. Not only would this infinite-Reynolds-number 'fixed point' account for the high-Reynolds asymptotics, but also it can be a suitable starting point for finite-Reynolds corrections, as discussed in § 3.1.3.3, bridging the range of intermediate Reynolds numbers. As we saw there also, many of the arguments that have been developed are quite robust and do not depend upon the existence of a 'strong anomaly', with merely a 'weak anomaly' sufficing.

Another reason for the importance of the 'ideal turbulence' theory is that there is already a large amount of discussion of the infinite-Reynolds-number limit in the fluid mechanics literature. Much of this work is based on less rigorous mathematics (e.g. Cantwell 2019; van Kuik 2022), or employs formal asymptotics whose time-range of validity is severely restricted by the underlying ansatz of smooth (and stable) boundary layers (see as examples Bar-Lev & Yang (1975) or the theory of Prandtl (1905), as recently reviewed by Nguyen van yen *et al.* 2018). The Onsager theory can help to put this other body of work on better mathematical footing and, in some cases, to emend and correct it.

In particular, the LES modelling community discusses frequently the infinite-Reynolds-number limit, as mentioned already in § 3.1.1. One segment of that community espouses the method of implicit LES which proposes to calculate turbulent flows at very high Reynolds numbers via numerical discretisations of dissipative weak solutions of the Euler equations (Hoffman & Johnson 2010; Grinstein, Margolin & Rider 2011; Kronbichler & Persson 2021) and in fact the ideas of Onsager (1949) are often cited explicitly in this context (e.g. see Hoffman *et al.* 2015; Fehn *et al.* 2022). The work on implicit LES is very interesting as a kind of 'computational experiment', but the method still lacks a complete mathematical and physical foundation. For example, in the presence of solid walls, it is not yet clear from mathematical theory what are the appropriate boundary conditions for either velocity or for pressure. The open question regarding the pressure boundary was already mentioned in § 4.2.3. As for velocity, natural boundary conditions for smooth Euler solutions are vanishing wall-normal velocity, the so-called 'no-penetration condition'. However, we have seen that a sufficiently strong form of that condition for weak Euler solutions, (4.25), implies weak-strong uniqueness and thus no limiting drag at infinite Reynolds number. Another issue in implicit LES which calls for serious attention by mathematicians is convergence. So far as I know, none of the various numerical schemes that have been proposed to compute dissipative Euler solutions have been proved to converge under grid refinement. An alternative point of view is that

Onsager's theory connects best to LES through the notion of 'coarse-grained solutions' as in (3.5) or (3.8), which may be interpreted as 'effective field theories' with running parameters depending upon the scale ℓ. In that case, convergence is not a relevant issue, as discussed already in § 3.1.3, and it is better to focus on getting answers independent of the arbitrary resolution scale ℓ. Methods from theoretical physics may be useful here, e.g. see Goldenfeld, McKane & Hou (1998). LES is an area that intersects strongly with Onsager's theory and provides fertile ground for interaction between modellers, mathematicians and physicists.

More broadly, the concept of 'ideal turbulence' has importance beyond fluid turbulence proper, as it gives an example of an 'effective field theory' in physics (Schwenk & Polonyi 2012; Liu & Glorioso 2018), but of a novel type which corresponds to an ideal Hamiltonian system for which naïvely conserved quantities are vitiated by anomalies. There is evidence to support such an 'ideal turbulence' description also for systems beyond just high-Reynolds incompressible fluids, e.g. nearly collisionless plasma kinetics is plausibly described by dissipative weak solutions of the ideal Vlasov equation (Eyink 2015, 2018*a*; Bardos *et al.* 2020). Thus, Onsager's theory helps to deepen the connections between fluid mechanics and other areas of physics.

5.3. *Last words*

I leave this essay hopefully having explained what Onsager's 'ideal turbulence' theory is all about, which mathematical results are now established and which conjectures are still open, and how observations so far confirm the theory. As in any living area of science, this review may soon be outdated, either by new theoretical breakthroughs or by novel observations that call for revision of the theory. After working on the subject for more than thirty years, I am still fascinated by the high-Reynolds-number limit of turbulence and I am looking forward to seeing further progress in the next couple of decades, possibly by readers of this essay!

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