DIRECT PRODUCTS OF NORMED LINEAR SPACES

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In this paper we shall study properties of a locally convex space (l.c.s.) which guarantee that it is a direct product of normed linear spaces or Banach spaces. The conditions will be given both as properties of the original space itself and as properties of the dual, and will take the form of a completeness condition and the existence of sub-basic sets of pseudo-norms with certain properties (a set of pseudo-norms is *basic* if the set of unit balls of its members is a base of neighbourhoods of 0. A set $\mathfrak{P}$ is *sub-basic* if

$$\{r_1 p_1 + \ldots + r_n p_n : r_i > 0, p_i \in \mathfrak{P}\}$$

is basic.)

1. Independent sets of pseudo-norms. We begin this section by disposing of the problem of when an l.c.s. is a projective limit of normed linear spaces. We shall need part of the result in what follows.

In **(2)**, a net $\{x_\alpha\}$ is said to be 0-*Cauchy* if for any continuous pseudo-norm p on E, there is an α_0 such that for $\alpha, \beta \geqslant \alpha_0$, $p(x_\alpha - x_\beta) = 0$. A l.c.s. is 0-*complete* if every 0-Cauchy net is convergent.

THEOREM 1. *A l.c.s. E is a projective limit of normed linear spaces if and only if it is* 0-*complete.*

Proof. Sufficiency follows the proof of **(4**, § 19, 9.(1)**)**, where it is shown that a complete l.c.s. is a projective limit of Banach spaces. One simply does not complete the resulting normed linear spaces. It is then easy to construct a 0-Cauchy net in E for any element in the projective limit which converges to the given element. Necessity is straightforward.

Now we proceed to the direct product situation.

Definition. Let $\mathfrak{P}$ be a set of continuous pseudo-norms on E. $\mathfrak{P}$ is *independent* if for any finite subset F of $\mathfrak{P}$, $p \in \mathfrak{P}$, $p \notin F$, for all $x \in E$, there is an $x_F \in E$ such that $p(x - x_F) = 0$, $q(x_F) = 0$ for all $q \in F$.

Our basic result is the following:

THEOREM 2. *A l.c.s. E is a direct product of normed linear spaces if and only if it is* 0-*complete and has an independent sub-basic set $\mathfrak{P}$ of pseudo-norms.*

Proof. Necessity is clear (using Theorem 1). For sufficiency, we embed E in the product of normed linear spaces in the usual way, i.e., for each $p \in \mathfrak{P}$, let $E_p = E/p^{-1}(0)$, ϕ_p: $E \rightarrow E_p$ the natural map, and topologize E_p with the

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norm $p \circ \phi_p^{-1}$. Then $x \to \langle \phi_p(x) \rangle$ is a topological isomorphism of E onto a subspace of $\Pi_{p \in \mathfrak{P}} E_p$. If $\langle x_p \rangle \in \Pi E_p$, choose $y_p \in \phi_p^{-1}(x_p)$ for each $p \in \mathfrak{P}$, and for each finite subset F of $\mathfrak{P}$, there is an $x_F \in E$ such that $p(x_F - y_p) = 0$ for each $p \in F$. This is possible because of independence. Then it is clear that if the set of finite subsets of $\mathfrak{P}$ is ordered by inclusion, $\{x_F\}$ is a 0-Cauchy net and $\{\langle \phi_p(x_F) \rangle\} \to \langle x_p \rangle$. Since E is 0-complete, $\langle x_p \rangle$ is in the image fo E under the embedding.

One result of Theorem 2 is that a l.c.s. with an independent sub-basic set of pseudo-norms is complete if and only if it is 0-complete and sequentially complete. This is not true in general, as is shown in **(3)**.

Another immediate application of the theorem is the following generalization of **(1**, Chapter IV, § 1, Ex. 11**)**.

COROLLARY 2.1. *Let E be a l.c.s. Then E is weakly 0-complete if and only if $E = \mathbf{R}^\alpha$ for suitable cardinal α.*

Proof. Let $\mathfrak{U}$ be a Hamel basis of E'. Then clearly $\{|u|: u \in \mathfrak{U}\}$ is an independent set of pseudo-norms which is sub-basic for the weak topology. For $u \in E'$, $E_{|u|}$ as defined in the proof of Theorem 2 is just $\mathbf{R}$, so E is $\mathbf{R}^\alpha$ in the weak topology. But by **(4**, § 22, 5.(3)**)**, $\mathbf{R}^\alpha$ also has the Mackey topology and since the original topology is between the weak and Mackey topologies, $E = \mathbf{R}^\alpha$. This proves the necessity and the sufficiency is obvious.

2. Independence and duality. In this section we develop the dual notion to independence which will in turn suggest a further area of investigation. Henceforth $\mathfrak{P}$ will be a set of continuous pseudo-norms on E and $\mathfrak{Q}$ a set of absolutely convex equicontinuous subsets of E'. We define a natural correspondence between $\mathfrak{Q}$ and $\mathfrak{P}$ as follows.

For $p \in \mathfrak{P}$,

$$Q_p = \{u \in E': |u| \leqslant p\} = \{x \in E: p(x) \leqslant 1\}^0.$$

If $Q \in \mathfrak{Q}$, p_Q is the pseudo-norm on E whose unit ball is Q^0. We shall call a subspace of co-dimension at most 1 a *hyperspace.*

Definition . $\mathfrak{Q}$ is *independent* if for each finite $\mathfrak{F} \subseteq \mathfrak{Q}$, $Q \in \mathfrak{Q}$, $Q \notin \mathfrak{F}$, and closed hyperspace M of E', there is a closed hyperspace M_0 of E' such that:

(1) $$M_0 \supseteq \cup \mathfrak{F},$$

(2) $$M_0 \cap Q = M \cap Q.$$

(Compare with **(2**, Cor. 4.7**)**.)

THEOREM 3. *The following statements are equivalent:*

(a) *$\mathfrak{Q}$ is independent;*

(b) *the corresponding set $\mathfrak{P} = \{p_Q: Q \in \mathfrak{Q}\}$ of pseudo-norms is independent;*

(c) *for all closed hyperspaces M of E', finite $\mathfrak{F} \subseteq \mathfrak{Q}$, and $Q \notin \mathfrak{F}$, $Q \in \mathfrak{Q}$,*

(3) $$M \cap Q = \operatorname{cl} \mathfrak{L}[(M \cap Q) \cup (\cup \mathfrak{F})] \cap Q.$$

($\mathfrak{L}(A)$ is the linear span of A.)

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Proof. (a) $\Rightarrow$ (c). The left side of (3) is always contained in the right, so we need only show the reverse inclusion. But if M_0 is a closed hyperspace satisfying (1) and (2), then $M_0 \supseteq \mathfrak{L}((M \cap Q) \cup (\cup \mathfrak{F}))$ so that

$$[\mathrm{cl}\,\mathfrak{L}((M \cap Q) \cup (\cup \mathfrak{F}))] \cap Q \subseteq \mathrm{cl}(M_0) \cap Q$$
$$= M_0 \cap Q = M \cap Q.$$

(c) $\Rightarrow$ (b). Let F be a finite subset of $\mathfrak{P}$, $p \in \mathfrak{P}$, $p \notin F$, $x \in E$. If $p(x) = 0$, then set $x_F = 0$. Otherwise we may choose a $u_0 \in Q_p$ with $u_0(x) \neq 0$ by the Hahn–Banach theorem. Let $M = x^\perp = \{u \in E'\colon u(x) = 0\}$, which is a closed hyperspace in E. Let $\mathfrak{F} = \{Q_q\colon q \in F\}$. By (3) we may choose $x_F \in E$ such that $u_0(x_F) \neq 0$ and $(M \cap Q_p) \cup (\cup \mathfrak{F}) \subseteq x_F{}^\perp$. In fact, we can assume that $u_0(x_F) = u_0(x)$. For all $q \in F$, $Q_q \subseteq x_F{}^\perp$ so that $q(x_F) = 0$. If $v \in Q_p$, then $v = m + ru_0$ for some $m \in M$, $r \in \mathbf{R}$, so $m = v - ru_0 \in (1 + |r|)Q_p$, $m \in M = (1 + |r|)M$, so $m \in (1 + |r|)(Q_p \cap M) \subseteq (1 + |r|)(Q_p \cap x_F{}^\perp)$ and $v(x_F) = m(x_F) + ru_0(x_F) = ru_0(x) = v(x)$. Therefore $x - x_F \in Q_p{}^\perp$, so $p(x - x_F) = 0$.

(b) $\Rightarrow$ (a). Suppose $\mathfrak{P}$ is independent and M is a closed hyperspace in E'; then $M = x^\perp$ for some $x \in E$. Let $Q \in \mathfrak{Q}$, $F \subseteq \mathfrak{Q}$ finite, $Q \notin \mathfrak{F}$. Choose x_F such that $p_Q(x_F - x) = 0$, $p_{Q'}(x_F) = 0$ for all $Q' \in \mathfrak{F}$, so $Q' \subseteq x_F{}^\perp$. $p_Q(x_F - x_0)$ implies that $u(x_0) = u(x)$ for all $u \in Q_p$, so (1) and (2) are satisfied for $M_0 = x_F{}^\perp$.

The above characterization suggests the following:

Definition. $\mathfrak{Q}$ is *quasi-independent* if for all $Q \in \mathfrak{Q}$

$$\mathfrak{L}(\cup(\mathfrak{Q} \sim \{Q\})) \cap Q = \{0\}.$$

A set of pseudo-norms is quasi-independent if its corresponding set of equicontinuous subsets is quasi-independent.

Analogously to Theorem 3 we have

THEOREM 4. *The following statements are equivalent:*

(a) $\mathfrak{Q}$ *is quasi-independent;*

(b) *for all* $Q \in \mathfrak{Q}$ *and closed hyperspaces* $M \subseteq E'$, *there is a* (*not necessarily closed*) *hyperspace* $M_0 \subseteq E'$ *satisfying* (2) *and*

$$M_0 \supseteq \cup (\mathfrak{Q} \sim \{Q\}); \tag{4}$$

(c) *for all* $Q \in \mathfrak{Q}$ *and all subspaces* $M \subseteq E'$, *there is a subspace* $M_0 \subseteq E'$ *satisfying* (2) *and* (4).

Proof. (a) $\Rightarrow$ (c). Let M be a subspace of E' and for fixed $Q \in \mathfrak{Q}$, set

$$M_0 = \mathfrak{L}((M \cap Q) \cup (\cup (\mathfrak{Q} \sim \{Q\}))).$$

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If $u \in M_0 \cap Q$, then $u = rv + w$, $v \in M \cap Q$, $w \in \mathfrak{L}(\bigcup(\mathfrak{Q} \sim \{Q\}))$, $r \in \mathbf{R}$. Hence $w \in (1 + |r|)Q$, so by assumption $w = 0$. Therefore $u = rv \in M$ and $u \in M \cap Q$. Hence $M_0 \cap Q \subseteq M \cap Q$ and the reverse inclusion is obvious, as is (4).

(c) $\Rightarrow$ (b). Suppose M is a closed hyperspace in E', $Q \in \mathfrak{Q}$. If $Q \subseteq M$, then we are finished, so let $u \in Q \sim M$, and let M_0 be a subspace satisfying (2) and (4). Then $u \notin M_0$ and there is a maximal subspace (hence hyperspace) M_1 containing M_0 but not u. Clearly $M_1 \supseteq \bigcup (\mathfrak{Q} \sim \{Q\})$, $M_1 \cap Q \supseteq M_0 \cap Q$. If $v \in M_1 \cap Q$, then $v = m + ru$, $m \in M$, $r \in \mathbf{R}$. As usual, then

$$(1 + |r|)^{-1}m \in M \cap Q = M_0 \cap Q \subseteq M_1 \cap Q,$$

so $r \neq 0$ implies that $u \in M_1$, a contradiction.

(b) $\Rightarrow$ (a). Suppose that for some $Q \in \mathfrak{Q}$, $u \in \mathfrak{L}(\bigcup(\mathfrak{Q} \sim \{Q\})) \cap Q$. If $u(x) \neq 0$ for some $x \in E$, then take $M = x^{\perp}$ and let M_0 satisfy (2) and (4). Then $u \in \mathfrak{L}(\bigcup(\mathfrak{Q} \sim \{Q\})) \subseteq M_0$ and $u \in Q \sim M = Q \sim M_0$, so $u \notin M_0$, a contradiction.

COROLLARY 4.1. *An independent set of pseudo-norms is quasi-independent.*

Proof. Let $\mathfrak{P}$ be independent, $\mathfrak{Q}$ correspond to $\mathfrak{P}$. If $\mathfrak{F}$ is any finite subset of $\mathfrak{Q}$, then Theorem 4 (b) $\Rightarrow$ (a) and Theorem 3 (b) $\Rightarrow$ (a) imply that $\mathfrak{F}$ is quasi-independent. But every member of $\mathfrak{L}(\bigcup(\mathfrak{Q} \sim \{Q\}))$ is a member of some $\mathfrak{L}(\bigcup(\mathfrak{F} \sim \{Q\}))$; hence the result.

We now prove a version of Theorem 2 for quasi-independence.

COROLLARY 4.2. A *l.c.s. E is a direct product of Banach spaces if and only if it is complete and has a quasi-independent sub-basic set of pseudo-norms.*

Proof. Necessity follows from Corollary 4.1 and Theorem 2. For sufficiency, suppose that E is complete, $\mathfrak{P}$ a sub-basic and quasi-independent set of pseudo-norms, $\mathfrak{Q}$ corresponding to $\mathfrak{P}$. Let M be a closed hyperspace in E', $p \in \mathfrak{P}$, and let M_0 be a hyperspace satisfying (2) and (4) for $Q = Q_p$. If we can show that M_0 is closed, we are finished, since then $\mathfrak{P}$ will be independent by Theorem 3. By the Grothendieck theorem (**4**, § 21, 9.(6)) and completeness of E it is enough to show that M_0 is weakly closed in A for each equicontinuous subset A of E'. But if A is such a set, then there are scalars, $r, r_1, \ldots, r_n$ and $p_1, \ldots, p_n \in \mathfrak{P}$ such that $p \neq p_i$ for all i and

$$A \subseteq B = \mathbf{C}\left(r\, Q_p \cup \bigcup_{i=1}^{n} r_i\, Q_{p_i}\right)$$

($\mathbf{C}(X)$ is the convex hull of X) and it is enough to show that $M_0 \cap B$ is weakly closed. Now

$$M_0 \cap B = \mathbf{C}\left(r(M_0 \cap Q_p) \cup \bigcup_{i=1}^{n} r_i(M_0 \cap Q_{p_i})\right)$$
$$= \mathbf{C}\left(r(M \cap Q_p) \cup \bigcup_{i=1}^{n} r_i\, Q_{p_i}\right)$$

since $M_0 \supseteq r_i Q_{pi}$ for all i. Now $r(M \cap Q_p)$ and $r_i Q_{pi}$ are weakly compact and convex, so by (**4**, § 20, 6.(5)), $M_0 \cap B$ is weakly compact, hence closed.

Corollary 4.2 says essentially that if a complete topology on E is uniform convergence on "linearly independent" equicontinuous subsets of E', then E is a product of Banach spaces. Note that while the definition of independence includes a sort of completeness assumption on E (the existence of the x_F), quasi-independence does not.

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