

A DEFORMATION OF THE ALEXANDER POLYNOMIALS OF KNOTS YIELDING LENS SPACES

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For a knot K in a homology 3-sphere Σ , by $\Sigma(K; p/q)$, we denote the resulting 3-manifold of p/q -surgery along K . We say that the manifold or the surgery is of *lens type* if $\Sigma(K; p/q)$ has the same Reidemeister torsion as a lens space.

We prove that, for $\Sigma(K; p/q)$ to be of lens type, it is a necessary and sufficient condition that the Alexander polynomial $\Delta_K(t)$ of K is equal to that of an (i, j) -torus knot $T(i, j)$ modulo $(t^p - 1)$.

We also deduce two results: If $\Sigma(K; p/q)$ has the same Reidemeister torsion as $L(p, q')$, then

- (1) q' or $\bar{q}' \equiv \pm qi^2 \pmod{p}$.
- (2) The multiple of $\Delta_K(t^k)$ over $k \in \langle i \rangle$ is $\pm t^m$ modulo $(t^p - 1)$, where $\langle i \rangle$ is the subgroup in $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$ generated by i . Conversely, if a subgroup H of $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$ satisfying that the product of $\Delta_K(t^k)$ ($k \in H$) is $\pm t^m$ modulo $(t^p - 1)$, then H includes i or j .

Here, i, j are the parameters of the torus knot above.

1. INTRODUCTION

For a knot K in a homology 3-sphere Σ , by $\Sigma(K; p/q)$, we denote the resulting 3-manifold of p/q -surgery along K . Throughout this paper, we assume that $p \geq 2$ and $q \neq 0$ without loss of generality. Then $\Sigma(K; p/q)$ is a homology lens space whose first homology $H_1(\Sigma(K; p/q); \mathbf{Z})$ is isomorphic to the cyclic group $\mathbf{Z}/p\mathbf{Z}$ of order p . We study the question “Which $\Sigma(K; p/q)$ is a lens space?” by algebraic methods.

We let $(\mathbf{Z}/p\mathbf{Z})^\times$ denote the group, with respect to the multiplication, of invertible elements in the ring $\mathbf{Z}/p\mathbf{Z}$. For an element i , by \bar{i} we denote its inverse. In our study, its quotient group $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$ plays an important role. Note that two lens spaces $L(p, q_1)$ and $L(p, q_2)$ are homeomorphic to each other up to orientation if and only if $q_1 = q_2$ or $q_1 = \bar{q}_2$ in $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$.

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We say that a polynomial $f(t) \in \mathbf{Z}[t]$ is *symmetric* if $t^d f(t^{-1}) = f(t)$ (thus $f(0) \neq 0$ if $d > 0$), where d is the degree of $f(t)$, or $f(t) = c \ (\in \mathbf{Z})$. The Alexander polynomial $\Delta_K(t)$ is usually defined in the Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$ up to multiplication as a unit (a trivial unit) $\pm t^m$, where m is an integer. But, in this paper, we always take a regularisation such that $\Delta_K(t) \in \mathbf{Z}[t]$, the symmetry $t^{\deg} \Delta_K(t^{-1}) = \Delta_K(t)$ holds, and $\Delta_K(1) = 1$. Here, \deg is the degree of $\Delta_K(t)$, which is always even. We often use $t^{-(\deg/2)} \Delta_K(t)$ in $\mathbf{Z}[t, t^{-1}]$.

We often treat with Alexander polynomials in the quotient ring of type $\mathbf{Z}[t, t^{-1}]/I$, where I is an ideal like $(t^p - 1)$. For two elements $f(t), g(t)$ in $\mathbf{Z}[t, t^{-1}]$ or in $\mathbf{Z}[t, t^{-1}]/I$, by $f(t) \doteq g(t)$, we denote that $f(t) = \pm t^m g(t)$ for an integer m .

In [12, 13, 14], the first author defined the concept ‘‘lens type’’: Roughly speaking, we say that a homology lens space M with $H_1(M; \mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z}$ is of (p, q) -lens type (or simply of lens type) if the Reidemeister torsion of M is same with that of a lens space $L(p, q)$ (for some q). See Section 2 for the precise definition. Every lens space is of lens type. We adopt the definition of the Reidemeister torsion with a combinatorial ‘‘Euler structure’’ following Turaev [30, 31].

Our first main theorem is on a deformation of the Alexander polynomial $\Delta_K(t)$ of a knot K in a homology 3-sphere Σ which has a lens type surgery, that is, a Dehn surgery yielding a lens type manifold.

THEOREM 1.1. *Let $\Delta_K(t)$ be the regularised Alexander polynomial of a knot K in a homology 3-sphere Σ . If a Dehn surgery $\Sigma(K; p/q)$ is of lens type, then there exist positive integers i, j and a symmetric polynomial $R(t)$ such that the triple p, i, j are mutually coprime, $ijq \equiv \pm 1 \pmod{p}$ and $\Delta_K(t)$ satisfies*

$$t^{-(\deg/2)} \Delta_K(t) = t^{-((i-1)(j-1))/2} \frac{(t^{ij} - 1)(t - 1)}{(t^i - 1)(t^j - 1)} - t^{-(p+1+r)/2} (t^p - 1)(t - 1)R(t)$$

in $\mathbf{Z}[t, t^{-1}]$, where \deg and r are the degrees of $\Delta_K(t)$ and $R(t)$, respectively.

Conversely, if $\Delta_K(t)$ has such a deformation as above, then $\Sigma(K; p/q)$ is of lens type, that is, $\Sigma(K; p/q)$ has the same Reidemeister torsion as that of a lens space.

The first term in the right hand-side of the equality is the Alexander polynomial of the (i, j) -torus knot $T(i, j)$:

$$\Delta_{T(i,j)}(t) = \frac{(t^{ij} - 1)(t - 1)}{(t^i - 1)(t^j - 1)}.$$

Thus, for lens surgery along a torus knot in S^3 ([19], see also [12]), Theorem 1.1 holds with $R(t) = 0$. The integers i, j in Theorem 1.1 are not unique. Such ambiguity will be considered in Lemma 3.4 in Section 3.

Next, we study the integer i (and j) that appeared in Theorem 1.1 more, and deduce the following two theorems.

THEOREM 1.2. *If a Dehn surgery $\Sigma(K; p/q)$ is of lens type, then it is of (p, qi^2) -lens type, that is, $\Sigma(K; p/q)$ has the same Reidemeister torsion as $L(p, qi^2)$, where i is the integer that appeared in Theorem 1.1*

Theorem 1.2 extends Fintushel-Stern’s in [5, Proposition 1], which says that if a lens space $L(p, r)$ is the integral Dehn surgery on a knot, then r or $-r$ is a quadric residue. As the integer i in Theorem 1.2, we can take the other integer j that appeared in Theorem 1.1. In fact, $ijq \equiv \pm 1$ implies $qj^2 \equiv \bar{q}i^2 \pmod{p}$ and thus $L(p, qj^2) = L(p, qi^2)$.

THEOREM 1.3. *Suppose that $\Sigma(K; p/q)$ is of lens type as above.*

- (1) *Let $\langle i \rangle$ (and $\langle j \rangle$, respectively) denote the cyclic subgroup in $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$ generated by i (and j) that appeared in Theorem 1.1. Then, the Alexander polynomial $\Delta_K(t)$ of K satisfies*

$$\prod_{k \in \langle i \rangle} \Delta_K(t^k) \doteq 1, \quad \text{and} \quad \prod_{k \in \langle j \rangle} \Delta_K(t^k) \doteq 1 \quad \text{in } \mathbf{Z}[t, t^{-1}]/(t^p - 1).$$

- (2) *Any subgroup H of $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$ satisfying*

$$\prod_{k \in H} \Delta_K(t^k) \doteq 1 \quad \text{in } \mathbf{Z}[t, t^{-1}]/(t^p - 1)$$

includes i or j (possibly both i and j).

In [34], the second author announced an example of Theorem 1.3, the case of the famous non-torus lens surgery along the pretzel knot $P(-2, 3, 7)$: 18-surgery ($\langle i \rangle = \{1, 5, 11\}$) and 19-surgery ($\langle i \rangle = \{1, 7, 11\}$), see Section 5.

In some cases, a weaker version of Theorem 1.3 possibly tells us that $\Sigma(K; p/q)$ is neither of lens type nor a lens space. For this purpose, we state it as a corollary:

COROLLARY 1.4. *If $\Sigma(K; p/q)$ is of lens type, then the Alexander polynomial $\Delta_K(t)$ of K satisfies*

$$\prod_{k \in (\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}} \Delta_K(t^k) \doteq 1 \quad \text{in } \mathbf{Z}[t, t^{-1}]/(t^p - 1).$$

The paper is organised as follows: In Section 2, we set up the preliminaries and re-define the concept *lens type* precisely, as a sharpened version of that in the first author’s previous papers [12, 13]. We also review the surgery formula due to Turaev [29, 30, 31] and Sakai [26], which connects Alexander polynomial and Reidemeister torsion. In Section 3, we prove Theorem 1.1 and 1.2. In Section 4, we prove Theorem 1.3 and study minimal subgroups satisfying the assumption of Theorem 1.3(2). In Section 5, we demonstrate some examples: two examples from rational lens surgeries along torus knots, and the famous non-torus lens surgeries $(P(-2, 3, 7); 18)$ and $(P(-2, 3, 7); 19)$ in S^3 . We also study two families of lens surgery (Fintushel-Stern’s knots K_n in [5], and Berge’s

knots $k(a, b)$ in [3, 33]; $P(-2, 3, 7) = K_2 = k(2, 3)$). At the end of the paper, we shall refer to some recent works in this subject.

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2. PRELIMINARIES

In this section, we prepare definitions and notations, and review the results from the first author’s results [12, 13]. Our method is a number theoretic study of Reidemeister torsion in the cyclotomic fields $\mathbf{Q}(\zeta)$. For the definition of Reidemeister torsion, see [18, 29, 30, 31].

2.1. 3-MANIFOLD OF LENS TYPE. For an oriented closed 3-manifold M , the concept that M is of *lens type* was originally defined in [12, 13], and was applied in [14]. Here we sharpen the definition slightly for the purpose in this paper.

Let M be a homology lens space whose first homology $H_1(M; \mathbf{Z})$ ($H_1(M)$ for short) is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ generated by an element t . We take a ring homomorphism $\psi_d : \mathbf{Z}[H_1(M)] \rightarrow \mathbf{Q}(\zeta)$, where the subindex d ($d \geq 2$) is the order of the image $\zeta = \psi_d(t)$ of t (thus d is a divisor of p), and $\mathbf{Q}(\zeta)$ is the d -th cyclotomic field. The Reidemeister torsion $\tau^{\psi_d}(M)$ of M associated to ψ_d is defined, via the simplicial chain complex of the cyclic Abelian covering space \widetilde{M} as $\mathbf{Z}[H_1(M)]$ -module, as an element in $\mathbf{Q}(\zeta)$ up to multiplication by $\pm\zeta^m$ for an integer m , which we call a *trivial unit*.

We shall study $\tau^{\psi_d}(M)$ of a fixed $M = \Sigma(K; p/q)$ with various divisors d of p . To avoid the ambiguity of the multiplicity of trivial units, we fix an oriented basis of simplicial chain complex of \widetilde{M} . Up to now, it has been known that such fixing the basis corresponds to fixing “Euler structure” of the 3-manifold defined by Turaev ([30, 31]), (In such a sense, it might be better that we introduce a new notation such as $\tau^{\psi_d}(M, \mathbf{e})$, where, by \mathbf{e} , we denote the specified Euler structure, but we keep the shorter notation $\tau^{\psi_d}(M)$.)

DEFINITION 2.1: We say that a homology lens space M with $H_1(M) \cong \mathbf{Z}/p\mathbf{Z}$ generated by t is of *lens type* if there exists ε, m, i and j such that, for every divisor d of p ($d \geq 2$) and ψ_d , the Reidemeister torsion satisfies

$$\tau^{\psi_d}(M) = \varepsilon\zeta^m(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1},$$

where $\zeta = \psi_d(t)$ is the primitive d -th root of unity,

- (1) $\varepsilon = 1$ or -1 , and m is a positive integer independent from d ,
- (2) i and j are positive integers independent from d , and
- (3) i and j are coprime to p .

Furthermore, more precisely, a 3-manifold M of lens type is of (p, q) -lens type if $i\bar{j} \equiv q$ or \bar{q} in $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$.

The Reidemeister torsion of the lens space $L(p, q)$ is known:

$$\tau^{\varphi_d}(L(p, q)) = (\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1},$$

for a good choice of φ_d , up to the multiplicity by trivial units. See also the next subsection ($L(p, q)$ is the $p/(-q)$ -surgery along the unknot). Each lens space $L(p, q)$ is of (p, q) -lens type.

This definition is sharper, at the condition (1), than that in the first author’s previous papers [12, 13] and the previous joint work [14].

2.2. SURGERY FORMULA FOR REIDEMEISTER TORSIONS. In the case that M is the p/q -surgery $\Sigma(K; p/q)$ on a knot K in a homology 3-sphere Σ , we take the oriented meridian of K as the generator t of $H_1(M) \cong \mathbf{Z}/p\mathbf{Z}$.

THEOREM 2.2. (Turaev [29]; Sakai [26].) *In the case $M = \Sigma(K; p/q)$, t is a meridian of K and $\psi_d(t) = \zeta$, we have*

$$\tau^{\psi_d}(M) = \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}.$$

3. PROOFS OF THEOREM 1.1 AND 1.2

Suppose that $\Sigma(K; p/q)$ is of lens type. Then, by Definition 2.1 and Turaev’s theorem 2.2, it holds that

$$(3.1) \quad \Delta_K(\zeta) = \varepsilon \zeta^m \frac{(\zeta - 1)(\zeta^{\bar{q}} - 1)}{(\zeta^i - 1)(\zeta^j - 1)} \quad \text{in } \mathbf{Z}(\zeta),$$

for any divisor d of p , where $\varepsilon = \pm 1$, m, i, j (i and j are coprime to p) are positive integers, all independent from d . In this section, we let ζ_d denote the d -th primitive root of unity.

Now, we show the following lemma.

LEMMA 3.1. *In the situation above,*

- (1) $\bar{q} \equiv \varepsilon ij \pmod{p}$, and
- (2) $2m - \text{deg} \equiv -(i - 1)(j - 1) + (1 - \varepsilon)ij \pmod{p}$,

where deg is the degree of the regularised Alexander polynomial $\Delta_K(t)$.

PROOF: (1) We took the inverse \bar{q} of q in $(\mathbf{Z}/p\mathbf{Z})^\times$ before, but here we re-define \bar{q} as a positive integer (a lift) satisfying $q\bar{q} \equiv 1 \pmod{p}$. We also fix positive integers \bar{i} and \bar{j} that satisfy $i\bar{i} \equiv j\bar{j} \equiv 1 \pmod{p}$. We set

$$F(t) := \varepsilon t^m \frac{t^{i\bar{i}} - 1}{t^i - 1} \cdot \frac{t^{j\bar{j}\bar{q}} - 1}{t^j - 1}.$$

Note that $F(t)$ is a polynomial ($F(t) \in \mathbf{Z}[t]$), because

$$\frac{t^{kl} - 1}{t^k - 1} = 1 + t^k + t^{2k} + \dots + t^{(l-1)k}$$

for $(k, l) = (i, \bar{i}), (j, \bar{j}\bar{q})$.

By the assumption of lens type-ness (3.1), for each divisor d of p we have $\Delta_K(\zeta_d) = F(\zeta_d)$ in $\mathbf{Z}[\zeta_d]$. This implies $\Delta_K(t) - F(t)$ is in the kernel of the natural projection $\pi_d : \mathbf{Z}[t] \rightarrow \mathbf{Z}[\zeta_d]$ (defined by $\pi_d(f(t)) = f(\zeta_d)$), which is the ideal $\langle \Phi_d(t) \rangle$ generated by the d -th cyclotomic polynomial $\Phi_d(t)$.

Since cyclotomic polynomials are irreducible and

$$\prod_{d|p, d \geq 2} \Phi_d(t) = \frac{t^p - 1}{t - 1},$$

there exists $g(t) \in \mathbf{Z}[t]$ such that

$$\Delta_K(t) - F(t) = \frac{t^p - 1}{t - 1} \cdot g(t).$$

By substituting $t = 1$, we have

$$1 - \varepsilon \bar{i} \bar{j} \bar{q} = p \cdot g(1) \equiv 0 \pmod{p}.$$

This means $\bar{q} \equiv \varepsilon ij \pmod{p}$.

(2) By the symmetry of $\Delta_K(t)$, we have $\zeta^{\deg} \Delta_K(\zeta^{-1}) = \Delta_K(\zeta)$ for the p -th primitive root of unity ζ . By Equation (3.1), we have $\zeta^{\deg - m - 1 - \bar{q} + i + j} = \zeta^m$. Here we used $(\zeta^{-k} - 1) = -\zeta^{-k}(\zeta^k - 1)$. By the congruence (1), we have the congruence. □

Lemma 3.1(1) implies Theorem 1.2;

PROOF OF THEOREM 1.2: If we assume that $\Sigma(K; p/q)$ is of lens type as Equation (3.1), then the integers ε, i, j satisfy $\bar{j} \equiv \varepsilon qi \pmod{p}$ by Lemma 3.1(1). Thus $i\bar{j} = qi^2$ in $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$. □

Before starting the proof of Theorem 1.1, we control the indexes i, j, ε and m . First, we may assume the sign ε is $+1$: Because of the equality

$$\varepsilon \zeta^m \frac{(\zeta - 1)(\zeta^{\varepsilon ij} - 1)}{(\zeta^i - 1)(\zeta^j - 1)} = (-\varepsilon) \zeta^{m-i} \frac{(\zeta - 1)(\zeta^{(-\varepsilon)(-i)j} - 1)}{(\zeta^{(-i)} - 1)(\zeta^j - 1)},$$

we can change the sign ε by changing i to $Np - i$ (or j to $Np - j$) and m , where N is a sufficiently large integer such that $Np - i > 0$. Next, we may assume that i and j are coprime (denoted by $(i, j) = 1$): Because of the prime number theorem of an arithmetic progression, for given integers i, j both coprime to p , there exist positive integers i' and j' such that $i' \equiv i, j' \equiv j \pmod{p}$ such that $(i', j') = 1$. These changes do not change the class of i, j in $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$.

PROOF OF THEOREM 1.1: We assume that $(i, j) = 1$ and $\varepsilon = 1$, by changing the indexes as above, if it is needed. Now, we set

$$H(t) := t^m \frac{(t-1)(t^{ij}-1)}{(t^i-1)(t^j-1)} \quad \text{in } \mathbf{Z}[t].$$

Then, by Lemma 3.1(1) ($\bar{q} \equiv ij \pmod{p}$), we have $\Delta_K(\zeta_d) = F(\zeta_d) = H(\zeta_d)$ for any divisor d of p , and also $\Delta_K(1) = H(1) = 1$. Thus, by the same argument as in the proof of Lemma 3.1, $\Delta_K(t) - H(t)$ is divisible by $(t^p - 1)$.

If p is odd, then Lemma 3.1(2) ($2m - \text{deg} \equiv -(i-1)(j-1) \pmod{p}$) implies that $m - (\text{deg}/2) \equiv -((i-1)(j-1))/2 \pmod{p}$. This congruence holds even if p is even, but we postpone the proof until the final part, as Claim 3.2. Then,

$$\mathcal{D}(t) := t^{-(\text{deg}/2)} \Delta_K(t) - t^{-((i-1)(j-1))/2} \frac{(t-1)(t^{ij}-1)}{(t^i-1)(t^j-1)} \quad (\text{in } \mathbf{Z}[t, t^{-1}])$$

is also divisible by $(t^p - 1)$, that is, there exists $h(t) \in \mathbf{Z}[t, t^{-1}]$ such that

$$\mathcal{D}(t) = (t^p - 1) \cdot h(t).$$

Since $\mathcal{D}(t)$ satisfies $\mathcal{D}(t^{-1}) = \mathcal{D}(t)$, $h(t)$ is divisible by $(t - 1)$. Thus we reset $\mathcal{D}(t) = (t^p - 1)(t - 1)k(t)$, where $k(t) \in \mathbf{Z}[t, t^{-1}]$. By the symmetry $\mathcal{D}(t^{-1}) = \mathcal{D}(t)$ again, $k(t)$ has to be in the form required in the statement. \square

Finally, we prove the following:

CLAIM 3.2. Even if p is even, $m - (\text{deg}/2) \equiv -((i-1)(j-1))/2 \pmod{p}$.

PROOF OF CLAIM 3.2: If the claimed congruence does not hold, by Lemma 3.1(2), $m - (\text{deg}/2) \equiv -((i-1)(j-1) + p)/2 \pmod{p}$. Consider

$$\mathcal{E}(t) := t^{-(\text{deg}/2)} \Delta_K(t) - t^{-((i-1)(j-1))/2} \frac{(t-1)(t^{ij}-1)}{(t^i-1)(t^j-1)}$$

in $\mathbf{Z}[t, t^{-1}]$. By the same argument as the proof of Lemma 3.1, we have $\mathcal{E}(t) = 0$ in $\mathbf{Z}[t, t^{-1}]/I_{p/2}$, where $I_{p/2}$ is the ideal generated by $(t^{p/2} - 1)$.

Let $p = 2^r p'$, where p' is odd, be the decomposition of p . Let ζ be the 2^r -th primitive root of unity, thus $\zeta^{p/2} = -1$. Because of the lens type-ness (the indexes has been controlled), we have

$$\Delta_K(\zeta) = \zeta^m \frac{(\zeta-1)(\zeta^{ij}-1)}{(\zeta^i-1)(\zeta^j-1)}$$

Thus we have

$$\begin{aligned} \mathcal{E}(\zeta) &= \zeta^{-(\text{deg}/2)} \Delta_K(\zeta) - \zeta^{-((i-1)(j-1))/2} \frac{(\zeta-1)(\zeta^{ij}-1)}{(\zeta^i-1)(\zeta^j-1)} \\ &= -2 \zeta^{-((i-1)(j-1))/2} \frac{(\zeta-1)(\zeta^{ij}-1)}{(\zeta^i-1)(\zeta^j-1)} \end{aligned}$$

and $\mathcal{E}(\zeta)/2$ has norm ± 1 : $|N(\mathcal{E}(\zeta)/2)| = 1$.

On the other hand, because $\mathcal{E}(1/t) = \mathcal{E}(t)$, there exist $g(t)$ such that $\mathcal{E}(t) = (t^{p/2} - 1)(t - 1)g(t)$. Substituting $t = \zeta$, we have

$$\frac{1}{2}\mathcal{E}(\zeta) = \frac{1}{2}(\zeta^{p/2} - 1)(\zeta - 1)g(\zeta) = -(\zeta - 1)g(\zeta).$$

Then, because $|N(\zeta - 1)| = 2$ and that the norm of an algebraic integer $g(\zeta) \in \mathbf{Z}[\zeta]$ is a non-zero integer, thus $|N(\mathcal{E}(\zeta)/2)| = 2|N(g(\zeta))| \geq 2$. We have a contradiction. \square

For the *norm* in a Galois extension over \mathbf{Q} , see [32]. The norm of $\Delta_K(\zeta)$ in $\mathbf{Q}(\zeta)$ plays an important role in our study, see [12, 14].

COROLLARY 3.3. *Let $\Delta_K(t)$ be the Alexander polynomial of a knot K in a homology 3-sphere Σ . If $\Sigma(K; p/q)$ is of lens type, then there exist positive integers i, j such that the triple p, i, j are mutually coprime, $ijq \equiv \pm 1 \pmod{p}$ and $\Delta_K(t)$ satisfies*

$$\Delta_K(t) \doteq \frac{(t^{ij} - 1)(t - 1)}{(t^i - 1)(t^j - 1)} \quad \text{in } \mathbf{Z}[t, t^{-1}]/(t^p - 1).$$

We shall use this Corollary in Section 4.

We have already remarked on the non-uniqueness of the indexes i, j, m in Theorem 1.1, but, the theorem is independent of the ambiguity, by the following lemma.

LEMMA 3.4. *Let i and j (respectively i' and j') be positive integers such that the triple p, i, j (and p, i', j') are mutually coprime. We define $H(t)$ and $H'(t)$ as follows:*

$$H(t) := t^{-((i-1)(j-1))/2} \frac{(t^{ij} - 1)(t - 1)}{(t^i - 1)(t^j - 1)}, \quad H'(t) := t^{-((i'-1)(j'-1))/2} \frac{(t^{i'j'} - 1)(t - 1)}{(t^{i'} - 1)(t^{j'} - 1)}.$$

(1) *If $i \equiv i', j \equiv j' \pmod{p}$, then*

$$\frac{(i - 1)(j - 1)}{2} \equiv \frac{(i' - 1)(j' - 1)}{2} \pmod{p}.$$

(2) *If $i \equiv \pm i', j \equiv \pm j' \pmod{p}$, then $H(t) - H'(t)$ is divisible by $(t^p - 1)$.*

PROOF: (1) We divide the proof into two cases.

CASE 1. (p is odd). Then the homomorphism $2 : \mathbf{Z}/p\mathbf{Z} \rightarrow \mathbf{Z}/p\mathbf{Z}$ defined by $2(x) = 2x$ is an isomorphism. Thus $i \equiv i', j \equiv j'$ implies $(i - 1)(j - 1) \equiv (i' - 1)(j' - 1)$ and also the required congruence.

CASE 2. (p is even). Then both i and j are odd, because of the coprime-ness. We only have to prove the congruence in the case $j = j'$. The difference is

$$\frac{(i - 1)(j - 1)}{2} - \frac{(i' - 1)(j - 1)}{2} = \frac{(i - i')}{2} \cdot (j - 1) \equiv 0,$$

because $(j - 1)$ is even.

(2) First, we treat with the case $i \equiv i', j \equiv j' \pmod{p}$. We have $H(1) = H'(1)$. By the assumption and the congruence (1), for any divisor $d (\geq 2)$ of p , we have also $H(\zeta_d) = H'(\zeta_d)$. Thus $H(t) - H'(t)$ is divisible by $(t^p - 1)$.

Next, we treat with the case that $i \equiv -i' \pmod{p}$ and $j' = j$. By the deformation

$$t^{-((i'-1)(j-1))/2} \frac{(t^{i'j} - 1)(t - 1)}{(t^{i'} - 1)(t^j - 1)} = t^{-((i'-1)(j-1))/2 + i'(j-1)} \frac{(t^{(-i')j} - 1)(t - 1)}{(t^{(-i')} - 1)(t^j - 1)}$$

and that

$$-\frac{(i' - 1)(j - 1)}{2} + i'(j - 1) = -\frac{((-i') - 1)(j - 1)}{2} \equiv -\frac{(i - 1)(j - 1)}{2}$$

by (1), we can go back to the previous case. The other cases can be proved by combination of the cases above. □

4. PROOF OF THEOREM 1.3

First, we remark that, if $q = 1$ (that is, the case of integral surgery), then $\langle i \rangle = \langle j \rangle$, because we have $j = \pm i$ by Lemma 3.1(1). On the other hand, in the case of rational surgery, there exists a case that $\langle i \rangle \neq \langle j \rangle$ (possibly $\langle i \rangle \subset \langle j \rangle$, $\langle j \rangle \subset \langle i \rangle$, or $(\langle i \rangle \not\subset \langle j \rangle$ and $\langle i \rangle \not\subset \langle j \rangle)$), see examples in Section 5.

PROOF OF (1): Suppose that $\Sigma(K; p/q)$ is of lens type. By Corollary 3.3, there exist positive integers i, j and an integer m (according to the indexes of the lens type-ness), such that

$$\Delta_K(t) = t^m \frac{(t^{ij} - 1)(t - 1)}{(t^i - 1)(t^j - 1)} \text{ in } \mathbf{Z}[t, t^{-1}]/(t^p - 1).$$

Because of the symmetry between i and j , we only prove the theorem in the case of i . We let I denote the order of i in the group $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$, thus I is equal to the cardinality of the subgroup $\langle i \rangle$. We shall take the multiple of $\Delta_K(t^k)$'s over $k \in \langle i \rangle$. If $k' \equiv k \pmod{p}$, any polynomial $f(t)$ satisfies $f(t^{k'}) \equiv f(t^k) \pmod{(t^p - 1)}$. Our regularised $\Delta_K(t)$ satisfies $\Delta_K(t^{-k}) = t^{-k \cdot \text{deg}} \Delta_K(t^k)$. Thus, we identify $\langle i \rangle$ (as a subset of $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$) with the set $\{1, i, i^2, \dots, i^{I-1}\}$ in \mathbf{Z} naturally. We define an integer M by $M := m(1 + i + i^2 + \dots + i^{I-1})$.

By the fractional reductions, it holds that, modulo $(t^p - 1)$,

$$\begin{aligned} \prod_{k \in \langle i \rangle} \Delta_K(t^k) &= t^M \frac{(t^{ij} - 1)(t - 1)}{(t^i - 1)(t^j - 1)} \cdot \frac{(t^{i^2j} - 1)(t^i - 1)}{(t^{i^2} - 1)(t^{ij} - 1)} \cdot \dots \\ &= t^M \frac{(t - 1)(t^{i^Ij} - 1)}{(t^j - 1)(t^{i^I} - 1)}. \end{aligned}$$

Now, we set a polynomial $F(t)$ by

$$F(t) := \begin{cases} t^M & \text{if } i^I = 1 \text{ in } (\mathbf{Z}/p\mathbf{Z})^\times, \\ t^{M-j+1} & \text{if } i^I = -1 \text{ in } (\mathbf{Z}/p\mathbf{Z})^\times. \end{cases}$$

Then, for any divisor d of p , it is easy to show

$$\prod_{k \in \langle i \rangle} \Delta_K(\zeta_d^k) = F(\zeta_d),$$

where ζ_d is the d -th primitive root of unity, and $F(1) = 1$. By the same argument with in the proof of Lemma 3.1 and Theorem 1.1, we have the conclusion. \square

Before starting the proof of Theorem 1.3(2), we recall Franz’s number-theoretical lemma ([6]), which was used in the classification of lens spaces, see also [12, 14]. It says that there is no relation of a special fractional type among the elements of $(\zeta^x - 1)^{\pm 1}$ type in $\mathbf{Q}(\zeta)$.

LEMMA 4.1. (Franz [6]) *Let ζ be a primitive n -th root of unity. Let a_i ($i \in (\mathbf{Z}/n\mathbf{Z})^\times$) be an integer satisfying the following conditions:*

- (1) $a_{-i} = a_i$,
- (2) $\sum_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} a_i = 0$,
- (3) $\prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} (\zeta^i - 1)^{a_i} = 1$.

Then $a_i = 0$ for all $i \in (\mathbf{Z}/n\mathbf{Z})^\times$.

PROOF OF (2): Suppose that a subgroup H of $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$ satisfies that

$$\prod_{k \in H} \Delta_K(t^k) \doteq 1 \quad \text{in } \mathbf{Z}[t, t^{-1}]/(t^p - 1).$$

Let ζ be the p -th primitive root of unity. By Corollary 3.3,

$$\prod_{k \in H} \Delta_K(\zeta^k) = \frac{\prod_{k \in H} (\zeta^{kij} - 1) \cdot \prod_{k \in H} (\zeta^k - 1)}{\prod_{k \in H} (\zeta^{ki} - 1) \cdot \prod_{k \in H} (\zeta^{kj} - 1)} \doteq 1$$

This equation and Franz’s lemma implies (see [12, 14]):

$$\{kij, k \ (k \in H)\} = \{ki, kj \ (k \in H)\} \quad \text{in } (\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$$

that is, $ijH \cup H = iH \cup jH$, where xH is the coset of the group $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$ by H . Thus, it holds that either $H \ni i$ or $ijH \ni i$ (thus $H \ni j$). Possibly both of them hold. We have the theorem. \square

Since H is a subgroup, $H \ni i$ implies $H \supset \langle i \rangle$. Thus, on the minimal subgroups H (with respect to the inclusion relation) in $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$ that satisfy

$$(4.2) \quad \prod_{k \in H} \Delta_K(t^k) \doteq 1 \quad \text{in } \mathbf{Z}[t, t^{-1}]/(t^p - 1),$$

we have the following:

COROLLARY 4.2. *Let K be a knot in a homology 3-sphere Σ with the Alexander polynomial $\Delta_K(t)$, and suppose that $\Sigma(K; p/q)$ is of lens type. If a subgroup H is minimal among the subgroups in $(\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ that satisfy the equation (4.2), then it holds that $H = \langle i \rangle$ or $H = \langle j \rangle$ (possibly both of them hold).*

Thus, the number of minimal subgroups H of $(\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ that satisfy the equation (4.2) is at most two.

By using Corollary 4.2, in some cases, for given $\Delta_K(t)$ and p , we can prove that $\Sigma(K; p/q)$ is not of lens type for any q :

EXAMPLE 4.3. Let K be a knot whose Alexander polynomial is the 429-th cyclotomic polynomial ($429 = 3 \cdot 11 \cdot 13$):

$$\Delta_K(t) = \frac{(t^{429} - 1)(t^{13} - 1)(t^{11} - 1)(t^3 - 1)}{(t^{143} - 1)(t^{39} - 1)(t^{33} - 1)(t - 1)}$$

and the coefficient $61/q$. In this case, in the group $(\mathbf{Z}/61\mathbf{Z})^\times/\{\pm 1\}$ ($\cong \mathbf{Z}/30\mathbf{Z}$, generated by 2), there are three distinct minimal subgroups: $\langle 3 \rangle$ of order 5, $\langle 11 \rangle$ of order 2, and $\langle 13 \rangle$ of order 3, and all of them satisfy the equation (4.2). Thus, any $(K; 61/q)$ is not of lens type.

5. EXAMPLES

First, we demonstrate two examples of lens surgery along the torus knot ([19]) with rational coefficient.

EXAMPLE 5.1. Let K be the torus knot $T(2, 3)$ in S^3 and $p/q := 17/3$. Then $(K; p/q)$ is the lens space $L(17, 5)$. We can take $i = 2, j = 3$ as the indexes. In this case, in the group $(\mathbf{Z}/17\mathbf{Z})^\times/\{\pm 1\}$ ($\cong \mathbf{Z}/8\mathbf{Z}$, generated by 3), it holds that $\langle i \rangle \neq \langle j \rangle$ and $\langle i \rangle \subset \langle j \rangle$.

EXAMPLE 5.2. Let K be the torus knot $T(2, 5)$ in S^3 and $p/q := 31/3$. Then $(K; p/q)$ is the lens space $L(31, 18)$. We can take $i = 2, j = 5$ as the indexes. In this case, in the group $(\mathbf{Z}/31\mathbf{Z})^\times/\{\pm 1\}$ ($\cong \mathbf{Z}/15\mathbf{Z}$, generated by 3). The order of $\langle i \rangle$ (or $\langle j \rangle$) is 5 (or 3, respectively), it holds that $\langle i \rangle \neq \langle j \rangle$ and $\langle i \rangle, \langle j \rangle \neq (\mathbf{Z}/31\mathbf{Z})^\times/\{\pm 1\}$.

Next, we study the famous lens surgery along non-torus knots:

EXAMPLE 5.3. ([34]) 18-surgery (and 19-surgery also) along the pretzel knot $P(-2, 3, 7)$ is the lens space $L(18, 5)$ (and $L(19, -7)$, respectively)([5]). We let $\Delta_P(t)$ denote the Alexander polynomial of $P(-2, 3, 7)$:

$$\Delta_P(t) = t^{10} - t^9 + t^7 - t^6 + t^5 - t^4 + t^3 - t + 1.$$

It holds that

$$\Delta_P(t) \cdot \Delta_P(t^5) \cdot \Delta_P(t^{11}) = t^{13} \quad \text{in } \mathbf{Z}[t]/(t^{18} - 1),$$

corresponding to $(\mathbf{Z}/18\mathbf{Z})^\times/\{\pm 1\} \cong \mathbf{Z}/3\mathbf{Z}$ (generated by 5). And it holds that

$$\Delta_P(t) \cdot \Delta_P(t^7) \cdot \Delta_P(t^{11}) = 1 \quad \text{in } \mathbf{Z}[t]/(t^{19} - 1).$$

In this case ($p = 19$), in $(\mathbf{Z}/19\mathbf{Z})^\times/\{\pm 1\} (\cong \mathbf{Z}/9\mathbf{Z}$, generated by 3), 7 generates the minimal cyclic subgroup of order 3.

Note that $\Delta_P(-t)$ is known as Lehmer’s polynomial in the number theory, see [8, 16, 27].

The former, 18-surgery along $P(-2, 3, 7)$ (as K_2) belongs to the family K_n of lens surgery found by Fitushel and Stern in [5]. The latter, 19-surgery (as $k(2, 3)$) belongs to the family $k(a, b)$ of lens surgery ([1, 3], see also [33]). Recently, Baker named this family Type VII in [2]. Here, we study the Alexander polynomial of them.

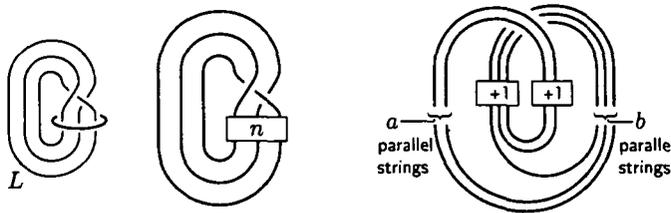


Figure 1: K_n and $k(a, b)$ ($K_2 = k(2, 3) = P(-2, 3, 7)$)

EXAMPLE 5.4. Let K_n be a knot in Figure 1, where n is an integer with $n \geq 2$, and the rectangle with n means the n times right full-twist. $9n$ -surgery along K_n is a lens space $-L(9n, 3n + 1)$. Note that

$$-L(9n, 3n + 1) = L(9n, 6n - 1) = -L(9n, 6n + 1) = L(9n, 3n - 1).$$

The Alexander polynomial $\Delta_{K_n}(t)$ of K_n is:

$$\Delta_{K_n}(t) = t^{3n-1} + \frac{(t^{3n} - 1)^2(t - 1)}{t^3 - 1}.$$

The order is $2(3n - 1)$. (The authors first calculated the two-variable polynomial $\Delta_L(t_1, t_2)$ by using the method in [10, 11], and got $\Delta_{K_n}(t)$.)

In this case, we can deform $\Delta_{K_n}(t)$ as

$$t^{-(3n-1)} \Delta_{K_n}(t) = t^{-9n^2} \frac{(t^{(3n+1)(6n+1)} - 1)(t - 1)}{(t^{3n+1} - 1)(t^{6n+1} - 1)} + t^*(t^{9n} - 1)(t - 1)f(t),$$

where $f(t)$ is a symmetric polynomial and the index $*$ is decided as in Theorem 1.1. It holds

$$\Delta_{K_n}(t) \cdot \Delta_{K_n}(t^{3n+1}) \cdot \Delta_{K_n}(t^{6n+1}) = t^{-3} \quad \text{in } \mathbf{Z}[t, t^{-1}]/(t^{9n} - 1).$$

In $(\mathbf{Z}/9n\mathbf{Z})^\times/\{\pm 1\}$, $(3n + 1)$ generates the minimal cyclic subgroup of order 3.

EXAMPLE 5.5. Let $k(a, b)$ be the knot in Figure 1, where (a, b) is a pair of coprime positive integers. It is known that $(a^2 + ab + b^2)$ -surgery along $k(a, b)$ is a lens space $-L(a^2 + ab + b^2, a\bar{b})$. Note that $(a\bar{b})^3 = 1$ in $(\mathbf{Z}/(a^2 + ab + b^2)\mathbf{Z})^\times$, because $a^3 \equiv b^3 \pmod{a^2 + ab + b^2}$.

In the case $(a, b) = (2, 2n + 1)$, the coefficient is $a^2 + ab + b^2 = 4n^2 + 8n + 7$, and

$$a\bar{b} = 2n^2 + 3n + 2, \quad (a\bar{b})^2 = 2n^2 + 5n + 4.$$

We set

$$(i, j) = \begin{cases} (2n^2 + 3n + 2, 2n^2 + 5n + 4) & \text{if } n \text{ is odd} \\ (2n^2 + 5n + 5, 2n^2 + 3n + 3) & \text{if } n \text{ is even} \end{cases}.$$

Then the pair (i, j) becomes coprime. By some calculus, we get

$$\Delta_{k(2,2n+1)}(t) = \frac{(t - 1) \left\{ \begin{array}{l} (t^{2n^2+5n+4} + 1)(t^{(2n+1)(n+1)} - 1) \\ + t^{2n^2+3n+2}(t^{2n+1} - 1) \end{array} \right\}}{(t^{2n+1} - 1)(t^{2n+3} - 1)}.$$

It satisfies

$$\Delta_{k(2,2n+1)}(t) \cdot \Delta_{k(2,2n+1)}(t^i) \cdot \Delta_{k(2,2n+1)}(t^j) \doteq 1$$

mod $(t^{4n^2+8n+7} - 1)$. See [34] in the case $n = 2$.

The authors conjecture that for any coprime positive pair (a, b) , for the cubic root $\omega = a\bar{b}$ in $(\mathbf{Z}/(a^2 + ab + b^2)\mathbf{Z})^\times / \{\pm 1\}$,

$$\Delta_{k(a,b)}(t) \cdot \Delta_{k(a,b)}(t^\omega) \cdot \Delta_{k(a,b)}(t^{\omega^2}) \doteq 1 \pmod{(t^{a^2+ab+b^2} - 1)}.$$

This conjecture is true in the cases that $a^2 + ab + b^2$ is a power of an odd prime. It is proved by Theorem 1.2 and that $(\mathbf{Z}/(a^2 + ab + b^2)\mathbf{Z})^\times / \{\pm 1\}$ is a cyclic group.

For other lens surgeries, see Berge’s work [3, 4], and [1, 2, 17]. For researches on Alexander polynomials of knots from the algebraic view points, see [7, 20, 23] and so on.

On the recent researches. In [9], Ichihara, Saito and Teragaito calculated the Alexander polynomials of knots in homology 3-spheres yielding $L(p, q)$ which include Berge’s knots in S^3 . Their form of the Alexander polynomial includes three parameters p, q and u (satisfying $q \equiv \pm u^2 \pmod{p}$), which is derived from the “dual knot” K^* in $L(p, q)$ ([3, 24]). Then we raise a problem.

PROBLEM 5.6. Calculate the Alexander polynomial of a Berge’s knot in S^3 with Berge’s original parameters in [4] itself.

Recently in [28], Tange reproved our Theorem 1.1 by using the “correction term invariant” due to Ozsváth and Szabó [15, 21, 22].

PROBLEM 5.7. Are Saito’s parameter u in [24, 25], Tange’s parameter h in [28], and Berge’s parameter B in [4], regarded as in $(\mathbf{Z}/p\mathbf{Z})^\times / \{\pm 1\}$, always equal to our parameter i or j for general p (the coefficients of the lens surgeries)?

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