# PART IV

# RELATIVISTIC STELLAR DYNAMICS

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# DYNAMICS OF RELATIVISTIC STELLAR SYSTEMS*

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**Abstract.** Relativistic stellar-dynamical systems and their possible occurrence in nature are discussed. Features of the equilibrium models that have been constructed for spherical star clusters in general relativity are delineated. The results of studies of the stability of relativistic spherical clusters are reviewed. It is noted that the results, while not conclusive, indicate that realistic spherical clusters are stable against gravitational collapse if their central redshifts $z_c \lesssim 0.55$ and unstable if $z_c \gtrsim 0.55$. More work is needed on this and other problems.

## 1. Introduction

About a decade ago, exciting developments in astronomy, involving, for example, the cosmic blackbody radiation, quasars, radio sources, and galaxies exhibiting violent activity, helped to nurture a renewed interest in possible astrophysical applications of general relativity. This interest flowered into a sustained outburst of work on relativistic models for the universe and for individual systems such as stars, clusters of stars, and black holes.

The fact that no individual relativistic systems had yet been discovered – or course, it had long been generally accepted that the overall structure of the universe requires a relativistic description – did not deter speculation about them. And one might say rightly so, if only because of the present experimental situation, which associates relativistic neutron stars with pulsars and black holes with the unseen components in certain binary-star systems.

There is as yet still no direct evidence for the existence of relativistic stellar systems (relativistic clusters of stars). And, depending on one's prejudices, he may or may not hope that the relativist's luck has run out as far as the arrival of evidence for the existence of yet another class of relativistic objects is concerned. In any case, at least a few relativists have expended considerable effort trying to discern the properties of relativistic stellar systems, and we shall here review the current theoretical situation.

## 2. General Features of Relativistic Stellar Systems

Relativistic effects should be important for a star cluster if the gravitational redshift of a photon emitted from the center of the cluster and received at infinity is greater than, say, a hundredth or so. Generally, except for a rather special class of clusters discussed below, such a significant redshift requires the ratio (we use units in which $G=c=1$)

$$\frac{2M}{R} \approx 0.01 \, \frac{M/(10^{11}\, m_{\odot})}{R/(1 \text{ pc})} \tag{1}$$

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to exceed about a hundredth, where $M$ and $R$ are the total mass-energy and the mean size of the system. This clearly requires either a very large mass density or, if the volume is moderate or large, a very large mass.

If one supposes that clusters are not born in a relativistic state, the question immediately arises whether they have been around long enough to permit evolution to such a state. This is a nagging question. It seems clear, on the one hand, that evolution pictured as driven by 2-body encounters will not do the trick: the associated evolution time-scales are simply too long, at least for the types of stellar systems that are known to exist (see, for example, Fackerell *et al.*, 1969). On the other hand, computer simulations of the evolution of stellar systems (see, for example, the contributions to this volume by Hénon and by Spitzer) often point to a scenario involving rapid evolution – compared with evolution expected on the basis of 2-body encounters – of a stellar system's central regions toward ever increasing ratio (1). It has been suggested that what one is seeing here is a phenomenon driven by many-particle interactions and analogous to a thermal runaway, or gravothermal catastrophe, of the type proposed by Antonov (1962) and Lynden-Bell and Wood (1968).

We shall return to the question of thermal runaways later, but for the moment let us suppose that nature does in fact have some way of making relativistic clusters. Once a cluster becomes relativistic, it should evolve rapidly. If the stars are of normal size, direct collisions between stars can be expected to occur rather frequently and to dominate the evolution. (For example, a relativistic cluster with $10^{12}$ solar-type stars in 1 pc will produce $\sim 10^8$ collisions per year !) One is then faced with the possibility that the center of the cluster is turned into one supermassive object. If the cluster 'stars' are neutron stars or black holes, then energy-loss via emission of gravitational radiation during close binary encounters dominates the evolution (Greenstein, 1969), and it is interesting to speculate about the possibility of observing such radiation.

If a relativistic cluster is not to evolve catastrophically rapidly on a dynamical time-scale, the evolution time-scale must be large compared with the time a typical star takes to traverse the system. For a given value of the ratio (1), this requirement places a lower limit on the size, and hence on the mass, of the system (Fackerell *et al.*, 1969). For example, if the ratio $2M/R \sim 0.1$, the lower limit on $R \sim 0.01$ pc for a cluster of solar-type stars. For $2M/R \sim 0.1$ and $R \sim 1$ pc, $M \sim 10^{12}\, m_\odot$ and the collision time-scale $\sim 10^4$ yr. If the evolution time-scale is sufficiently greater than the star crossing-time, then the cluster should evolve quasistatically; and at any moment it should be able to be approximated by a collisionless near-equilibrium state.

## 3. Fundamental Equations of Relativistic Stellar Dynamics

To date, almost all formal studies of relativistic stellar systems have adopted descriptions in terms of a one-particle distribution function, $\mathcal{N}$. An observer at a spacetime event $(\mathbf{x}, t)$ determines $\mathcal{N}$ there by measuring, in his local Lorentz frame, the number $\mathrm{d}N$ of stars occupying a volume $\mathrm{d}^3x$ in physical space, occupying a volume $\mathrm{d}^3p$ in

3-momentum space and having rest masses in the range $\mathrm{d}m$:

$$\mathcal{N} = \mathrm{d}N/\mathrm{d}m\ \mathrm{d}^3p\ \mathrm{d}^3x. \tag{2}$$

If the geometry of spacetime is described by the line element (latin indices run over 0, 1, 2, 3)

$$\mathrm{d}s^2 = g_{ab}\ \mathrm{d}x^a\ \mathrm{d}x^b \tag{3}$$

in a particular curvilinear coordinate system ($x^0$, $x^1$, $x^2$, $x^3$), it can be shown that Equation (2) is equivalent to

$$\mathcal{N} = \mathrm{d}N/\mathrm{d}\mathscr{V}_p\ \mathrm{d}\mathscr{V}_x, \tag{4}$$

with

$$\mathrm{d}\mathscr{V}_p = -\mathrm{d}p_0\ \mathrm{d}p_1\ \mathrm{d}p_2\ \mathrm{d}p_3/\sqrt{-g}, \qquad \mathrm{d}\mathscr{V}_x = (p^0/m)\sqrt{-g}\ \mathrm{d}x^1\ \mathrm{d}x^2\ \mathrm{d}x^3. \tag{5}$$

Here $p_a$ and $p^a$ are the covariant and contravariant components of a star's 4-momentum, $g$ is the determinant of the metric tensor, and $m$ is a star's rest mass. The distribution function $\mathcal{N}$ is an invariant in that all observers at a given event agree on its value for a given group of stars. The distribution function determines a smoothed-out stress-energy tensor

$$T^{ab} = \int (\mathcal{N}/m)\ p^a p^b\ \mathrm{d}\mathscr{V}_p. \tag{6}$$

The stress-energy tensor in turn determines the geometry via Einstein's field equations,

$$G^{ab} = 8\pi T^{ab}, \tag{7}$$

where $G^{ab}$ is the Einstein field tensor. If, for one reason or another, a cluster can be approximated as collisionless, the circle begun by Equations (6) and (7) is completed by demanding that the geometry determines the distribution function via the Boltzmann-Liouville, or collisionless Boltzmann, equation

$$\begin{aligned} \mathscr{D}\mathcal{N} &\equiv \frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\partial\mathcal{N}}{\partial x^a} + \frac{\mathrm{d}p_a}{\mathrm{d}s}\frac{\partial\mathcal{N}}{\partial p_a} \\ &= \frac{p^a}{m}\frac{\partial\mathcal{N}}{\partial x^a} - \frac{1}{2m}\frac{\partial g^{bc}}{\partial x^a}\ p_b p_c\ \frac{\partial\mathcal{N}}{\partial p_a} = 0. \end{aligned} \tag{8}$$

The operator $\mathscr{D}$ is the derivative with respect to proper time along the path of a star through phase space.

## 4. Relativistic Equilibrium Configurations

Generalizations of familiar Newtonian methods enable one to break the circle formed by Equation (6)–(8) and to construct collisionless equilibrium configurations, for

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which there is no explicit dependence on time $t = x^0$ (see, for example, Fackerell, 1968; and Zel'dovich and Novikov, 1971). To date, only spherically symmetric configurations have been constructed – almost always numerically.

Under the assumption of spherical symmetry, the line element (3) takes the form

$$ds^2 = -e^{\nu(r,t)}\,dt^2 + e^{\lambda(r,t)}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{9}$$

in Schwarzschild coordinates $(t, r, \theta, \varphi)$. At equilibrium, when the metric functions $\nu$ and $\lambda$ are time independent, the Boltzmann-Liouville Equation (8) implies that $\mathcal{N}$ is generally (there are contrived exceptions – see Zel'dovich and Novikov, 1971) a function of the isolating integrals

$$m, E = p_0, \quad \text{and} \quad J = [p_\theta^2 + (p_\varphi/\sin\theta)^2]^{1/2} \tag{10}$$

along stellar orbits. The integral $E$ is the energy of a star as measured at infinity, and $J$ is the magnitude of its angular momentum. If $\mathcal{N}$ does not depend on $J$, the cluster has an isotropic velocity distribution and an effective isotropic pressure

$$P = T_r^r = T_\theta^\theta = T_\varphi^\varphi. \tag{11}$$

Zel'dovich and Podurets (1965) constructed numerical equilibrium models by taking $\mathcal{N}$ to be that appropriate to a truncated isothermal cluster of identical stars,

$$\mathcal{N} = Ae^{-E/kT}\,\delta(m - m_0)\,H(E_{CUT} - E). \tag{12}$$

Here $A$ is a normalization constant, $k$ is Boltzmann's constant, $T$ is the constant temperature as measured from infinity (the temperature as measured locally in the cluster is not constant but increases toward the center) and $m_0$ is the rest mass of a star; $H(x)$ is the step function

$$\begin{aligned} H(x) &= 1 \qquad x \geqslant 0 \\ &= 0 \qquad x < 0, \end{aligned} \tag{13}$$

so that the constant $E_{CUT}$ is the maximum energy of a star in the cluster. Fackerell (1968, 1970, 1971) and Ipser (1969) have also constructed isothermal models as well as a variety of other isotropic models that obey polytropic relations of the form

$$-T_0^0 = \alpha P^{n/(n+1)} + P/(\Gamma - 1), \tag{14}$$

where $\alpha$, $n$, and $\Gamma$ are constants and $P$ is the isotropic pressure (4.3).

Certain features appear to be common to all these types of models. For example, as the central redshift $z_c$ – a convenient measure of the importance of general relativity that can take any value in the range $(0, \infty)$ – increases from zero, the ratio (1) at first increases to a maximum value typically $\sim 0.3$ and thereafter oscillates about its maximum. More importantly (and we shall shortly see why), the fractional binding energy,

$$\mathcal{E}_B \equiv (M_0 - M)/M_0, \tag{15}$$

$M_0$ = rest mass-energy of cluster, $M$ = total mass-energy of cluster, also at first increases to a maximum value and thereafter oscillates about its maximum. Remarkably, the maximum value of $\mathscr{E}_B$ and the corresponding $z_c$ show little variation from one type of model to another: for all isothermal and polytropic models that have been studied, $\mathscr{E}_B$ has a maximum value $\sim 0.0355 \pm 0.005$ at a central redshift $z_c \sim 0.53 \pm 0.03$.

## 5. The Stability of Relativistic Stellar Systems

Hoyle and Fowler (1967) proposed a quasar model in which the quasar lies at the center of a massive relativistic star cluster and derives its redshift from the gravitational field of the cluster. The viability of such a model clearly depends crucially on whether equilibrium clusters become unstable against gravitational collapse before their redshifts reach values as large as $z_c \sim 3$. The type of instability referred to here involves a gross overall instability of the gravitational field of a cluster. Having evolved to a point at which such an instability sets in, a cluster would subsequently collapse on a time scale of the order of the star transit-time across the cluster. One is consequently dealing here with a dynamical instability.

Methods for studying the dynamical stability of relativistic spherical clusters to small spherical perturbations were developed by Ipser and Thorne (1968). They used the fact that collisions between stars can be neglected in a study of dynamical stability if the evolution time-scale is assumed to be large compared with the dynamical time-scale. Briefly, their analysis begins with the expression

$$\mathrm{d}s^2 = -\exp[\nu_A(r)+\nu_B(r,t)]\,\mathrm{d}t^2 + \exp[\lambda_A(r)+\lambda_B(r,t)]\,\mathrm{d}r^2 + \\ + r^2(\mathrm{d}\theta^2 + \sin^2\theta)\,\mathrm{d}\varphi^2 \qquad (16)$$

for the line element of a slightly perturbed spherical cluster, where the subscripts $A$ and $B$ refer to equilibrium quantities and small perturbations of those quantities, respectively. The distribution function is split in a particular way into an equilibrium part, $\mathscr{N}_A = F(m, E_A = e^{\nu_A} p_0, J)$, and a perturbed part, $\mathscr{N}_B = f(x^a, p^a)$, so that

$$\mathscr{N} = \mathscr{N}_A + \mathscr{N}_B = F(m, E_A, J) + f(x^a, p^a). \qquad (17)$$

Equations (6)–(8) are then linearized in $\nu_B$, $\lambda_B$, and $f$. The perturbation $f$ is split into its even and odd parts,

$$f_\pm \equiv \tfrac{1}{2}[f(x^a, p^0, p^\alpha) \pm f(x^a, p^0, -p^\alpha)], \qquad (18)$$

as a function of spatial momenta $p^\alpha$ ($\alpha = 1, 2, 3$), and a dynamical equation of the form

$$(1/-F_E)\,\partial^2 f_-/\partial t^2 = \mathscr{T} f_- \qquad (19)$$

is derived. In this equation

$$F_E \equiv [\partial F(m, E_A, J)/\partial E_A]_{m,J}, \qquad (20)$$

and $\mathscr{T}$ is an integro-differential operator in phase space. It is self-conjugate for well-behaved odd functions $h$, $g$ which vanish outside the phase space of the equilibrium cluster; that is

$$\int h\mathcal{T}g\,\mathrm{d}\mathcal{V}_p\,\mathrm{d}\mathcal{V}_x=\int g\mathcal{T}h\,\mathrm{d}\mathcal{V}_p\,\mathrm{d}\mathcal{V}_x. \tag{21}$$

Attention is restricted to clusters satisfying the physically reasonable condition

$$F_E\leqslant 0, \tag{22}$$

which implies that there are fewer stars at high energies than at low energies. It then follows from Equations (19) and (21) that a cluster is dynamically stable to spherical perturbations if and only if $\mathcal{T}$ is positive-definite,

$$\int g\mathcal{T}g\,\mathrm{d}\mathcal{V}_p\,\mathrm{d}\mathcal{V}_x>0, \tag{23}$$

for all well-behaved odd perturbation functions $g$.

One can attempt determine a cluster's stability by inserting various trial functions in the integral in Equation (23). Ipser (1969) followed this path for trial functions of the simple form

$$g=F_E p^r C(r). \tag{24}$$

His study of a wide variety of isothermal and polytropic clusters reveals a pattern that changes strikingly little from one cluster type to another: at small redshift $\mathcal{T}$ is positive-definite over the subspace of functions (24); but $\mathcal{T}$ ceases to be positive-definite at a redshift very near that at which the fractional binding energy peaks. Consequently, the clusters studied are unstable if $z_c\gtrsim 0.55$.

This behavior, which tends to be supported by Fackerell's (1970) stability calculations, tempts one to adopt the general picture that a spherical cluster born at low $z_c$ will evolve in the direction of increasing $z_c$ and binding energy along a sequence of initially stable states, and will become unstable near $z_c=0.55$. There is an impediment to adopting this picture, however. It arises from work by Bisnovatyi-Kogan and Zel'dovich (1969), who studied the so-called $\gamma$-law cluster models. Such a model consists of a nearly constant-density core surrounded by an extended mantle in which the density of total mass-energy, $-T_0^0(r)$, and the mass energy inside radius $r$, $M(r)$, have the behavior

$$-T_0^0(r)\sim\frac{\gamma}{1+6\gamma+\gamma^2}\frac{1}{2\pi r^2},\qquad M(r)\sim\frac{2\gamma r}{1+6\gamma+\gamma^2}. \tag{25}$$

The constant $\gamma$ is the ratio of the isotropic pressure to the density in the core and mantle. The join between the core and mantle occurs at a radius

$$r_{\mathrm{core}}\sim(\gamma/2\pi\varrho_c)^{1/2},\qquad \varrho_c\equiv-T_0^0(r=0). \tag{26}$$

Surrounding the mantle is an envelope in which the sensity drops to zero. The redshift a photon experiences in traveling from the center to the outer edge, $R_m$, of the mantle is

$$z_c \sim (R_m/r_{\rm core})^{2\gamma} - 1. \tag{27}$$

Bisnovatyi-Kogan and Zel'dovich (1969) pointed out that, in principle, Equations (25)–(27) for sufficiently small $\gamma$ allow the construction of models with arbitrarily large redshifts and arbitrarily *small* ratios $2M(r)/r$ everywhere. Hence, they argued, by choosing $\gamma$ very small, one could construct large-redshift models that must be stable, since the models have very small ratios $P/(-T_0^0)$ and $2M(r)/r$ and hence are locally nearby Newtonian everywhere.

Bisnovatyi-Kogan and Thorne (1970) showed that the operator $\mathscr{T}$ is in fact positive-definite for all trial functions of the form (24) if and only if $\gamma \lesssim 0.117$. Add to this the fact that the fractional binding energy of the pure-mantle models (infinite $\varrho_c$, $z_c$, and $R_m$) peaks as a function of $\gamma$ at $\gamma \sim 0.025$. Then, remembering the way isothermal and polytropic clusters behave, one is strongly drawn to the conjecture that $\gamma$-law models are stable if $\gamma \lesssim 0.025$.

What will be the properties of stable $\gamma$-law models? Note that for each value of $\gamma$ and $\varrho_c$, a minimum size $R$, and hence total mass $M$, of the system is needed to produce a desired redshift. For example, for $\gamma = 0.025$,

$$R \gtrsim (1+z_c)^{20}/(15\varrho_c^{1/2}), \qquad M \gtrsim 0.05\,R. \tag{28}$$

Hence if $z_c = 1$, then $R \gtrsim 3\times10^5$ pc for a massive object with $\varrho_c \sim 10^{12}\,m_\odot\,\mathrm{pc}^{-3}$; and $R \gtrsim 3\times10^4$ pc for a less massive object with the large density $\varrho_c \sim 10^{14}\,m_\odot\,\mathrm{pc}^{-3}$ (corresponding to, say, $10^8$ stars in $\sim 0.01$ pc). Though these radii may not be unmanageable, the corresponding total masses, $M \gtrsim 3\times10^{17}\,m_\odot$ and $3\times10^{16}\,m_\odot$, might be. And matters only get worse as $z_c$ is increased. At very small $\gamma$ one runs into problems with the very large radii ($\gg$ Hubble radius) needed to produce large redshifts. Conversely, if models can be stable for $\gamma$ near 0.1, then the possibility of large redshifts becomes more attractive.

Obviously, more work is needed to conclusively demonstrate whether reasonable spherical clusters can be stable at central redshifts significantly larger than $z_c = 0.55$.

Up until now we have not raised the question of dynamical stability to nonspherical perturbations. Nonspherical perturbations, in contrast with spherical perturbations, take on an added significance in general relativity in that the associated motion of the cluster's overall gravitational field generates gravitational radiation. If a cluster reached a point of onset of instability to nonspherical perturbations, its subsequent unstable motion might produce copious amounts of gravitational waves. Unfortunately, the gravity-wave experimentalist is out of luck here, at least regarding clusters with $F_E \leqslant 0$. One can show (Ipser, 1975) that, just as in Newtonian theory, so also in general relativity, a spherical isotropic cluster with $F_E \leqslant 0$ is dynamically stable to nonspherical perturbations.

On occasion, there appear analyses expressing interest in some sort of secular, as opposed to dynamical, stability of stellar systems. One type of such analyses seeks to determine whether there exist equilibrium states for which the system's entropy, defined in one way or another, is a maximum subject to certain constraints, usually those of fixed mass-energy and stars. For example, Antonov (1962) and Lynden-Bell and Wood (1968) studied the Boltzmann entropy of Newtonian systems for which there are no limits on the velocities of the constituent particles, but which are held in by boxes of finite size in physical space. They found that a Newtonian spherical equilibrium configuration is a local entropy-maximum, when compared with all neighboring configurations having the same energy, mass, and confining radius, if and only if it is isothermal and the confining radius is less than the critical value $-0.335\, M_0^2/\mathscr{E}$, where $M_0$ and $\mathscr{E}$ are the mass and energy of the configuration. They argued that confined systems with radii greater than the critical value have no states of locally maximum entropy to which they can evolve, and should experience a thermal runaway, or gravothermal catastrophe. Presumably, a thermal runaway is driven by encounters – perhaps collective in nature – and involves evolution to states of ever increasing entropy and density contrast, on a time scale that is fairly rapid and is determined by the driving mechanism but is nevertheless long compared with the dynamical time-scale (hence the term secular).

Even if some sort of thermal runaway is actually important for confined systems, it is not immediately clear what the situation is for realistic star clusters, which are not held in by boxes. In fact, recent calculations (Ipser, 1974, 1975) in Newtonian theory and in general relativity suggest that perhaps secular-stability criteria involving entropy maxima are not very useful for realistic self-bound stellar systems. The calculations derive a criterion for the Boltzmann entropy,

$$S \equiv -k \int \mathcal{N} \ln \mathcal{N}\, \mathrm{d}\mathscr{V}_p\, \mathrm{d}\mathscr{V}_x, \tag{29}$$

of a self-bound spherical equilibrium cluster to be a maximum, when compared with the entropy of all neighboring configurations having the same energy and stars. According to the criterion, $S$ is a maximum if and only if the cluster has a truncated isothermal distribution function and the corresponding isentropic gas sphere – which has the same radial distribution of density and pressure as the cluster – is dynamically stable to spherical perturbations. One finds, however, that this criterion cannot be satisfied for a cluster unless its distribution function is so heavily truncated that the cluster is unrealistic. Hence one is led to conclude that no realistic clusters are Boltzmann-entropy maxima; and, further, that at present it is not clear how the occurrence of any sort of thermal runaway can be predicted or pinpointed by appealing to entropy arguments.

## 6. Conclusion

In this paper we have reviewed aspects of the current theoretical situation pertaining to relativistic stellar systems. Several important problems remain at an unsatisfactory stage of solution, and others have not even been tackled yet. It seems clear that relativistic stellar dynamics offers fertile ground for future study.

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# DISCUSSION

*Contopoulos:* What is the role of gravitational radiation in your systems?

(1) How large is the gravitational radiation due to the graininess of the system?

(2) What is the role of non-spherical perturbations of spherical clusters?

*Ipser:* For spherical perturbations of spherical systems there is no gravitational radiation. Nonspherical perturbations do involve gravitational radiation. I have recently been able to show that, just as in Newtonian theory, so also in general relativity, spherical systems are stable to nonradial perturbations if the distribution function is a decreasing function of the energy of a star. Also, the incoherent gravitational radiation emitted due to the graininess of the cluster is not important compared with other processes.

*Bardeen:* There is one example of a rotating relativistic 'stellar-dynamical' system – the relativistic version of the cold MacLaurin disk. The structure of such disks was calculated numerically for models with all central redshifts from zero to infinity by Bardeen and Wagoner a few years ago. The binding energy increases monotonically with redshift, so one suspects the disks are stable against overall collapse even when infinitely relativistic. However, they are certainly violently unstable to fragmentation. They do exhibit such interesting phenomena as the ergotoroids mentioned by Ipser.

*Ipser:* I agree that your rapidly rotating disks are probably stable to overall collapse.