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CATEGORICAL EQUIVALENCE OF FINITE GROUPS

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We determine the minimal relational sets related to finite groups. With the help of this result we prove that two categorically equivalent finite groups are weakly isomorphic.

1. INTRODUCTION

In [**1**] Bergman and Berman define the notion of categorically equivalent algebras. In [**6**] we introduce minimal relational sets and point out the relationship between categorical equivalence of algebras and minimal relational sets corresponding to the algebras. In this paper we shall describe the minimal relational sets assigned to finite groups, see Theorem 3.1. Moreover, by using this description, in Corollary 3.3 we shall show that two categorically equivalent finite groups are weakly isomorphic.

2. DEFINITIONS AND EARLIER RESULTS

A *relational set* is a set equipped with some relations. All relational sets occuring in this paper are of a fixed type. *Morphisms* between relational sets of the same type are relation preserving maps. *Product* and *retract* of relational sets of a fixed type are meant as usual. A morphism r from a relational set to itself is called *idempotent*, if $r^2 = r$. A relational set is *finite* if its base is finite and its relations are finitary. Note that this definition allowes finite relational sets to have infinitely many relations. Relational sets here will be denoted by boldface type capital letters and their base sets by the same capital ones.

In the next section we shall need the following simple lemma.

LEMMA 2.1. *Let* $\mathbf{G}$ *be a finite relational set and* $S \subseteq G$. *Then the minimal* S*-containing idempotent images of* $\mathbf{G}$ *are isomorphic.*

PROOF: Let $r : \mathbf{G} \to \mathbf{G}$ and $q : \mathbf{G} \to \mathbf{G}$ be idempotent morphisms such that $r(G)$ and $q(G)$ are minimal S-containing. By finiteness of G there exists some positive integer m such that the m-th power of rq and qr are idempotent. Then $S \subseteq (rq)^m(G) \subseteq r(G)$ and $S \subseteq (qr)^m(G) \subseteq q(G)$. By the minimality of $r(G)$ and

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$q(G)$, $(rq)^m(G) = r(G)$ and $(qr)^m(G) = q(G)$. Thus, $r(\mathbf{G})$ and $q(\mathbf{G})$ are isomorphic via $q|_{r(G)}$. $\square$

Let $\mathbf{G}$ and $\mathbf{H}$ be finite relational sets of the same type. A pair $(\mathbf{H}, f)$ is called a $\mathbf{G}$-*coloured* relational set, if f is a partially defined map from $\mathbf{H}$ to $\mathbf{G}$. If f does not extend to a fully defined morphism from $\mathbf{H}$ to $\mathbf{G}$ then $(\mathbf{H}, f)$ is called $\mathbf{G}$-*nonextendible.* For two $\mathbf{G}$-coloured relational sets $(\mathbf{K}, g)$ and $(\mathbf{H}, f)$ we write $(\mathbf{K}, g) \subseteq (\mathbf{H}, f)$ and say that $(\mathbf{K}, g)$ is *contained* in $(\mathbf{H}, f)$, if $K \subseteq H$, $u_{\mathbf{K}} \subseteq u_{\mathbf{H}}$ for each relational symbol u in the given type and $g \subseteq f$. A $\mathbf{G}$-*obstruction* is a $\mathbf{G}$-coloured, $\mathbf{G}$-nonextendible relational set $(\mathbf{H}, f)$, where H is finite and every $\mathbf{G}$-coloured relational set $(\mathbf{K}, g)$ properly contained in $(\mathbf{H}, f)$ is $\mathbf{G}$-extendible.

A $\mathbf{G}$-obstruction $(\mathbf{H}, f)$ is a *minimal obstruction*, if for every morphism $g : \mathbf{G} \to \mathbf{G}$ where $(\mathbf{H}, gf)$ is also a $\mathbf{G}$-obstruction there exists a morphism $g' : \mathbf{G} \to \mathbf{G}$ such that $(\mathbf{H}, g'gf) = (\mathbf{H}, f)$. A pair formed by two distinct elements a and b in G is called a *minimal pair*, if for every morphism $g : \mathbf{G} \to \mathbf{G}$ with $g(a) \neq g(b)$ there exists a morphism $g' : \mathbf{G} \to \mathbf{G}$ such that $g'g(a) = a$ and $g'g(b) = b$. We call a relational set $\mathbf{R}$ a *minimal relational set* of $\mathbf{G}$ if there exists a minimal $\mathbf{G}$-obstruction $(\mathbf{H}, f)$ (a minimal pair (a, b)) such that $\mathbf{R}$ is one of the idempotent images of $\mathbf{G}$ that contain the range of f (a and b) and have minimum cardinality. By Lemma 2.1, in this definition 'have minimum cardinality' can be replaced by 'are minimal with respect to containment'. The following is a special case of [**6**, Theorem 1.14].

THEOREM 2.2. *Let $\mathbf{G}$ be a finite relational set and suppose that $\mathbf{G}$ is a retract of the product of some of its retracts $\mathbf{G}_i$, $i \in I$. Then every minimal relational set of $\mathbf{G}$ is a retract of $\mathbf{G}_i$ for some $i \in I$.*

A variety of algebras is considered to be a category here; the objects are the algebras in the variety and the morphisms are the homomorphisms between them. Two algebras A and B are called categorically equivalent, if there is a categorical equivalence between the varieties they generate that sends A to B. Two algebras A and B are called weakly isomorphic, if there exists an isomorphism that maps A to an algebra with a base set and term operations coinciding with the ones of B. Note that weakly isomorphic algebras are categorically equivalent. We say that a relational set $\mathbf{G}$ is a relational set for an algebra A if the base sets of $\mathbf{G}$ and A are the same and the set of morphisms from finite powers of $\mathbf{G}$ to $\mathbf{G}$ equals the set of term operations of A.

Throughout the paper in notation we do not differentiate between an algebra and its base set. The following theorem appears in [**6**] as part of Theorem 2.5.

THEOREM 2.3. *Two finite algebras A and B are categorically equivalent if and only if there exist finite relational sets, $\mathbf{A}$ for A and $\mathbf{B}$ for B such that $\mathbf{A}$ and $\mathbf{B}$ are of the same type and have the same minimal relational sets up to isomorphism.*

The interested reader can find a different characterisation of categorically equivalent algebras in McKenzie [**4**].

3. Minimal relational sets assigned to finite groups

Let G be a finite group of exponent m with $m \geqslant 2$. Let p_i, $i \in I$, denote the pairwise distinct prime divisors of m. Let q_i be the highest p_i-power dividing m. So $m = \prod_{i \in I} q_i$. Let $G_i = \{g : g^{q_i} = 1\}$. Let $\mathbf{G} = (G, R)$ a relational set of a fixed type for the group G. For example, R can be taken to be the set of subgroups of finite powers of G. Define $\mathbf{G}_i$ to be the relational set on G_i by restricting the relations of $\mathbf{G}$ to G_i, $i \in I$.

Theorem 3.1. *Let G be a finite group of exponent $m = \prod_{i \in I} q_i$, with $m \geqslant 2$. Then the minimal relational sets of $\mathbf{G}$ are the $\mathbf{G}_i$, $i \in I$. These relational sets are pairwise nonisomorphic.*

Proof: The unary term operations of G are of the form $\alpha_l : g \mapsto g^l$ for some positive integer l. The map α_l is idempotent if and only if $g^{l^2} = g^l$ for all $g \in G$, that is, m divides $(l-1)l$. Since l and $l-1$ are mutually prime, the latter condition holds if and only if there exist mutually prime positive integers P and Q such that $m = PQ$, P divides $l-1$ and Q divides l. Observe, that for such P and Q, $\alpha_l(G) = \{g : g^P = 1\}$. Moreover, if l is a positive integer such that q_i divides $l-1$ and m/q_i divides l then $\alpha_l(G) = G_i$. It is clear now that the idempotent images of G form a Boolean-lattice-ordered set with respect to containment. We call this lattice L. In the lattice L the top element is G, the bottom element is $\{1\}$ and the atoms are just the G_i, $i \in I$.

We claim that the minimal relational sets of $\mathbf{G}$ are the $\mathbf{G}_i$, $i \in I$. First, we show that the $\mathbf{G}_i$ are indeed minimal relational sets. Let $i \in I$. Obviously, $1 \in G_i$. Choose $a \in G_i$ such that its order equals p_i. The elements 1 and a form a minimal pair in $\mathbf{G}$ since for each n with $\alpha_n(1) \neq \alpha_n(a)$ there exists an s such that $\alpha_s\alpha_n(1) = 1$ and $\alpha_s\alpha_n(a) = a$. Moreover, $\mathbf{G}_i$ is a minimal relational set with respect to the minimal pair $(1, a)$ as G_i is an atom of L.

It remains to prove that $\mathbf{G}$ has no other minimal relational sets. It looks complicated to describe the minimal $\mathbf{G}$-obstructions and the related minimal relational sets of $\mathbf{G}$. Instead, we show that $\mathbf{G}$ is a retract of the product $\prod_{i \in I} \mathbf{G}_i$. Then by applying Theorem 2.2, we get that each minimal relational set is a retract of some of the $\mathbf{G}_i$, $i \in I$, whence it coincides with one of them.

To prove that $\mathbf{G}$ is a retract of $\prod_{i \in I} \mathbf{G}_i$ let l_i be integers such that q_i divides $l_i - 1$

and m/q_i divides l_i, $i \in I$. Define the maps

$$r : \prod_{i\in I} G_i \to G, \quad (g_i)_{i\in I} \mapsto \prod_{i\in I} g_i$$

and

$$e : G \to \prod_{i\in I} G_i, \quad g \mapsto \left(\alpha_{l_i}(g)\right)_{i\in I}.$$

Observe, that r is a morphism from $\prod_{i\in I} \mathbf{G}_i$ to $\mathbf{G}$ and e is a morphism from $\mathbf{G}$ to $\prod_{i\in I} \mathbf{G}_i$. Moreover, since m divides $\left(\sum_{i\in I} l_i\right) - 1$, $r\Big(\left(\alpha_{l_i}(g)\right)_{i\in I}\Big) = g$ for all $g \in G$. Thus, re equals the identity on G, that is, r is a retraction from $\prod_{i\in I} \mathbf{G}_i$ to $\mathbf{G}$.

To prove that the $\mathbf{G}_i$ are pairwise nonisomorphic observe that the only morphism between $\mathbf{G}_k$ and $\mathbf{G}_j$, $k \neq j$, is the constant 1. This is due to the fact that for any morphism $\iota : \mathbf{G}_k \to \mathbf{G}_j$ the morphism $\iota\alpha_{l_k} : \mathbf{G} \to \mathbf{G}_j$ is of the form α_l for some l and α_l maps any g in G_k to an element whose order divides the order of g. ∎

Next we shall show how to reconstruct the relational set $\mathbf{G}$ from its minimal relational sets.

Theorem 3.2. *Let G be a finite group of exponent $m = \prod_{i\in I} q_i$, with $m \geqslant 2$. The relational set $\prod_{i\in I} \mathbf{G}_i$ has a minimal idempotent image containing set*

$$S = \bigcup_{i\in I} \Big\{ \Big(1, \dots, 1, \underset{i}{\underbrace{g}}, 1, \dots, 1\Big) : g \in G_i \Big\}$$

and isomorphic to $\mathbf{G}$.

Proof: Consider the morphisms e and r defined in the end of the proof of Theorem 3.1. Since e is a coretraction it is an embedding of $\mathbf{G}$ into $\prod_{i\in I} \mathbf{G}_i$. The image $e(G)$ certainly contains S. Moreover, $er : \prod_{i\in I} \mathbf{G}_i \to \prod_{i\in I} \mathbf{G}_i$ is an idempotent morphism with image $e(\mathbf{G})$.

Let $t : \prod_{i\in I} \mathbf{G}_i \to \prod_{i\in I} \mathbf{G}_i$ be an arbitrary idempotent morphism whose image contains S and lies in $e(G)$. We want show that $t\Big(\prod_{i\in I} G_i\Big) = e(G)$, which will conclude the proof of the theorem. Let $s = rte$. Observe that s is an idempotent morphism from $\mathbf{G}$ to $\mathbf{G}$. Moreover, $\bigcup_{i\in I} G_i \subseteq s(G)$. This means, that in the lattice L of idempotent images of G, $s(G)$ is greater than or equal to the join of atoms. Since L is Boolean, $s(G) = G$. So s is the identity map on G. Hence t maps onto $e(G)$, that is, $t\Big(\prod_{i\in I} G_i\Big) = e(G)$. ∎

COROLLARY 3.3. *Finite groups are categorically equivalent if and only if they are weakly isomorphic.*

PROOF: Suppose that G and G' are categorically equivalent finite groups. We shall prove that G and G' are weakly isomorphic. Assume that G is at least two element; otherwise the claim is obvious. Then by Theorem 2.3 there exist two relational sets $\mathbf{G}$ for G and $\mathbf{G}'$ for G' such that $\mathbf{G}$ and $\mathbf{G}'$ are of the same type and have the same minimal relational sets up to isomorphism. Without loss of generality we assume that these minimal sets actually are the same, say $\mathbf{G}_i$, $i \in I$. Let

$$S = \bigcup_{i \in I} \Big\{ \Big(1, \ldots, 1, \underbrace{g}_{i}, 1, \ldots, 1\Big) : \ g \in G_i \Big\}.$$

By the previous theorem, both $\mathbf{G}$ and $\mathbf{G}'$ are mimimal S-containing idempotent images of the product of these minimal relational sets. Hence, by Lemma 2.1, $\mathbf{G}$ is isomorphic to $\mathbf{G}'$. Thus, G and G' are weakly isomorphic. ∎

4. CONCLUDING REMARKS

There is a long standing conjecture of Suzuki apparent in [**5**]: the abstract subalgebra lattices of finite powers of a finite group determine the group up to isomorphism. Since, by [**4**], the subalgebra lattices of related powers of categorically equivalent algebras are isomorphic the truth of Suzuki's conjecture would imply our result in Corollary 3.3. On the other hand, there is a concrete version of Suzuki's conjecture, stated as a conjecture by Hulanicki and Świerzkowski in [**2**]. Namely, that two weakly isomorphic finite groups are isomorphic. In other words, if two group operations on the same finite set generate the same clone of operations then the corresponding groups are isomorphic. This would mean that the concrete subalgebra lattices of finite powers of a finite group determine the group up to isomorphism.

It is interesting to note that in [**3**] Lüders proved that the abstract relational clone of any finite algebra determines the algebra up to categorical equivalence. Relational clones are slightly more complicated structures than the sequences of subalgebra lattices of finite powers. For the exact definition of relational clones see [**3**]. Here, it is enough to know that the relational clone of an algebra is an algebra whose base set is the set of subalgebras of finite powers of the algebra and whose set of operations contains the usual meet of subalgebras besides others. Hence, if the relational clones of two algebras are isomorphic then so are the subalgebra lattices of the related finite powers. Lüder's result combined with Corollary 3.3 gives that the abstract relational clone of a finite group determines the group up to weak isomorphism.

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