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EXISTENCE OF SOLUTIONS IN A SINGULAR BIHARMONIC NONLINEAR PROBLEM

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In this work we prove the existence and uniqueness of positive solutions of the nonlinear singular boundary value problem

$$\Delta^2 u - \frac{q(x)}{u^{\delta}} = 0, \quad x \in \Omega$$

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0,$$

where $0 < \sigma < 1$.
Extensions of the above results to the case of $\Delta^2 u - f(x,u) = 0$ with appropriate singularity built into f are also given.

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1. Introduction

In this work we prove the existence and uniqueness of positive solutions of the nonsingular boundary value problem

$$\begin{aligned} \Delta^2 u - \frac{q(x)}{u^{\sigma}} &= 0, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} &= 0, \end{aligned} \tag{1}$$

where $0 < \sigma < 1$. Our main result is summarized in the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume $q(x) \in C^{\alpha}(\bar{\Omega})$ with $0 < \alpha < 1$, and $q(x) > 0$ in $\bar{\Omega}$. Then there exists a unique classical solution u of (1) with $u \in C^{4+\alpha}(\Omega) \cap C^3(\bar{\Omega})$. Moreover $u \geqq \delta\varphi_1$ for some $\delta > 0$, where $\varphi_1 > 0$ is the first eigenfunction of the negative Laplacian operator subject to zero Dirichlet boundary conditions.*

Note. An extension of the above result to the case of $\Delta^2 u - f(x,u) = 0$ with appropriate singularity built into f can be proved by a similar technique. This is discussed in the Remark at the end of section 3.

There is an extensive literature for the problem

$$\Delta u + \frac{q(x)}{u^\sigma} = 0,$$
$$u\big|_{\partial\Omega} = 0, \tag{2}$$

When $n=1$ the ordinary differential equation problem was first treated by Taliaferro [**10**]. Generalization of the nonlinearity $q(x)/u^\sigma$ to $f(x,u)$, where $f(x,u)$ is a non-negative function that is non-increasing in u for every $x \in \Omega$, $\lim_{u\to 0} f(x,u) = \infty$ and $\lim_{u\to\infty} f(x,u) = 0$, has been obtained by Gatica, Hernandez and Waltman [7], for radially symmetric solutions in a ball. Thus they treat

$$y'' + \frac{n-1}{x}\, y' + f(x, y) = 0. \tag{3}$$

In [**6**] the same authors treat in detail the general one-dimensional problem, the radially symmetric problem, and the behaviour of the solutions near the degeneracy point on the boundary. In this latter reference, only $\sigma > 0$ is required. The result for (2) in an n-dimensional domain was obtained by Lazer and McKenna [**8**].

The singular fourth order equation

$$y^{iv} = f(t, y, y'), \quad 0 < t < 1, \tag{4}$$

where y satisfies some given boundary conditions and f is singular at y', has been treated by O'Regan [**9**]. Eloe and Henderson [**3**] have also discussed fourth and higher order singular problems with different boundary conditions. Neither of these references treats the boundary conditions discussed here. Moreover they use techniques which are restricted to ordinary differential equations only.

In [**6**], the authors approximate the singular problem (2) by a sequence of regular elliptic problems. As monotone iteration technique is then employed to show the existence of a solution for each regular problem. These solutions are also monotonically ordered, which allows them to extract a convergent sequence which is the solution to (3). Similar arguments are used in [**8**]. These proofs depend heavily on the use of the Maximum Principle.

The system of two second order equations, formed by rewriting equations (1), is not quasi-monotone. Hence the monotone iteration technique fails. The idea of approximating the original singular problem by a sequence of regular problems still proves to be fruitful. A compactness argument will be employed to replace the monotone iteration technique. The major hurdle is the establishment of a uniform bound for the solutions of each of the regular problems. Once accomplished, this allows extraction of a convergent subsequence whose limit will satisfy the original singular problem.

Section 2 contains a useful *a priori* estimate, which is stated in Lemma 2.1. It is crucial in establishing a uniform bound for solutions of the sequence of regular problems. In Section 3 we prove Theorem 1.1 using Schauder's fixed point theorem.

In this paper $C(\bar{\Omega})$ will denote the space of continuous functions on Ω with the supremum norm, denoted by $\|\cdot\|$. All other norms involved will be appropriately indicated.

2. A useful lemma

We first introduce some definitions. Let $\lambda_1>0$ be the first eigenvalue and $\varphi_1>0$ on Ω be the corresponding eigenfunction for the negative Laplacian operator with zero Dirichlet boundary conditions, i.e., they satisfy

$$\begin{aligned}\Delta\varphi_1+\lambda_1\varphi_1&=0,\\ \varphi_1|_{\partial\Omega}&=0.\end{aligned}\tag{5}$$

Assume φ_1 is normalized so that $\|\varphi_1\|=1$. From Hopf's lemma, there exists a $\delta>0$ such that $|\nabla\varphi_1|\geqq\delta$ for all $x\in\partial\Omega$.

Next define ϕ_0 to be the function that satisfies

$$\begin{aligned}-\Delta\phi_0&=1 \quad \text{in } \Omega,\\ \phi_0|_{\partial\Omega}&=0.\end{aligned}\tag{6}$$

By the maximum principle $\phi_0(x)>0$ in Ω.

Further, we let $G(x,y)$ be the Green's function corresponding to the negative Laplacian operator subject to zero Dirichlet boundary conditions. It is then known that G is non-negative, and if

$$\begin{aligned}-\Delta w&=h(x),\\ w|_{\partial\Omega}&=0,\end{aligned}\tag{7}$$

the solution of (7) with $h\in\mathscr{C}(\bar{\Omega})$ is given by

$$w(x)=\int_\Omega G(x,y)h(y)\,dy.\tag{8}$$

In particular

$$\varphi_1(x)=\lambda_1\int_\Omega G(x,y)\varphi_1(y)\,dy$$

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and

$$\phi_0(x) = \int_\Omega G(x, y)\, dy,$$

which, as a consequence of the normalization of φ_1, leads to

$$\varphi_1 \leqq \lambda_1 \phi_0. \tag{9}$$

The following lemma is a crucial *a priori* estimate which will be useful in proving our main theorem.

Lemma 2.1. *Given* $0 < \sigma < 1$, *there exists a constant* $C > 0$, *which depends on* σ, *such that for all* $x \in \Omega$,

$$\int_\Omega \frac{G(x, y)}{\varphi_1^\sigma} dy \leqq C.$$

Proof. Let

$$w_\varepsilon(x) = \int_\Omega \frac{G(x, y)}{(\varphi_1 + \varepsilon)^\sigma}.$$

Hence by the definition of the Green's function G,

$$\begin{aligned} -\Delta w_\varepsilon &= \frac{1}{(\varphi_1 + \varepsilon)^\sigma}, \\ w_\varepsilon\big|_{\partial\Omega} &= 0. \end{aligned} \tag{10}$$

Further let $z = C\varphi_1^\sigma$, C being a constant to be chosen. For $x \in \Omega$, a simple calculation shows that

$$-\Delta z = \varphi_1^{\sigma-2}[C\sigma(1-\sigma)|\nabla\varphi_1|^2 + C\lambda_1\sigma\varphi_1^2].$$

C is now chosen to be large enough so that

$$C\sigma(1-\sigma)|\nabla\varphi_1|^2 + C\lambda_1\sigma\varphi_1^2 \geqq \varphi_1^{2(1-\sigma)}$$

in Ω. This is always possible because $|\nabla\varphi_1| \geqq \delta > 0$ at the boundary of Ω. Thus

$$\varphi_1^{\sigma-2}[C\sigma(1-\sigma)|\nabla\varphi_1|^2 + C\lambda_1\sigma\varphi_1^2] \geqq \varphi_1^{-\sigma} \geqq \frac{1}{(\varphi_1 + \varepsilon)^\sigma},$$

so

$$-\Delta z \geqq -\Delta w_\varepsilon \quad \text{in} \quad \Omega,$$

$$z = w_\varepsilon = 0 \quad \text{on} \quad \partial\Omega.$$

The maximum principle then implies $w_\varepsilon(x) \leqq z(x) \leqq C$ in $\bar{\Omega}$.

For fixed x, as ε decreases to 0, $G(x, y)/(\varphi_1 + \varepsilon)^\sigma$ increase to $G(x, y)/\varphi_1^\sigma$ which is defined everywhere, except at $y = x$ and may be on the boundary of Ω. By Fatou's Lemma

$$\int_\Omega \liminf \frac{G(x, y)}{(\varphi_1 + \varepsilon)^\sigma} \leqq \liminf \int_\Omega \frac{G(x, y)}{(\varphi_1 + \varepsilon)^\sigma} \leqq C.$$

Thus

$$\int_\Omega \frac{G(x, y)}{\varphi_1^\sigma} \leqq C$$

and the lemma follows.

3. Proof of Theorem 1

Suppose u is a classical solution of (1), hence $\|u\| \leqq K_1$ for some $K_1 > 0$. Define $q_0 = \min_{\bar{\Omega}} q(x) > 0$ and let $w = u - \delta\varphi_1$, where $\delta < q_0/\lambda_1^2 K_1^\sigma$. Then $\Delta^2 w \geqq 0$ in Ω, and $w = \Delta w = 0$ on $\partial\Omega$. A repeated application of the Maximum Principle gives $w \geqq 0$, and so $u \geqq \delta\varphi_1$ on $\bar{\Omega}$.

Uniqueness can be easily established as follows. Assume there are two classical solutions of (1) u and $\hat{u}$, which are in $C^{4+\alpha}(\Omega) \cap C^3(\bar{\Omega})$. Hence, there exists $\delta > 0$ such that $u \geqq \delta\varphi_1$ and $\hat{u} \geqq \delta\varphi_1$ in Ω. Define $z = u - \hat{u}$. By the mean value theorem,

$$\Delta^2 z = -\sigma \frac{q(x)}{\tilde{u}^{(\sigma+1)}} z$$

where $\tilde{u} \geqq \delta\varphi_1$ in Ω. We then multiply the above equation by z and integrate over a smooth domain Ω' compactly contained in Ω, After applying the Divergence Theorem twice, we get

$$\int_{\Omega_1} \left[(\Delta z)^2 + \sigma \frac{q(x) z^2}{\tilde{u}^{\sigma+1}} \right] + \int_{\partial\Omega_1} \left[z \frac{\partial \Delta z}{\partial n} \right] - \int_{\partial\Omega_1} \left[\Delta z \frac{\partial z}{\partial n} \right] = 0$$

where n is a unit outward normal on $\partial\Omega$. Taking the limit as $\Omega' \to \Omega$, the second and the last term on the left-hand side of the above equation vanish. Since $\int_\Omega (\Delta z)^2 < \infty$, then $\int_\Omega \sigma\, (q(x) z^2/\tilde{u}^{\sigma+1})$ is well defined. Hence

$$\int_\Omega \left[(\Delta z)^2 + \sigma \frac{q(x)z^2}{\tilde{u}^{\sigma+1}} \right] = 0,$$

which implies $z=0$ in Ω. We have therefore proved the uniqueness of the solution in the class of $C^{4+\alpha}(\Omega) \cap C^3(\bar{\Omega})$.

We now turn to the question of the existence of a solution. Consider the ε-approximation problems,

$$\begin{aligned} \Delta^2 u &= \frac{q(x)}{(u+\varepsilon)^\sigma}, \\ u|_{\partial\Omega} &= \Delta u|_{\partial\Omega} = 0, \end{aligned} \tag{11}$$

where $\varepsilon > 0$. This can be cast as a system of equations:

$$\begin{aligned} \Delta u + \lambda_1 v &= 0, \\ \Delta v + \frac{1}{\lambda_1} \frac{q(x)}{(u+\varepsilon)^\sigma} &= 0, \\ u|_{\partial\Omega} = v|_{\partial\Omega} &= 0. \end{aligned} \tag{12}$$

Let

$$\mathscr{A} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathscr{C}(\bar{\Omega}) \times \mathscr{C}(\bar{\Omega}) : \delta\varphi_1 \leqq u \leqq K_1,\ \delta\varphi_1 \leqq v \leqq K_2 \right\}$$

and define

$$T_\varepsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda_1 \int_\Omega G(x,y) v(y)\, dy \\ \dfrac{1}{\lambda_1} \displaystyle\int_\Omega \frac{G(x,y)}{(u(y)+\varepsilon)^\sigma} q(y)\, dy \end{pmatrix}. \tag{13}$$

To proceed, we need the following lemma.

Lemma 3.1. *There exist K_1, K_2 and δ such that T_ε maps $\mathscr{A}$ into $\mathscr{A}$.*

To avoid loss of continuity, we assume the lemma holds for the time being. It is then easy to see that $\mathscr{A}$ is closed and convex. Since v and $q/(u+\varepsilon)$ are continuous in $\bar{\Omega}$, they are bounded in the $L^p(\Omega)$ norm for any $1<p<\infty$, the solutions $T_\varepsilon(u\quad v)$ are then in $W^{2,p}$ by the regularity estimate of second order elliptic equations, [2].

By Sobolev's imbedding theorem, $T_\varepsilon(u\quad v)$ is in $C^{1+\beta}(\bar{\Omega})$ for any $0<\beta<1$. Thus the map T_ε is compact.

We can now apply Schauder's fixed point theorem, which leads to the existence of a fixed point $(u_\varepsilon\ \ v_\varepsilon)$ of T_ε. That is to say

$$u_\varepsilon = \lambda_1 \int_\Omega G(x, y) v_\varepsilon(y)\, dy, \tag{14}$$

$$v_\varepsilon = \frac{1}{\lambda_1} \int_\Omega \frac{G(x, y)}{(u_\varepsilon(y) + \varepsilon)^\sigma} q(y)\, dy. \tag{15}$$

With u_ε and v_ε in $C^{1+\beta}(\bar{\Omega})$ and $q \in C^\alpha(\bar{\Omega})$, we can bootstrap the regularity given $u_\varepsilon \in C^{4+\alpha}(\bar{\Omega})$ and $v_\varepsilon \in C^{2+\alpha}(\bar{\Omega})$. It is then easy to see that $(u_\varepsilon\ \ v_\varepsilon)$ satisfies equations (12).

Since $(u_\varepsilon\ \ v_\varepsilon) \in \mathscr{A}$, then

$$\frac{1}{(u_\varepsilon(y) + \varepsilon)^\sigma} \leqq \frac{1}{\delta^\sigma \varphi_1^\sigma},$$

Thus there is a uniform bound of u_ε and v_ε in $C^{4+\alpha}(\bar{\Omega}_1)$ and $C^{2+\alpha}(\bar{\Omega}_1)$ respectively for any Ω_1 that is compactly contained in Ω. Take $0 < \alpha_1 < \alpha$. By standard compactness results, [**1**], we can extract a convergent subsequence of u_ε and v_ε in $C^{2+\alpha_1}(\bar{\Omega}_1)$ as $\varepsilon \to 0$. Let u and v be the corresponding limit. It is clear that u and v satisfy equation (12) with $\varepsilon = 0$ in Ω_1.

Let Ω_k, $k = 1, 2, \ldots$ be a sequence of smooth domains compactly contained in Ω with $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega$ and $\Omega_k \to \Omega$. The usual diagonal subsequence argument will generate u and v satisfying equation (12) on Ω with $\varepsilon = 0$, since at the boundary of Ω, $u = v = 0$. We therefore conclude that u satisfies equations (1). According to our construction, $u \in C^{4+\alpha_1}(\Omega)$. We can bootstrap the regularity of this solution so that $u \in C^{4+\alpha}(\Omega)$. Hence we have a classical solution in $C^2(\bar{\Omega}) \cap C^{4+\alpha}(\Omega)$.

To show that $u \in C^3(\bar{\Omega})$, we note that there exists some constants M_1 and M_2 [**5**] such that

$$\int_\Omega \frac{|G_x(x, y)|}{\varphi_1^\sigma} \leqq \int_\Omega \left(\frac{M_1}{|x - y|^{n-1}} + M_2 \right) \frac{1}{(\varphi_1(y))^\sigma} dy < \infty.$$

The last inequality can be proved by covering $\partial\Omega$ with finitely many open sets, mapping each of them locally to 'straighten' its boundary, and using the fact that $|\nabla \varphi_1| \geqq \delta > 0$ on the boundary. Hence $v = \Delta u$ can be differentiated once more even at the boundary of Ω by the dominated convergence theorem. By a standard result ([**4**, Lemma 1, p. 7]) this third derivative of u is continuous.

This completes the proof of Theorem 1.1.

Finally we give the proof of Lemma 3.1.

Proof of Lemma 3.1. Let $q_0 = \min_{\bar{\Omega}} q(x)$, $q_\infty = \max_{\bar{\Omega}} q(x)$, $m_0 = \max_{\bar{\Omega}} \phi_0(x)$, and C be as defined in Lemma 2.1. We choose K_1 such that

$$\frac{(q_\infty m_0 C)^{1/\sigma}}{q_0} < \frac{K_1^{1/\sigma}}{\lambda_1^2(K_1+1)^\sigma},$$

which is always possible since $\sigma \in (0,1)$. Now choose δ such that

$$\frac{(q_\infty m_0 C)^{1/\sigma}}{K_1^{1/\sigma}} < \delta < \frac{q_0}{\lambda_1^2(K_1+1)^\sigma}.$$

Then

$$\frac{q_\infty m_0 C}{\delta^0} < K_1$$

and

$$\delta < \frac{q_0}{\lambda_1^2(K_1+1)^\sigma}.$$

Finally choose

$$K_2 = \frac{K_1}{\lambda_1 m_0}.$$

With such choices of K_1, K_2, and δ in $\mathscr{A}$, we prove T_ε maps $\mathscr{A}$ into $\mathscr{A}$ by the following calculations. Without loss of generality, we take $\varepsilon \leqq 1$.

Lower bounds for T_ε:

$$T_\varepsilon\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda_1 \int_\Omega G(x,y)v(y)\,dy \\ \dfrac{1}{\lambda_1}\int_\Omega \dfrac{G(x,y)}{(u(y)+\varepsilon)^\sigma} q(y)\,dy \end{pmatrix} \geqq \begin{pmatrix} \delta\lambda_1 \int_\Omega G(x,y)\varphi_1(y)\,dy \\ \dfrac{q_0}{\lambda_1(K_1+1)^\sigma}\int_\Omega G(x,y)\,dy \end{pmatrix}$$

$$\geqq \begin{pmatrix} \delta\varphi_1(x) \\ \dfrac{q_0}{\lambda_1(K_1+1)^\sigma}\phi_0(x) \end{pmatrix}.$$

Using equation (9), we get

$$T_\varepsilon\begin{pmatrix} u \\ v \end{pmatrix} \geqq \begin{pmatrix} \delta\varphi_1(x) \\ \dfrac{q_0}{\lambda_1^2(K_1+1)^\sigma}\varphi_1(x) \end{pmatrix}$$

$$\geqq \begin{pmatrix} \delta\varphi_1(x) \\ \delta\varphi_1(x) \end{pmatrix}. \tag{16}$$

Upper bounds for T_ε:

$$
\begin{aligned}
T_\varepsilon\begin{pmatrix}u\\v\end{pmatrix}&=\begin{pmatrix}\lambda_1\int_\Omega G(x,y)v(y)\,dy\\ \dfrac{1}{\lambda_1}\int_\Omega \dfrac{G(x,y)}{(u(y)+\varepsilon)^\sigma}q(y)\,dy\end{pmatrix}\\
&\leqq\begin{pmatrix}K_2\lambda_1\int_\Omega G(x,y)\,dy\\ \dfrac{q_\infty}{\lambda_1}\int_\Omega \dfrac{G(x,y)}{(\delta\varphi_1)^\sigma}dy\end{pmatrix}\\
&\leqq\begin{pmatrix}K_2\lambda_1 m_0\\ \dfrac{q_\infty}{\lambda_1\delta^\sigma}\int_\Omega \dfrac{G(x,y)}{\varphi_1^\sigma}dy\end{pmatrix}\\
&\leqq\begin{pmatrix}K_2\lambda_1 m_0\\ \dfrac{q_\infty C}{\lambda_1\delta^\sigma}\end{pmatrix}\\
&\leqq\begin{pmatrix}K_2\lambda_1 m_0\\ \dfrac{K_1}{\lambda_1 m_0}\end{pmatrix}\leqq\begin{pmatrix}K_1\\K_2\end{pmatrix}.
\end{aligned}
\tag{17}
$$

Thus T_ε maps $\mathscr{A}$ into $\mathscr{A}$. We have therefore completed the proof of Lemma 3.1.

Remark. The same technique can be employed to prove the existence of solution for the equations

$$
\begin{aligned}
&\Delta^2 u-f(x,u)=0,\\
&u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=0,
\end{aligned}
\tag{18}
$$

where

$$\frac{q_0}{u^\sigma}\leqq f(x,u)\leqq\frac{q_\infty}{u^\gamma}+C_1\quad\text{for}\quad x\in\Omega\quad\text{and}\quad u>0.$$

Here q_0, q_∞, and C_1 are positive constants, and $0<\sigma<\gamma<1$. The only modifications in the above proof are in the definitions of equations (13) and the proof in Lemma 3.1.

Equations (13) are changed to

$$T_\varepsilon\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\lambda_1\int_\Omega G(x,y)v(y)\,dy\\ \dfrac{1}{\lambda_1}\int_\Omega G(x,y)f(x,u(y)+\varepsilon)\,dy\end{pmatrix}$$

for $(u_\varepsilon\ \ v_\varepsilon)$ in $\mathscr{A}$. Now choose C, which depends on γ, as in Lemma 2.1. K_1 and δ are positive constants satisfying:

$$\left(\frac{1}{\dfrac{K_1}{q_\infty m_0 C} - \dfrac{C_1 m_0}{q_\infty C}}\right)^{1/\gamma} < \delta < \frac{q_0}{\lambda_1^2 (K_1+1)^\sigma},$$

and $K_2 = K_1/\lambda_1 m_0$. Then Lemma 3.1 holds. Hence the solution to the singular equation (18) exists. If we further assume $f(x,u)$ to be monotone decreasing in u for all x, then we can recover the uniqueness proof.

REFERENCES

1. Robert A. Adams, *Sobolev Spaces* (Academic Press, 1975).

2. S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, *Comm. Pure Appl. Math.* **12** (1959), 623–727.

3. P. W. Eloe and J. Henderson, Nonlinear boundary value problems and *apriori* bounds on solutions. *SIAM J. Math. Anal.* (1984), 642–647.

4. Avner Friedman, *Partial Differential Equations of Parabolic Type* (Prentice-Hall, 1964).

5. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer-Verlag, 1983).

6. Gaston E. Hernandez, J. A. Gatica, A. M. Fink and P. Waltman, Approximations of solutions of singular second order boundary value problems, *SIAM J. Math Anal.* **22** (1989), 440–462.

7. Gaston E. Hernandez, J. A. Gatica and P. Waltman, Radially symmetric solutions of a class of singular elliptic equations, *Proc. Edinburgh Math. Soc.* **33** (1989), 169–180.

8. A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary-value problem. *Proc. Amer. Math. Soc.* **111** (1991), 721–730.

9. Donal O'Regan, Fourth (and higher) order singularity boundary value problems, *Nonlinear Anal.* **14** (1990), 1001–1038.

10. S. Taliaferro, A nonlinear singular boundary value problem, *Nonlinear Anal.* **3** (1979), 897–904.

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