DECOMPOSITION OF GRAPHS INTO TWO-WAY INFINITE PATHS

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1. Introduction. We consider undirected graphs in which two vertices may be joined by more than one edge and in which a vertex may be joined to itself by one or more edges. G will always denote a graph. The set of vertices [edges] of G will be denoted by $V(G)$ $[E(G)]$. G is *finite* or *infinite* according as $V(G) \cup E(G)$ is finite or infinite. The *degree*, $d(\xi)$ or $d_G(\xi)$, of a vertex ξ of G is $2a + b$, where a is the number of edges joining ξ to itself and b is the number of those joining ξ to other vertices. A vertex is *even* [*odd*] if it has even [odd] finite degree. A *subgraph* of G is a graph H such that $V(H) \subset V(G)$, $E(H) \subset E(G)$, and each edge of H joins the same vertices in H as in G. A *component* of G is a maximal connected subgraph. A collection of subgraphs of G are *disjoint* [*edge-disjoint*] if no two have a common vertex [edge]. $|A|$ denotes the cardinal number of the set A. G will be called *relevant* if $|E(G)| = \aleph_0$ and G has no odd vertex or finite component. A *decomposition* of G is a set of edge-disjoint subgraphs of G whose union is G. A *path-sequence* of G is a finite or infinite sequence whose terms are alternately vertices and edges of G, starting and ending if at all with a vertex, such that each edge appearing in the sequence appears just once and joins the vertices immediately preceding and following it. An infinite path-sequence is *one-way infinite* [*two-way infinite*] if it has [has not] a first or last term. The subgraph formed by the terms of a path-sequence [n-way infinite path-sequence] is a *path* [*n-way infinite path* $(n = 1, 2)$]. (The same path can be both one-way and two-way infinite.) It follows from (**1**, Theorem 2) that a relevant graph is decomposable into two-way infinite paths, and Ore (**3**, p. 47, problem 3) asks in effect what is the minimum number of such paths into which it can be decomposed. This paper answers the question.

2. Limited graphs. Definitions. If X, Y are subsets of $V(G)$, $\bar{X}$ will denote $V(G) - X$, $X \circ Y$ will denote the set of those edges of G which join elements of X to elements of Y, $X\delta$ (or $X\delta_G$) will denote $X \circ \bar{X}$, and X^* will denote the subgraph of G defined by $V(X^*) = X$, $E(X^*) = X \circ X$. X is *inessential* if $X \circ V(G)$ is finite. $X\delta$ will be called a *cincture*; it is an *even* [*odd*] cincture if $|X\delta|$ is finite and even [odd]. Whenever two or more graphs are under consideration and one of them is denoted by G, the notations $d(\xi)$, $X\delta$ will mean $d_G(\xi)$, $X\delta_G$, and the notations $X \circ Y$, $\bar{X}$, X^* will also be interpreted relative to G. (But an odd vertex of a subgraph of G is one whose degree in

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that subgraph is odd.) If $A_1, \ldots, A_n$ are sets, $A_1 + \ldots + A_n$ denotes the set of those elements which belong to an odd number of the A_i. Two subgraphs H, K of G are *similar* (in symbols, $H \sim K$) if the symmetric differences $V(H) + V(K)$ and $E(H) + E(K)$ are finite. Clearly similarity is an equivalence relation and so subdivides the subgraphs of G into *similarity classes*. Let l be a non-negative integer. An *l-splitting* of G is a set $\{H_1, \ldots, H_l\}$ of disjoint infinite subgraphs of G such that G is the union of these and a finite subgraph: any finite subgraph H such that $G = H_1 \cup \ldots \cup H_l \cup H$ will be called a *completion* of the splitting. G is *l-limited* (in symbols, $\lim G = l$), if it has an l-splitting but no l'-splitting for $l' > l$. G is *limited* if it is l-limited for some non-negative integer l.

LEMMA 1. *If G is l-limited, there are l distinct similarity classes of subgraphs of G such that the subgraphs in any l-splitting of G belong one to each of these classes.*

Proof. Since the members of an l-splitting of G are clearly dissimilar, it suffices to prove that, if $\{H_1, \ldots, H_l\}$ and $\{K_1, \ldots, K_l\}$ are l-splittings of G, each H_i is similar to one of the K_j. To prove this, let the given splittings have completions H, K respectively. Since the subgraphs $H_i \cap K_j$ are disjoint and the union of these and the finite subgraph $H \cup K$ is G, which is l-limited, at most l of the $H_i \cap K_j$ can be infinite. But, since the H_i and K_j are infinite, there exists for each i a j and for each j an i such that $H_i \cap K_j$ is infinite. Hence there is a permutation $K_1', \ldots, K_l'$ of $K_1, \ldots, K_l$ such that $H_i \cap K_j'$ is infinite if and only if $i = j$. Since $V(H_i) + V(K_i')$, $E(H_i) + E(K_i')$ are contained in the sets of vertices and edges of the finite graph

$$H \cup K \cup \bigcup_{j \neq i} [(H_i \cap K_j') \cup (H_j \cap K_i')],$$

it follows that $H_i \sim K_i'$ $(i = 1, \ldots, l)$ as required.

DEFINITIONS. *The l similarity classes characterized by Lemma 1 will be called wings of G. If W is a wing of G and X is a subset of $V(G)$ such that $X^* \in W$ and $X\delta$ is finite, we shall call X a W-set and $X\delta$ a W-cincture.*

LEMMA 2. *If X is an inessential subset of $V(G)$, then*

$$|X\delta| \equiv \sum_{\xi \in X} d(\xi) \pmod 2.$$

Proof. This is so since an edge contributes 2, 1, or 0 to $\sum_{\xi \in X} d(\xi)$ according as it belongs to $X \circ X$, $X\delta$, or $\bar{X} \circ \bar{X}$, respectively.

COROLLARY 2A. *If, in addition, G is relevant, $X\delta$ is even.*

LEMMA 3. *If $X, Y \subset V(G)$, then $X\delta + Y\delta = (X + Y)\delta$.*

The proof is left to the reader.

LEMMA 4. *Let G be l-limited, $\{H_1, \ldots, H_l\}$ be an l-splitting of G, and W_i be* *the wing of G which includes H_i. Then* (i) *H_i is 1-limited and* (ii) *$V(H_i)$ is a W_i-set.*

Proof. (i) Obviously H_i has a 1-splitting. On the other hand, the elements of any r-splitting of H_i together with the H_j $(j \neq i)$ constitute an $(l + r - 1)$-splitting of G; hence r cannot exceed 1.

(ii) If H is a completion of the given splitting, $E(V(H_i)^*) - E(H_i)$ and $V(H_i)\delta$ are both contained in the finite set $E(H)$. Hence $V(H_i)^* \sim H_i \in W_i$ and $V(H_i)$ is a W_i-set.

LEMMA 5. *If G is limited and relevant and W is a wing of G, there exists a W-cincture and all W-cinctures have the same parity.*

Proof. The first assertion follows from Lemma 4(ii). Moreover, if X, Y are W-sets, write $X \cap Y = Z_1$, $X \cap \bar{Y} = Z_2$, $\bar{X} \cap Y = Z_3$, $\bar{X} \cap \bar{Y} = Z_4$, and $Z_i \circ Z_j = A_{ij}$. Then

$$(X + Y) \circ V(G) = \bigcup_{i=1}^{4} (A_{i2} \cup A_{i3}) \subset A_{22} \cup A_{33} \cup \bigcup_{i \neq j} A_{ij},$$

which is finite since $\bigcup_{i\neq j} A_{ij} = X\delta \cup Y\delta$, $A_{22} \cup A_{33} \subset E(X^*) + E(Y^*)$, and $X^* \sim Y^*$. Hence $X + Y$ is inessential and so, by Lemma 3 and Corollary 2A,

$$|X\delta| + |Y\delta| \equiv |X\delta + Y\delta| = |(X + Y)\delta| \equiv 0 \pmod 2.$$

Therefore $X\delta$ and $Y\delta$ have the same parity, and Lemma 5 is proved.

DEFINITIONS. *If G is limited and relevant, a wing W of G will be called odd or even according as the W-cinctures of G are odd or even respectively, and $p(G)$ will denote $\alpha + \frac{1}{2}\beta$, where α is the number of even wings of G and β is the number of odd ones. For an unlimited relevant G, we define $p(G)$ to be $\aleph_0$.*

LEMMA 6. *If G is limited and relevant, $p(G)$ is an integer.*

Proof. Let $\lim G = l$ and $\{H_1, \ldots, H_l\}$ be an l-splitting of G with completion H. Let W_i be the wing of G which includes H_i and let $X_i = V(H_i)$. Let $Y = V(G) - X_1 \cup \ldots \cup X_l$. By Lemma 3 and induction,

$$(1) \qquad X_1\delta + \ldots + X_l\delta = (X_1 + \ldots + X_l)\delta = (X_1 \cup \ldots \cup X_l)\delta = Y\delta.$$

Since $Y \circ V(G) \subset E(H)$, Y is inessential and hence $Y\delta$ is even by Corollary 2A. Moreover, $X_i\delta$ is a W_i-cincture by Lemma 4(ii). Therefore, by (1), an even number of the W_i are odd and $p(G)$ is an integer.

3. Solution of Ore's problem. *Definitions.* If H is a subgraph of G, $G - H$ will denote the subgraph obtained by omitting from G the edges of H and those vertices of H which are not incident with other edges. The path formed by the terms of a path-sequence will be said to be *derived* from it. When a path-sequence is denoted by a small letter, the corresponding capital letter will denote the path derived from it. For any edge λ appearing in a two-way

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infinite path-sequence s, the sub-sequence formed by the terms before [after] λ will be called a *tail* [*head*] of s. An *open* [*closed*] path-sequence is a finite path-sequence whose first and last terms are different [the same]. An *open* [*closed*] path is one derivable from an open [closed] path-sequence. Clearly an open path has just two odd vertices which are the end terms of every path-sequence from which it is derivable: the path will be said to *connect* these vertices. A $\xi\infty$*-path* is one derivable from a one-way infinite path-sequence with first or last term ξ.

LEMMA 7. *Let $\{H_1, \ldots, H_l\}$ be an l-splitting of G and $s_1, \ldots, s_k$ be two-way infinite path-sequences of G such that $G = S_1 \cup \ldots \cup S_k$. Then we can select a tail t_i and head u_i of s_i for $i = 1, \ldots, k$ such that each T_i and U_i is contained in one of the H_j and each H_j contains at least one T_i or U_i.*

Proof. Let H be a completion of $\{H_1, \ldots, H_l\}$. Since $E(H)$ is finite, s_i has a tail t_i and head u_i neither of which includes an edge of H. Clearly S_i is the union of T_i, U_i, and a finite subgraph F_i. Since the T_i and U_i are connected infinite graphs having no edges in common with H, each of them is contained in an H_j. It follows, since no H_j could be contained in the finite graph $F_1 \cup \ldots \cup F_k$, that each H_j contains at least one T_i or U_i.

LEMMA 8. *If G is decomposable into k two-way infinite paths, where k is a positive integer, then G is limited and relevant and $k \geqslant p(G)$.*

Proof. G is obviously relevant. Let $s_1, \ldots, s_k$ be two-way infinite path-sequences such that the S_i decompose G. If $\mathfrak{H}$ is a splitting of G, then by Lemma 7 there exist $2k$ non-empty subgraphs such that each member of $\mathfrak{H}$ contains at least one of them, whence $|\mathfrak{H}| \leqslant 2k$. Therefore G is limited and $\lim G \leqslant 2k$. Let $\lim G = l$ and $\{H_1, \ldots, H_l\}$ be an l-splitting of G. Let W_j be the wing of G which includes H_j and let $X_j = V(H_j)$. By Lemma 7, we can select for $i = 1, \ldots, k$ a tail t_i and head u_i of s_i such that each T_i and U_i is contained in one of the H_j and each H_j contains at least one T_i or U_i. If n_j is the number of the subgraphs $T_1, \ldots, T_k$, $U_1, \ldots, U_k$ which are contained in H_j, then $n_j \geqslant 1$. Moreover, if $n_j = 1$, then, for some i, one of T_i, U_i is contained in H_j whilst the other, together with all the T_h, U_h $(h \neq i)$, is contained in $\bigcup_{m \neq j} H_m$. Hence one of t_i, u_i has all its vertex-terms in X_j whilst the other, and the t_h, u_h $(h \neq i)$, have them all in $\bar{X}_j$. Therefore s_i includes an odd number of edges from $X_j\delta$ and each s_h $(h \neq i)$ includes an even number. Therefore $X_i\delta$ is odd, and hence, by Lemma 4(ii), W_j is odd. We have thus seen that $n_j \geqslant 1$ and can only be 1 if W_j is odd, from which it follows that $p(G) \leqslant \frac{1}{2}\sum n_j = k$.

LEMMA 9. *If G is finite and has k odd vertices, then k is even and G is decomposable into $\frac{1}{2}k$ open paths connecting all the odd vertices in pairs and a finite number of disjoint closed paths.*

Proof. By **(3**, Theorem 3.1.1**)**, each component of G which has no odd

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vertices is a closed path. Moreover, if a component of G has h (>0) odd vertices, then h is even by Lemma 2 or **(3**, Theorem 1.2.1**)**, and it follows from the proof (although not, strictly speaking, from the statement) of **(3**, Theorem 3.1.2**)** that the component is decomposable into $\frac{1}{2}h$ open paths connecting all its odd vertices in pairs. Lemma 9 follows from these remarks.

LEMMA 10. *If G is 1-limited and connected and has no odd vertex and $|E(G)| = \aleph_0$, then G is a two-way infinite path.*

Proof. This follows from **(3**, Theorem 3.2.2**)** if we show that, for any finite subgraph H of G, $G - H$ has just one infinite component. But this is clear, since $G - H$ is infinite but (since G is 1-limited) not the union of two disjoint infinite subgraphs.

LEMMA 11. *If $|E(G)| = \aleph_0$, $\xi \in V(G)$, $d(\xi)$ is odd or infinite, and G is 1-limited and connected and has no odd vertex except possibly ξ, then G is a $\xi\infty$-path.*

Proof. This follows from the proof of **(3**, Theorem 3.2.1**)**, since the above proof of Lemma 10 shows that G satisfies Condition δ_1 of that theorem.

LEMMA 12. *If G has no finite component and F_0 is a finite subgraph of G, then F_0 is contained in a finite subgraph H such that no component of $G - H$ is finite or includes more than one odd vertex of H.*

Proof. If a component of $G - F_0$ includes two odd vertices of F_0, let F_1 be the union of F_0 and a path connecting two such vertices in $G - F_0$. If a component of $G - F_1$ includes two odd vertices of F_1, let F_2 be the union of F_1 and a path connecting two such vertices in $G - F_1$; and so on. Since F_{i+1} has two less odd vertices than F_i, we must ultimately reach an F_n such that no component of $G - F_n$ includes more than one odd vertex of F_n. If H is the union of F_n and the finite components of $G - F_n$, then no component of $G - H$ is finite or includes more than one odd vertex of H. To see that H is finite, we must show that $G - F_n$ has only finitely many finite components. But this is clear, since each finite component of $G - F_n$, not being a component of G, must include a vertex of F_n.

LEMMA 13. *A relevant limited graph G is decomposable into $p(G)$ two-way infinite paths.*

Proof. Let $\lim G = l$. Let $\{H, \ldots, H_l\}$ be an l-splitting of G with completion H. By Lemma 12, H is contained in a finite subgraph K such that no component of $G - K$ is finite or includes more than one odd vertex of K. Moreover, $G - K$ is the union of the disjoint infinite subgraphs $(G - K) \cap H_i$, and cannot have more than l infinite components since $\lim G = l$. Hence the components of $G - K$ are precisely the infinite subgraphs $I_1, \ldots, I_l$, where $I_i = (G - K) \cap H_i$. Moreover, $\{I_1, \ldots, I_l\}$ is an l-splitting of G, and, if W_i denotes the wing of G which includes I_i, then I_i is 1-limited and $V(I_i)$

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is a W_i-set by Lemma 4. But $V(I_i)\delta = V(I_i \cap K)\delta_K$, which by Lemma 2 is even or odd according as I_i includes 0 or 1 odd vertices of K respectively. Hence I_i includes 0 or 1 odd vertices of K according as W_i is even or odd. In case W_i is odd, let ξ_i be the odd vertex of K in I_i. Since the degree in I_i of a vertex ξ is $d(\xi)$ if $\xi \notin V(K)$ and $d(\xi) - d_K(\xi)$ if $\xi \in V(K)$, I_i has no odd vertex except possibly, if W_i is odd, ξ_i, whose degree in I_i is then odd or infinite. We have seen that I_i is 1-limited. Hence, by Lemmas 10 and 11, I_i is a two-way infinite path if W_i is even and is a $\xi_i\infty$-path if W_i is odd.

Since a vertex of K which is not in one of the I_i has the same degree in K as in G, the odd vertices of K are precisely the ξ_i associated with the odd W_i. Therefore, by Lemma 9, K is decomposable into some open paths $P_1, \ldots, P_m$ connecting all these ξ_i in pairs and some disjoint closed paths $C_1, \ldots, C_n$, where m is half the number of ξ_i, i.e. of odd wings of G. If P_r connects ξ_h to ξ_i, let $P_r' = I_h \cup P_r \cup I_i$, which is a two-way infinite path since I_h is a $\xi_h\infty$-path and I_i a $\xi_i\infty$-path. Hence $C_1, \ldots, C_n, P_1', \ldots, P_m'$, and the I_i associated with the even W_i constitute a decomposition of G into $p(G)$ two-way infinite paths and n disjoint closed paths. Each of the closed paths, not being a component of G, has a vertex in common with at least one of the two-way infinite paths. Hence, by absorbing each closed path into an infinite path with which it has a vertex in common, we obtain ultimately a decomposition of G into $p(G)$ two-way infinite paths.

By Lemmas 8 and 13, $p(G)$ is the minimum number of two-way infinite paths into which a limited relevant graph G can be decomposed. Moreover, if G is unlimited and relevant, $p(G)$ still has this property, since then (i) $p(G) = \aleph_0$ by definition, (ii) G is decomposable into two-way infinite paths by Theorem 2 of **(1)**, and (iii) the number of paths in such a decomposition must be $\aleph_0$ by Lemma 8.

4. The maximum number of paths. In this section, G denotes a relevant graph. We shall call G *k-constricted* if all vertices have finite degree and $V(G)$ can be partitioned into an infinite sequence of disjoint finite subsets $X_1, X_2, \ldots$ such that $|X_i \circ X_{i+1}| = k$ $(i = 1, 2, \ldots)$ and $X_r \circ X_s = \emptyset$ whenever $|r - s| \geqslant 2$. If G is k-constricted for some positive integer k, the least such k is its *width* $w(G)$; if not, $w(G) = \aleph_0$. It is not very hard to show that $w(G)$ is even or $\aleph_0$ and that (if $\frac{1}{2}\aleph_0$ means $\aleph_0$) G cannot be decomposed into more than $\frac{1}{2}w(G)$ two-way infinite paths. But it was shown in **(2)** that, *for every α such that $p(G) \leqslant \alpha \leqslant \frac{1}{2}w(G)$, there exists a decomposition of G into α two-way infinite paths.* A somewhat similar result concerning decompositions of directed graphs into two-way infinite directed paths was also obtained in **(2)**, as well as a number of other theorems concerning decompositions of undirected and directed graphs into infinite paths. As, however, the details of all this are very lengthy and perhaps rather tedious, I have here confined myself to answering Ore's question.

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