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The Relativistic Force-free Equation

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Abstract: This paper is a preliminary report on ongoing work, in collaboration with Drs S. P. Goodwyn, A. J. Mestel, and G. A. E. Wright. The non-dissipative force-free condition should be a good approximation to describe the electromagnetic field in much of the pulsar magnetosphere, but we may plausibly expect it to break down in singular domains. The detailed properties of the solutions will be affected critically by the choice of equatorial boundary condition beyond the light-cylinder.

Keywords: pulsar magnetosphere: force-free fields

1 Introduction

With canonical values inserted for the neutron star dipole moment and the particle density, and with realistic pair production efficiency near the star in the wind zone, the electromagnetic energy density in the pulsar magnetosphere will be much greater than the kinetic energy density except for very high values of the particle γ-values. This suggests that over much of the magnetosphere, the *relativistic force-free equation* will be a good approximation for determining the magnetic field structure. Equally, experience with the analogous non-relativistic equation suggests that there are likely to be local domains in which non-electromagnetic forces (including possibly inertial forces) are required for force balance. An associated question is whether the dissipation-free conditions can be imposed everywhere, or whether the dynamics of the problem will itself demand local breakdown in the simple plasma condition $\mathbf{E}\cdot\mathbf{B} = 0$ in singular domains, in addition to the acceleration/pair production domain.

Of the various discussions of the axisymmetric force-free pulsar magnetosphere in the literature, which go back to the early days of pulsar theory (e.g. Scharlemann & Wagoner 1973), we refer particularly to the recent paper by Contopoulos, Kazanas, & Fendt (1999: CKF), which has been the principal stimulus, and to those by Michel (1973, 1974, 1991), Mestel & Wang (1979: MW), Mestel et al. (1985: MRWW), Fitzpatrick & Mestel (1988a, 1988b: FM), Mestel & Pryce (1992: MP), and Mestel & Shibata (1994: MS). (Much of the work is summarised in Mestel (1999).) All the models have within the light-cylinder (l-c) a 'dead zone', with field lines that close within the l-c and without any current flow along the (purely poloidal) field, and a 'wind zone', with poloidal field lines that cross the l-c, and with poloidal currents maintaining a toroidal field component. The spin-down of the star occurs through the action of the magnetic torques associated with the flow of current across the light-cylinder. The axisymmetric, aligned pulsar model is an idealisation of a realistic oblique rotator model, inferred from the observed pulsing. It may be that strictly aligned models will always be 'dead', with zero magnetospheric currents, as argued by Smith, Michel, & Thacker (2001). Study of the live axisymmetric model is nevertheless justified, since one can expect that similar problems will arise in the more complicated geometry of the oblique rotator.

2 The Relativistic Force-free Equation

We confine attention in this paper to steady, axisymmetric states, and with the magnetic axis parallel rather than anti-parallel to the rotation axis $\mathbf{k}$. The notation is as in the cited papers (e.g. FM). The cylindrical polar coordinate system (ϖ, ϕ, z) is based on $\mathbf{k}$; $\mathbf{t}$ is the unit azimuthal, toroidal vector. The poloidal magnetic field $\mathbf{B}_\mathrm{p}$ is conveniently described by the flux function $P(\varpi, z)$:

$$\mathbf{B}_\mathrm{p} = -\nabla P \times \left(\frac{\mathbf{t}}{\varpi}\right) = \frac{1}{\varpi}\left(\frac{\partial P}{\partial z},\ 0,\ -\frac{\partial P}{\partial \varpi}\right). \quad (1)$$

The star is assumed to have a poloidal field of dipolar form, defined by

$$P = -\frac{B_\mathrm{s} R^3}{2}\frac{\varpi^2}{(\varpi^2 + z^2)^{3/2}}, \quad (2)$$

with R the stellar radius and B_s the polar field strength. By Ampère's law, $\mathbf{B}_\mathrm{p}$ is maintained by the toroidal current density

$$\mathbf{j}_\mathrm{t} = (c/4\pi)(\nabla \times \mathbf{B}_\mathrm{p}) = (c/4\pi)(\nabla \times \mathbf{B})_\phi \mathbf{t} \quad (3)$$

$$= (c/4\pi)[\nabla^2 P/\varpi - (2/\varpi^2)\partial P/\partial \varpi]\mathbf{t}. \quad (4)$$

In the wind zone of an active magnetosphere there will be also a toroidal component $\mathbf{B}_\mathrm{t}$, conveniently written

$$\mathbf{B}_\mathrm{t} = B_\phi \mathbf{t} = -[4\pi S/c](\mathbf{t}/\varpi), \quad (5)$$

where S clearly must vanish on the axis. By Ampère's law, the field given by (5) is maintained by the poloidal current density

$$\mathbf{j}_\mathrm{p} = (c/4\pi)\nabla \times \mathbf{B}_\mathrm{t} = -\nabla S \times \mathbf{t}/\varpi. \quad (6)$$

Thus $S(\varpi, z)$ is a Stokes stream function, constant on the poloidal current lines, with $-2\pi S$ measuring the total

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outflow of charge between the axis and the current line S. In a steady state the total outflow of charge must be zero, so there must be a current closing streamline on which S vanishes.

We limit discussion here to the case in which the simple plasma condition $\mathbf{E}\cdot\mathbf{B}=0$ is supposed to hold from the rigidly rotating, perfectly conducting star out to the light-cylinder (the 'inner domain') and beyond, into the beginning of the 'outer domain' between the light-cylinder and infinity. Then all field lines corotate with the star, and

$$\begin{aligned}\mathbf{E} &= -(\alpha\varpi/c)\mathbf{t}\times\mathbf{B}\\ &= (\alpha\varpi/c)(-B_z, 0, B_x) = (\alpha/c)\nabla P, \end{aligned}\qquad (7)$$

where α is the star's rotation. This assumes that within the dead zone, there is no vacuum gap separating the domains of negatively and positively charged particles (Holloway & Pryce 1981: MRRW; FM; MP). In fact, in the open field line domain, there has to be near the star a locally non-trivial component of $\mathbf{E}$ along $\mathbf{B}_\mathrm{p}$, able to accelerate the primary electrons to γ-values high enough for pair production to occur. In a hypothetical steady state, the simple plasma condition $\mathbf{E}\cdot\mathbf{B}=0$ will again be set up in the electron–positron plasma, but now with $\mathbf{E}=-\tilde{\alpha}(P)(\varpi/c)\mathbf{t}\times\mathbf{B}$ — i.e. with the field lines in the wind zone having individual rotation rates $\tilde{\alpha}$ that differ somewhat from the rotation α of the star (cf. MS, Section 4). However, at least for the more rapid rotators this effect will be small, and so is left for inclusion in a generalisation of the present treatment.

By the Poisson–Maxwell equation, the Goldreich–Julian (Goldreich & Julian 1969: GJ) charge density maintaining the electric field (7) is

$$\begin{aligned}\rho_\mathrm{e} = \frac{\nabla\cdot\mathbf{E}}{4\pi} &= -\frac{\alpha}{2\pi c}\mathbf{k}\cdot\left[\mathbf{B}-\frac{1}{2}\mathbf{r}\times(\nabla\times\mathbf{B})\right]\\ &= -\frac{\alpha}{2\pi c}\left[B_z-\frac{1}{2}\varpi(\nabla\times\mathbf{B})_\phi\right].\end{aligned}\qquad (8)$$

With the rotation and magnetic axes aligned, the primary outflowing particles are the negatively charged electrons, yielding a negative current $\mathbf{j}_\mathrm{p}$ emanating from the polar cap. The stream function S defined by (6) begins by increasing from zero on the axis, so that B_ϕ is negative — the field lines are twisted backwards with respect to the axis $\mathbf{k}$. As the electric force density

$$\rho_\mathrm{e}\mathbf{E} = -\rho_\mathrm{e}(\alpha\varpi/c)\mathbf{t}\times\mathbf{B}_\mathrm{p} \qquad (9)$$

is purely poloidal, in a force-free magnetosphere the toroidal component of the magnetic force density $\mathbf{j}_\mathrm{p}\times\mathbf{B}_\mathrm{p}/c$ vanishes (the 'torque-free' condition), so $\mathbf{j}_\mathrm{p}$ must be parallel to $\mathbf{B}_\mathrm{p}$, yielding from (1) and (6) the functional relation

$$S = S(P), \quad \mathbf{j}_\mathrm{p} = \frac{\mathrm{d}S}{\mathrm{d}P}\mathbf{B}_\mathrm{p}: \qquad (10)$$

the poloidal current streamlines are identical with the poloidal field lines. The respective contributions of $\mathbf{B}_\mathrm{p}$ and $\mathbf{B}_\mathrm{t}$ to the poloidal force density are

$$\mathbf{j}_\mathrm{t}\times\mathbf{B}_\mathrm{p}/c = (\nabla\times\mathbf{B})_\phi(\mathbf{t}\times\mathbf{B}_\mathrm{p})/4\pi \qquad (11)$$

and

$$\mathbf{j}_\mathrm{p}\times\mathbf{B}_\mathrm{t}/c = \frac{4\pi S}{c^2\varpi}\frac{\mathrm{d}S}{\mathrm{d}P}(\mathbf{t}\times\mathbf{B}_\mathrm{p}) = -\frac{4\pi}{c^2\varpi^2}S\frac{\mathrm{d}S}{\mathrm{d}P}\nabla P, \qquad (12)$$

on use of (10) and (1). By (1), (8), (9), (11), and (12), the poloidal component of the force-free equation

$$\rho_\mathrm{e}\mathbf{E}+\mathbf{j}_\mathrm{t}\times\mathbf{B}_\mathrm{p}/c+\mathbf{j}_\mathrm{p}\times\mathbf{B}_\mathrm{t}/c = 0 \qquad (13)$$

reduces to

$$\frac{1}{\varpi}(\nabla\times\mathbf{B})_\phi\left[1-\left(\frac{\alpha\varpi}{c}\right)^2\right] + \left(\frac{\alpha\varpi}{c}\right)^2\frac{2B_z}{\varpi^2}+\left(\frac{4\pi}{c\varpi}\right)^2 S\frac{\mathrm{d}S}{\mathrm{d}P} = 0. \qquad (14)$$

Note that the first term in (14) combines part of the electric force with the force due to $\mathbf{B}_\mathrm{p}$; the second again comes from the electric force, while the third is due to $\mathbf{B}_\mathrm{t}$.

An essential part of the whole problem is the search for appropriate forms for the function $S(P)$, which should emerge from the construction of mutually consistent fields for the inner and outer domains, respectively, within and without the light-cylinder, and behaving properly near the star and at infinity. With the sign convention in (1), P decreases from zero on the axis, so that initially $\mathrm{d}S/\mathrm{d}P$ is negative, and the force density (12) due to the toroidal field $\mathbf{B}_\mathrm{t}$ acts toward the axis. However, since S vanishes on the current closing streamline, there must be an intermediate field line on which $\mathrm{d}S/\mathrm{d}P$ changes sign; below this field line the force (12) acts towards the equator. Closure via a sheet current is not excluded (cf. Section 4).

As in the earlier work, we define dimensionless coordinates $(x,\bar{z})=(\alpha/c)(\varpi,z)$, and normalise $P, S, \mathbf{E}, \mathbf{B}$ in terms of a standard light-cylinder field strength $B_\mathrm{lc}=(B_\mathrm{s}/2)(\alpha R/c)^3$:

$$\begin{aligned}P = \bar{P}B_\mathrm{lc}(c/\alpha)^2, \quad S = \bar{S}B_\mathrm{lc}(c^2/4\pi\alpha),\\ (\mathbf{B},\mathbf{E}) = B_\mathrm{lc}(\bar{\mathbf{B}},\bar{\mathbf{E}}).\end{aligned}\qquad (15)$$

(Once defined, the dimensionless quantities are again immediately written without the bars.) The normalised fields have the form

$$\mathbf{B}_\mathrm{p} = -\nabla P\times\mathbf{t}/x, \quad B_\phi = -S/x, \quad \mathbf{E} = \nabla P. \qquad (16)$$

As $(\varpi,z)\to 0$, P must reduce to $-x^2/(x^2+z^2)^{3/2}$, the normalised form of (2). The normalised form of the force-free equation (14) is

$$(x^2-1)\frac{\partial^2 P}{\partial x^2}+\frac{(1+x^2)}{x}\frac{\partial P}{\partial x}+(x^2-1)\frac{\partial^2 P}{\partial z^2} = S\frac{\mathrm{d}S}{\mathrm{d}P}. \qquad (17)$$

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Note that the light-cylinder $x = 1$ is a singularity of this differential equation.

For the non-active case, with $S = 0$, the solution of (17) — supposed to hold everywhere within the l-c — has been discussed in Michel (1973, 1991), in MW, and in MP, and will be referred to as the MMWP field. The field lines which reach the equator cross normally, forming a closed domain: the inner domain equatorial boundary condition is

$$B_x(x, 0) = 0, \tag{18}$$

with no equatorial current sheet. As the field must be non-singular at the l-c, (17) with $S = 0$ requires that $B_z = -\partial P/\partial x = 0$ when $x = 1$: the field lines cross the l-c normally, and there is a neutral point at the intersection $(1, 0)$ of the l-c and the equator.

The appropriate changes for the realistic case — with $S \neq 0$ — are discussed in Section 5.

3 The Outer Domain

Let us suppose provisionally that the simple 'perfect conductivity' condition (7) continues to hold everywhere beyond the l-c, so that there is no trans-field motion of the plasma. Then the field beyond the l-c must be topologically 'open', with no field lines crossing the equator. The CKF model has the field lines crossing the equator normally within the l-c, as in the MMWP and FM fields, but beyond the l-c, the boundary condition $B_z = 0$ is imposed at the equator. In the northern hemisphere $B_x > 0$, so by (7), near the equator the electric field xB_x is in the positive z-direction. As the basic field is dipolar, B_x must change sign at the equator, implying a locally positive current density j_ϕ and a magnetic force density that acts towards the equator. Thus the poloidal magnetic field pinches, as in familiar non-relativistic problems, but the combined electric force (9) and the $\mathbf{B}_\mathrm{p}$ force (11) is $[(x^2 - 1)(\nabla \times \mathbf{B})_\phi/4\pi]\mathbf{k}$ and so acts away from the equator, since $x > 1$. The proposed force-free equilibrium near the equator must therefore be maintained by the pinching effect of $\mathbf{B}_\mathrm{t}$. We have seen above that $S\,\mathrm{d}S/\mathrm{d}P$ does indeed become positive at low latitudes, so that (12) has the required direction.

Consider first the idealised case, with B_x, B_ϕ and E_z abruptly reversing sign at the equator, and so with positive sheet currents J_ϕ, J_x and positive surface charges σ. Over the surface $z = +\varepsilon$, there is a net electromagnetic stress $T_{ij}k_j$ where

$$T_{ij} = \frac{1}{4\pi}\left[-\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\delta_{ij} + (E_iE_j + B_iB_j)\right] \tag{19}$$

is the Maxwell stress tensor. The electric terms yield $(x^2B_x^2/8\pi)\mathbf{k}$ and the magnetic terms $-[(B_x^2 + B_\phi^2)/8\pi]\mathbf{k}$, so the net electromagnetic pressure, acting towards the equator, is

$$\left[B_\phi^2 - (x^2 - 1)B_x^2\right]/8\pi. \tag{20}$$

There is an equal electromagnetic pressure, also acting towards the equator on the surface $z = -\varepsilon$. Provided

$$|B_\phi/B_x| > \sqrt{x^2 - 1}, \tag{21}$$

then an equal particle pressure at the equator is both necessary and sufficient to maintain equilibrium.

The force-free equation (14) can be written succinctly as $\partial T_{ij}/\partial x_j = 0$. Near the equator, $|B_z| \ll |B_x|$ in this model, and the force-free condition becomes

$$\frac{\partial}{\partial x_3}T_{33} = \frac{1}{8\pi}\frac{\partial}{\partial z}\left[-B_\phi^2 + (x^2 - 1)B_x^2\right] = 0, \tag{22}$$

whence

$$\frac{1}{8\pi}\left[B_\phi^2 - (x^2 - 1)B_x^2\right] = p_\mathrm{eq} \tag{23}$$

where p_eq is independent of z. If (21) holds, then p_eq is identified as a particle pressure at the equator, where B_x and B_ϕ change sign.

More realistically, the transition to zero field at the equator will be continuous, with the particle pressure p increasing steadily across half of the thin equatorial sheet. The force-free condition (22) is replaced by

$$\frac{\partial}{\partial z}\left\{\frac{1}{8\pi}\left[-B_\phi^2 + (x^2 - 1)B_x^2\right] - p\right\} = 0, \tag{24}$$

and (23) by

$$\frac{1}{8\pi}\left[B_\phi^2 - (x^2 - 1)B_x^2\right] + p = p_\mathrm{eq}. \tag{25}$$

As the equator is approached, p steadily increases from the uniform value (implicit in the force-free assumption), which could in principle be zero, to the value p_eq. *Thus the model necessarily includes a thin domain with a non-force-free electromagnetic field.*

The limiting case, with the inequality sign in (21) replaced by equality, appears consistent with zero equatorial pressure p_eq. However, condition (14) assumes that all non-electromagnetic forces, including inertial forces, are small compared with the dominant terms in the Lorentz force. With the electromagnetic field subject to the condition (7), the kinematics of the flow has the form familiar from stellar wind theory, with $\mathbf{v}_\mathrm{p}$ parallel to $\mathbf{B}_\mathrm{p}$, and the rotation velocity given by

$$v_\phi = c\left(x + \frac{v_\mathrm{p}}{c}\frac{B_\phi}{B_\mathrm{p}}\right). \tag{26}$$

The energy integral in a pressure-free system is (MRWW; MS; Contopoulos 1995)

$$\gamma[1 - x(v_\phi/c)] = \text{constant}, \tag{27}$$

showing that γ would become infinite if $v_\phi = c/x$, $v_\mathrm{p} = c\sqrt{(x^2 - 1)}/x$; and from (26) this is equivalent to

$$|B_\phi/B_\mathrm{p}| = \sqrt{x^2 - 1}. \tag{28}$$

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Thus the seemingly exceptional case, with zero equatorial pressure and so with the force-free equation holding all the way to the equator, in fact requires that the flow outside the equatorial zone be so highly relativistic that the neglected inertial terms are not small, so violating an essential condition for the force-free approximation to hold. We conclude that in all cases, the equilibrium conditions for this model will require at least a local breakdown in force-free conditions. In analogous non-relativistic problems, Lynden-Bell (1996) has pointed out that a thermal pressure is again required to balance magnetic pinching forces, exerted locally by an otherwise force-free field. However, we again emphasise the important difference, that whereas in a non-relativistic problem the electric stresses are normally smaller than the magnetic by the factor $(v/c)^2$, in the present problem the opposing electric stresses exceed the pinching poloidal field stresses by the factor x^2, and equilibrium is possible only through the pressure exerted by the toroidal field.

In the domain near the equator, with $|B_x/B_z| \gg 1$, the relativistic flow likewise has $|v_x/v_z| \gg 1$, so that $v_x^2 + v_\phi^2 \approx c^2$, whence from (26) (remembering that B_ϕ is negative),

$$v_\phi/c = \frac{x - b\sqrt{(1+b^2-x^2)}}{(1+b^2)},$$
$$v_x/c = \frac{bx + \sqrt{(1+b^2-x^2)}}{(1+b^2)}, \tag{29}$$

where $b \equiv |B_\phi/B_x|$. (The algebraically allowed choice of the opposite signs before the two radicals would yield $v_\phi/c = 1$ at $x = 1$, implying infinite γ, and so is rejected.) The outflow of angular momentum from the star in both hemispheres across a closed surface Σ with local outward unit normal $\mathbf{n}$ is (e.g. Mestel 1999)

$$-\int (\varpi B_\phi/4\pi)\mathbf{B}_\mathrm{p}\cdot\mathbf{n}\,\mathrm{d}\Sigma = -(c/\alpha)^3 B_\mathrm{lc}^2 \int_0^{P_c} S(P)\,\mathrm{d}P, \tag{30}$$

on use of (1), (5), and (15).

4 The $S(P)$ Relation

As noted in Section 2, if there do exist magnetospheric models that are everywhere dissipationless, with the field force-free outside singular regions such as the equatorial sheet, then allowed relations $S(P)$ should emerge as part of the solution. An early model of the whole magnetosphere by Michel (1974, 1991) has no dead zone, but a poloidal field that is radial all the way from the star to infinity. In our notation, the Michel field is

$$P = P_c\left(1 - \frac{z}{(x^2+z^2)^{1/2}}\right), \quad B_x = -P_c\frac{x}{(x^2+z^2)^{3/2}},$$
$$B_z = -P_c\frac{z}{(x^2+z^2)^{3/2}}, \tag{31}$$

with $P_c < 0$ in the northern hemisphere, and with the critical field line $P = P_c$ coinciding with the equator. Michel's $S(P)$ relation is

$$S = -2P + \frac{P^2}{P_c} = -P_c\frac{x^2}{(x^2+z^2)}, \quad B_\phi = -\frac{S}{x}, \tag{32}$$

so that

$$\frac{\mathrm{d}S}{\mathrm{d}P} = -2\left(1 - \frac{P}{P_c}\right) = -\frac{2z}{(x^2+z^2)^{1/2}},$$
$$S\frac{\mathrm{d}S}{\mathrm{d}P} = 2P_c\frac{x^2 z}{(x^2+z^2)^{3/2}}. \tag{33}$$

One can easily verify that (17) is satisfied. Note that the poloidal field (31) is radial and (away from the equator) independent of the spherical polar angle θ, and so has no curl: the equilibrium condition is a balance between the electric force given by (8) and (9) and the force due to the toroidal field (32). However, since P_c changes sign on the equator, there are again both toroidal and poloidal equatorial sheet currents that respectively maintain the jumps in B_x and B_ϕ; also $B_\phi^2 - (x^2-1)B_x^2 = P_c^2/x^4$ when $z = 0$, so that by the above discussion, $p_\mathrm{eq} > 0$. Note also that $\mathrm{d}S/\mathrm{d}P = 0$ on the critical line $P = P_c$.

Near the star, the poloidal field is more plausibly taken to have a dipolar rather than a radial structure, together with an associated dead zone within the l-c, similar to that found in the MMWP, FM, and MS fields, and analogous to that in the non-relativistic wind problem (e.g. Mestel & Spruit 1987). The dead zone terminates at the point $(x_c, 0)$; the value of x_c (≤ 1) will be seen to be an extra parameter, fixing the global field structure. Within the dead zone the field lines close, crossing the equator normally so that the appropriate equatorial boundary condition is $\partial P \partial z \propto B_x(x,0) = 0$ for $x < x_c$. The dead zone is bounded by the separatrix field line P_c. In the wind zone outside of P_c, the wind flow is along the poloidal field, and so the equatorial boundary condition is $\partial P/\partial x \propto B_z = 0$. Note that this condition holds not just beyond the l-c but from $x_c < x < \infty$. Again there is a charge-current sheet maintaining the discontinuous sign changes in E_z, B_x, and B_ϕ. Thus the critical field line $P(x,0) = P_c$, extending along the equator from x_c outwards, is the continuation of the separatrix between the wind and dead zones.

Just outside the equatorial charge-current sheet and the separatrix, (17) holds all the way in from ∞. At the intersection $(1, 0)$ with the l-c and just outside the sheet, with $P_x \propto B_z = 0$ and the second derivatives P_{xx} and P_{zz} non-singular, (17) requires that the constant value of $S\,\mathrm{d}S/\mathrm{d}P$ along P_c must be zero. But $S(P_c) \neq 0$, because, as seen above, beyond the l-c the pinching force exerted by B_ϕ is necessary for equilibrium; we therefore require, as in the Michel field,

$$\left(\frac{\mathrm{d}S}{\mathrm{d}P}\right) = 0 \tag{34}$$

on the critical field line P_c. This condition is propagated inwards along P_c onto its continuation, the separatrix P_c

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between the wind and dead zones: the equatorial equilibrium conditions beyond the l-c impose a constraint on the global $S(P)$ relation. From (10), condition (34) requires that the poloidal volume current density $\mathbf{j}_\mathrm{p}$ falls to zero on P_c, but the finite value for $S(P_c)$ requires a poloidal current sheet at the equator beyond x_c and along the separatrix within it.

We note that CKF report that they were unable to construct a model without some of the return current in the form of a sheet current along the separatrix; in our notation, $S(P_c) \neq 0$. This appears to be due to the propagation along P_c into the inner domain of the equatorial equilibrium requirement found in the outer domain. From their Figure 3, it appears that for $x \gg 1$, the poloidal part of the CKF field does approximate quite closely to the Michel radial form, in both cases forced presumably by the equatorial boundary condition $B_z = 0$.

5 The Inner Domain

We have emphasised that beyond the neutral point $(x_c, 0)$, there is a finite thermal pressure on the equator, necessary for equilibrium. Likewise, it is not obvious that within the l-c, the boundary condition on the field line separatrix between the wind zone (labelled 2) and the dead zone (labelled 1) can be satisfied without a thermal pressure within the dead zone (Figure 1).

It is in fact easy to generalise (14) and (17) to include a pressure gradient: with gravity and inertia still negligible,

$$-\frac{\nabla P}{4\pi}\left\{\frac{1}{\varpi}(\nabla\times\mathbf{B})_\phi\left[1-\left(\frac{\alpha\varpi}{c}\right)^2\right]+\left(\frac{\alpha\varpi}{c}\right)^2\frac{2B_z}{\varpi^2}+\left(\frac{4\pi}{c\varpi}\right)^2 S\,\frac{\mathrm{d}S}{\mathrm{d}P}\right\}-\nabla p=0. \tag{35}$$

Thus $p = p(P)$ — the constant pressure surfaces must coincide with the poloidal field lines. In normalised form,

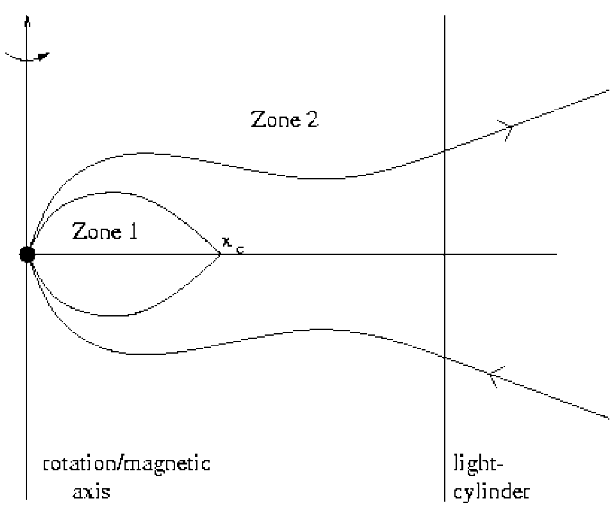

Figure 1 Schematic magnetosphere.

(35) becomes

$$(1-x^2)\frac{\partial^2 P}{\partial x^2}-\frac{(1+x^2)}{x}\frac{\partial P}{\partial x}+(1-x^2)\frac{\partial^2 P}{\partial z^2} = -S\,\frac{\mathrm{d}S}{\mathrm{d}P}-\frac{1}{2}x^2\,\frac{\mathrm{d}p}{\mathrm{d}P}. \tag{36}$$

Prima facie, there is no obvious objection to the adoption of the simplest case, with p the same constant on all the field lines within the dead zone and zero in the wind zone, i.e. with a discontinuity in both p and $S(P)$ on the separatrix. The equation for P within the wind zone then remains (17) (with signs reversed for convenience)

$$(1-x^2)\frac{\partial^2 P}{\partial x^2}-\frac{(1+x^2)}{x}\frac{\partial P}{\partial x}+(1-x^2)\frac{\partial^2 P}{\partial z^2} = -S\,\frac{\mathrm{d}S}{\mathrm{d}P}, \tag{37}$$

and in the dead zone we have the same equation with $S = 0$. However, along the separatrix P_c, extending inwards from the equatorial point $(x_c, 0)$, the equilibrium conditions require a discontinuity in B_p as well as those in p and $S(P)$. Writing

$$n_i = -[(\mathbf{t}\times(\mathbf{B}_\mathrm{p}/B_\mathrm{p})]_i \tag{38}$$

the unit normal to the separatrix, we require continuity of

$$\left[-(8\pi p+E^2+B_\mathrm{p}^2+B_\phi^2)\delta_{ij}+2E_iE_j+2B_iB_j\right]n_j \tag{39}$$

with $\mathbf{E}$ given by (7). The components of (39) parallel to $\mathbf{t}$ and to the separatrix are automatically zero. The component normal to P_c reduces to continuity of

$$8\pi p+B_\mathrm{p}^2[1-(\alpha\varpi/c)^2]+B_\phi^2, \tag{40}$$

i.e. to

$$8\pi p+B_{\mathrm{p}1}^2(1-x^2)=B_{\mathrm{p}2}^2(1-x^2)+B_{\phi 2}^2. \tag{41}$$

It is convenient to normalise p in units of $B_{\mathrm{lc}}^2/8\pi$.

For $x > x_c$, the wind zone 2 extends from the equator $z = 0$ to $z = \infty$; for $x < x_c$, zone 2 extends from $z_c(x)$ defined by the separatrix

$$P(x, z_c) = P_c \tag{42}$$

to $z = \infty$. The separatrix function (42) is not known *a priori* but must emerge as part of the solution by iteration. We assume provisionally that at the point $(x_c, 0)$ (referred to as N), the poloidal field $B_\mathrm{p} = 0$ both just outside and just inside the separatrix. From (41), the pressure of the toroidal field is balanced by the thermal pressure p, so that

$$p = S^2(P_c)/x_c^2. \tag{43}$$

For $x < x_c$, with use of (15) and (16), (41) then becomes

$$(\nabla P_1)^2 = (\nabla P_2)^2 + S^2(P_c)\frac{(1-x^2/x_c^2)}{(1-x^2)}. \tag{44}$$

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The discontinuity in $|\nabla P|$ grows from zero as x moves in from x_c but will become a small fraction of $|\nabla P|$ for small x. When $x_c < 1$, the function P has a simple analytical behaviour near the neutral point $(x_c,\ 0)$. The separatrix leaves $(x_c,\ 0)$ making an angle $\theta = 2\pi/3$ with the outward-pointing equator. To leading order, in both the dead and wind zones, $P_{xx} + P_{zz} = 0$, which has the local solution

$$\begin{aligned} P/P_c &= 1 + A_{2,1} R^{3/2} \sin(3\theta/2), \\ R^2 &= z^2 + (x - x_c)^2, \end{aligned} \tag{45}$$

where the coefficients $A_{2,1}$ apply respectively to the wind zone $0 < \theta < 2\pi/3$ and the dead zone $2\pi/3 < \theta < \pi$. It is seen that along $\theta = 0$, $P = P_c$, and on $\theta = \pi$, $P_\theta \propto P_z = 0$, as required. Across the separatrix $z = \sqrt{3}(x_c - x)$, P is continuous, while the jump condition (44) then yields

$$A_1^2 = A_2^2 + \frac{4S^2(P_c)}{9P_c^2 x_c(1 - x_c^2)}. \tag{46}$$

At this point, it is instructive to make a comparison with the analogous non-relativistic problem, in which the electric stresses are small by factors $O(v/c)^2$ and so are negligible, and also inertial forces are still neglected. Suppose that there is again a separatrix passing through the poloidal field neutral point at $(x_c,\ 0)$. The balance equation across the separatrix is now continuity of $8\pi p + B_{\rm p}^2 + B_\phi^2$ — the $(1 - x^2)$ factors in (41) are replaced by unity. If again p is negligible in the wind zone, then the constant value of p along the separatrix within the dead zone is again fixed by the condition at the neutral point — $p = S^2(P_c)/x_c^2$ in normalised form. The balance condition is then $S^2(P_c)(1 - x^2/x_c^2) + (\nabla P_2)^2 = (\nabla P_1)^2$. The presence of the factor $(1 - x^2/x_c^2)$ in the S^2 term enables $\mathbf{B}_{\rm p}$ to be continuous (and zero) at the neutral point $(x_c,\ 0)$, but with $S^2(P_c)$ still non-zero. (Clearly, in the non-relativistic problem, the numerical value of x_c is of no significance.)

By contrast, in the relativistic problem, near the l-c the electric field strength approaches the poloidal magnetic field strength, so that the factor $(1 - x^2)$ now appears multiplying both the $(\nabla P)^2$ terms in (44). If now one were to take $x_c = 1$, then the non-vanishing factor $(1 - x^2)$ would cancel, and (44) would reduce to

$$S^2(P_c) + (\nabla P_2)^2 = (\nabla P_1)^2. \tag{47}$$

along the separatrix. At the point $(x_c,\ 0)$, the simultaneous vanishing of ∇P_1 and ∇P_2 would then require that $S(P_c) = 0$ (and so also by (43) $p = 0$). But if beyond the l-c the equatorial toroidal component $B_\phi = -S(P_c)/x$ were zero, then the balance condition could not be satisfied (cf. (23)).

In fact, as noted in Section 7, the vanishing of S on the separatrix is appropriate to models with a radically different external equatorial boundary condition. In the present problem, the case $x_c = 1$, which yields (47), may perhaps be incorporated provided that for this limiting case, we allow $\nabla P_1 \neq 0$, i.e. the discontinuity in $B_{\rm p}$ across the separatrix persists at the equator. If $x_c = 1 - \epsilon$ with $\epsilon \ll 1$, then from (44), the equatorial B_z can be zero on both sides of the separatrix, without requiring that $S(P_c) = 0$. However, at a neighbouring separatrix point $x = x_c - X$ with $X \ll 1$, the second term in (44) will have climbed from zero at N to $S^2(P_c)[X/(X + \epsilon)] \simeq S^2(P_c)$ once $X \gg \epsilon$. Should the solutions with $S(P_c) \neq 0$ continue to exist as $x_c \to 1$, then the separatrix balance condition will imply a steeper and steeper local gradient in B_z, corresponding indeed to a discontinuity at $x_c = 1$.

An essential step in all the argument is the condition $p = p(P)$, following from (35). If instead the thermal pressure p were (illicitly) allowed to vary so as to balance the pressure $B_{\phi 2}^2 = S^2(P_c)/x^2$ exerted by the external toroidal field, then from (41) there would be no discontinuity in $B_{\rm p}$ at the equator or indeed anywhere along the separatrix.

It is appropriate also to re-emphasise just how crucial to the discussion is the outer domain equatorial boundary condition

$$B_z(x, 0) = 0 \tag{48}$$

for $x > 1$, for it is the consequent local equilibrium requirement $S(P_c) > 0$ that implies a poloidal sheet current within the l-c along the separatrix, leading to the conditions (41) and (44), with a non-vanishing $B_{\phi 2}$. It is then clear that a non-zero $p(P_c)$ is required for the balance condition to hold at the neutral point.

6 Model Construction

The two basic parameters of a pulsar are clearly the angular velocity and the dipole moment of the neutron star. The spin-down rate of an active pulsar will depend on the strength of the circulating poloidal current. In the Michel $S(P)$ relation (32), the form with $S \simeq -2P$ valid near the poles derives from taking the current density flowing along the axis as equivalent to the Goldreich–Julian electron charge density moving with the speed c, a reasonable upper limit, and indeed one that must be closely approached if the electrons are to generate an electron–positron pair plasma near the star (cf. e.g. MS, Section 3). We likewise consider $S(P)$ relations that all behave like the Michel form for small P.

The only other parameter introduced into the theory is the pressure in the dead zone, which we have taken as uniform. *Prima facie*, there should be a one-parameter family of possible functions $S(P;\ P_c)$, each fixing the global field function P and simultaneously the shape of the dead zone and its limit $x_c(P_c)$. The value of the required dead zone pressure $p(P_c)$ is given by (43). Intuitively, one expects a decrease in p to correspond to an increase in x_c and a decrease in $|P_c|$. It can also be convenient to think of x_c rather than P_c as the defining parameter.

Two approaches are being used in the ongoing project. The first follows the trail blazed by CKF: for each x_c we search for a function $S(P)$ which yields solutions for P in both inner and outer domains, satisfying the respective boundary conditions, and in particular with P and $\partial P/\partial x$ continuous on $x = 1$.

As noted in Section 4, the Michel relation $S = -2P + P^2/P_c$ yields the poloidal field (31) that is radial all the way from infinity to the origin. The asymptotic behaviour of the Michel field is acceptable; however, we need modified $S(P)$ relations compatible with the existence of the dead zone, for which there must be an equatorial neutral point $(x_c,\ 0)$ of $B_{\rm p}$.

The force-free equation

$$(x^2-1)P_{xx} + \frac{(1+x^2)}{x}P_x + (x^2-1)P_{zz} = S\,\frac{\mathrm{d}S}{\mathrm{d}P} \tag{49}$$

is valid in the whole of the wind zone beyond the l-c subject to the equatorial boundary condition $P_x = 0$ for $x \geq 1$. On $x = 1$, as noted in CKF, MRWW, and elsewhere, the requirement of non-singular P_{xx} and P_{zz} yields the constraint

$$2P_x = S\,\frac{\mathrm{d}S}{\mathrm{d}P}\,. \tag{50}$$

Thus with adoption of a trial $S(P)$ relation, the solution of (49) for $x > 1$, subject to finite behaviour at infinity and on the l-c $x = 1$, and with P a constant on the equator, will yield P and $\partial P/\partial x$ satisfying (50) on $x = 1$.

It is instructive to begin by studying how modest variations from the Michel $S(P)$ relation (32), when fed into (49), can change the shape of the poloidal field lines, in particular sometimes making the radial Michel field lines curve within the l-c so as to yield a poloidal field neutral point x_c. For computational purposes, the functions $P(x, z)$, $S(P)$ are renormalised in terms of the constant value P_c along the segment of the equator $z = 0$, extending now from a point $x_c < 1$, through the l-c $x = 1$ to ∞:

$$\bar{P} = P/P_c, \quad \bar{S} = S/P_c. \tag{51}$$

With our sign convention, $\bar{P}$ is positive and $\bar{S}$ negative; in the rest of this Section, the bar will be dropped, all quantities being assumed normalised. On the equator for $x \geq x_c$, $P = 1$. The normalised dipole at the origin now has the form $x^2/|P_c|(x^2+z^2)^{3/2}$.

In the initial attack, the Michel $S(P)$ relation $S = -2P + P^2$ is generalised to

$$S = -2P + P^2 + f(P) \tag{52}$$

where $f(P)$ is a simple polynomial. An 'outer solution' of (49), for the domain between $x = 1$ and ∞, subject to the equatorial boundary condition and with proper behaviour at infinity, is constructed by a relaxation method. This solution will in particular yield values for P and P_x on $x = 1$, related by (50). Just within $x = 1$, P and P_x must both be continuous, and the same $S(P)$ relation must hold.

In the simple problem with no poloidal currents (Michel 1973, 1991; MW; MP; Mestel 1999; Section 2 above), equation (49) with $S = 0$, subject by (50) to the condition $P_x = 0$ on $x = 1$, is best solved within $x = 1$ by a Fourier transform in z (cf. the discussion in Section 7). The requirement that the second derivatives be finite at the singularity $x = 1$ ensures that the initial value of P_x is not prescribable but must be zero, and so likewise its Fourier transform. Inward integration of the equation for each Fourier transform is performed. Link-up with the solutions valid near the origin yields transform coefficients which decay exponentially at large Fourier wave-number k, so that the construction of P from the back-transform converges rapidly.

In the present problem, with $S(P) \neq 0$, the Fourier technique may again be used to explore the initial smooth extension inwards of the constructed outer solution. The Fourier coefficients on $x = 1$, now fixed from the outer solution, again decay exponentially with large k. For suitably chosen trial functions $S(P)$, the technique does indeed yield a continued poloidal field with a neutral point N at $(x_c,\ 0)$.

We are interested in the family of global models, each with its defining parameter x_c, comprising the 'inner solution' for the domain from the star out to the l-c, and linking up smoothly with the outer domain solution, and with the same $S(P)$ relation in the wind zones in both domains. With x_c chosen, the associated trial function $S(P)$ is adopted as input into the first iteration in the search for the global solution.

Recall the constraints on the inner solution. The separatrix — defined by the renormalised $P = 1$ — demarcates the dead zone $x < x_c,\ 0 < z < z_c(x)$, from the two domains of the wind zone: $x < x_c,\ z_c(x) < z < \infty$, and $x_c < x < 1,\ \ 0 < z < \infty$. The same $S(P)$ relation must hold in the whole of the wind zone; in the dead zone, $S = 0$. The condition (44) holds on the separatrix, and near the origin P must reduce to the vacuum dipole form $x^2/|P_c|(x^2+z^2)^{3/2}$, with $|P_c|$ fixed by $P(x_c, 0) = 1$.

The suggested procedure is a generalisation of that used in CKF. With x_c chosen, then as seen, a first choice for $S(P)$ is available. A plausible first choice for the separatrix field line is adopted, and again a relaxation method is used to construct an inner solution, well-behaved on the l-c. The procedure is repeated, using the separatrix from the previously constructed field, until convergence is achieved. The values of $P(1, z)$ and $P_x(1, z)$ are related by (50), but will not agree with the values found from the outer solution. As in CKF, the inner and outer l-c values are used to derive a modified $S(P)$ relation. This $S(P)$ relation and the separatrix from the constructed model are input for the next iteration; the process is continued until effective convergence is achieved. The preliminary results are encouraging.

A second approach to the problem, being developed at Imperial College, attacks the fully non-linear problem globally using finite difference techniques. For a fixed value of P_c and initial estimates of P and $S(P)$, quasi-timestepping schemes are employed separately in the wind zones inside and outside the l-c and in the dead zone. The imposed boundary conditions in the three regions are a regularity condition at $x = 1$, continuity of P at $P = 1$,

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while as $x^2 + z^2 \to \infty$, the field becomes radial, so that $xP_x + zP_z \to 0$. With each time-step, new values of $S(P)$ are deduced to ensure continuity of ∇P across $x = 1$, while the dead zone boundary is advanced according to the jump condition on ∇P at $P = 1$. Once equilibrium is established, a self-consistent solution to the nonlinear elliptic problem has been found.

Provisional results for the cases $P_c = -3.0$ and -2.0 give $x_c = 0.2283$ and 0.4619, and normalised $S(P_c) = -0.9795$ and -0.9387 respectively. From (43), the corresponding values for p are 18.41 and 4.13, so that as expected lower pressure in the dead zone requires the neutral point to be further out. Outside the dead zone, the field quickly becomes quasi-radial. With x_c well below 1, it is not surprising that in both cases the $S(P)$ relation deviates only slightly from the Michel form.

It will be of particular interest to see what happens to the search for a global solution as x_c approaches 1. Will there be a maximum value for $x_c < 1$, or will there be a solution with non-vanishing spin-down at $x_c = 1$, with the gradient of $|B_z|$ becoming arbitrarily large, as suggested in Section 5?

A more detailed account of the present work, hopefully with much more extensive numerical results, will be submitted to the Royal Astronomical Society of London.

7 The Equatorial Boundary Condition in the Outer Domain

In this last Section we outline how a change in the equatorial physics in the outer domain can enforce qualitative changes in the global model, both within and without the l-c.

Within the dead zone the field lines cross the equator normally, so that $\partial P/\partial z = 0$ for $x < x_c$, $z = 0$, whereas the condition that the wind zone be perfectly conducting enforces a fully open field crossing the l-c, so that $\partial P/\partial x = 0$ for $x > x_c$, $z = 0$. This in turn yields a non-zero S both at the outer domain equator and on its continuation as the separatrix P_c between the wind and dead zones, so that much of the current returns to the star as a sheet. Simultaneously, the conditions of equilibrium require a gas pressure both at the equator beyond x_c and within the dead zone.

In the somewhat more complicated model of MS there is again a force-free domain extending beyond the l-c, but now the inner boundary condition $\partial P/\partial z \propto B_x(x, 0) = 0$ is supposed to hold in the outer domain $x > 1$ also. This is the appropriate approximation if there is trans-field, radial flow of gas along the equator, requiring a dynamically-driven macro-resistivity. The constraint (10) still holds away from the equator, but a local departure from torque-free conditions allows $S(P)$ to vary along the equator. Again (14) yields $S\,\mathrm{d}S/\mathrm{d}P = 0$ at the neutral point (1, 0), and so also along the separatrix P_c between the wind and dead zones within the l-c, but now this can be satisfied by the simple choice $S(P_c) = 0$: $S(P)$ can go to zero continuously, without there needing to be a sheet current along P_c, nor need we introduce a pressure into the dead zone.

These simplifications, however, come at a price: at least for the cases studied, with the equatorial boundary condition $\partial P/\partial z \propto B_x = 0$ holding everywhere, a solution that is well-behaved and continuous at the l-c blows up before it can reach infinity.

The method of solution used in MS is a generalisation of the Fourier integral technique used in MW and MP. Both within and beyond the l-c, P is written as

$$P = \frac{2}{\pi}\int_0^\infty f\cos kz\,\mathrm{d}k, \qquad f = \int_0^\infty P\cos kz\,\mathrm{d}z, \tag{53}$$

yielding

$$(x^2-1)f_{xx} + \frac{(x^2+1)}{x}f_x - (x^2-1)k^2 f = g(x,k), \tag{54}$$

with

$$g(x,k) \equiv \int_0^\infty S\frac{\mathrm{d}S}{\mathrm{d}P}\cos kz\,\mathrm{d}z. \tag{55}$$

To illustrate the breakdown, it is convenient to apply the JWKB method to study the asymptotic solutions of (54) for large Fourier variable k (e.g. Jeffreys & Jeffreys 1972). Away from the singularity at $x = 1$, equation (54) has the general asymptotic solution

$$f = \left(\frac{x}{x^2-1}\right)^{1/2}\Bigg[\frac{1}{k}\int_{\bar{x}}^{x}\frac{g(t,k)}{[t(t^2-1)]^{1/2}}\sinh k(x-t)\,\mathrm{d}t + \left(A_k e^{kx} + B_k e^{-kx}\right)\Bigg] \tag{56}$$

where a convenient lower limit $\bar{x}$ will be chosen subsequently. Near $x = 1$, we define the variable $u = k(x-1)$, so that (54) reduces to

$$uf_{uu} + f_u - uf = g(1,k)/2k. \tag{57}$$

The general solution of (57) that is non-singular at $u = 0$ ($x = 1$) is

$$f = \mathrm{I}_0(u)\left[d_k + \frac{g(1,k)}{2k}\int_0^u \mathrm{K}_0(v)\,\mathrm{d}v\right] - \mathrm{K}_0(u)\frac{g(1,k)}{2k}\int_0^u \mathrm{I}_0(v)\,\mathrm{d}v, \tag{58}$$

where I, K are modified Bessel functions in standard notation. Thus d_k is the Fourier cosine transform of $P(1, z)$. Matching of the solutions (56) and (58) in the overlap domain where $(x-1) \ll 1$ but $u = k(x-1) \gg 1$, with the help of the asymptotic expressions for Bessel functions (Abramowitz & Stegun 1965: AS, 9.7), yields the relations

$$A_k e^k (k/2)^{1/2} = \left(\frac{1}{2\pi}\right)^{1/2}\left[d_k + \frac{1}{2k}\int_0^{\bar{u}} \mathrm{K}_0(v) g(1,k)\,\mathrm{d}v\right] = \left(\frac{1}{2\pi}\right)^{1/2}\left[d_k + \frac{\pi}{4k}g(1,k)\right], \tag{59}$$

$$B_k e^{-k}(k/2)^{1/2} = -\frac{\pi^{1/2}}{2k}\int_0^{\bar{u}} \mathrm{I}_0(v) g(1,k)\,\mathrm{d}v. \tag{60}$$

Continuity of P on the l-c $u = 0$ requires that d_k in (58) be equal to the Fourier cosine transform on the l-c of the inner domain solution. If $S = 0$ — no poloidal currents — then $g = 0$, and from the known asymptotic form of the inner domain MMWP solution (MP, Section 2), continuity yields $A_k = -(\pi k/2)^{1/2}\exp(-2k)$, so that f_k blows up like $\exp k(x-2)$, as found in MP. We are now interested to know if an appropriate choice of $S(P)$ will yield $g(1,k)$ such that

$$d_k = -\frac{\pi}{4k} g(1,k), \tag{61}$$

so by (59) yielding $A_k = 0$, the requirement for finiteness at large x.

A simple example with S vanishing on the separatrix P_c is the one adopted in MP:

$$S = -2P + \frac{2P^2}{P_c}. \tag{62}$$

If the solution sought for P were to exist, then d_k — its Fourier cosine transform on the l-c — should be identical with d_k given by (61). Substitution of plausible forms for P in fact yields a sign discrepancy. Though in no sense a proof, the result does complement the conclusion in MP. There, imposition of continuity at the l-c yielded a singularity in the solution at less than two l-c radii; here, the requirement that there be no blow-up in the solution yields fundamental incompatibility at the l-c.

The resolution of the dilemma suggested in MP is that there has to be a breakdown in the simple plasma condition $\mathbf{E}\cdot\mathbf{B} = 0$, not only on the equator, but also in a thin, dissipative volume domain, idealised as a cylindrical shell symmetric about the rotation axis.

Now return to the problem of this paper, the search for a solution of (17), satisfying $\partial P/\partial x = -xB_x = 0$ for $x > 1$, $z = 0$, so that the separatrix field line P_c extends beyond the l-c and lies in the equator. The appropriate transform for P is as a Fourier sine integral:

$$\begin{aligned} P(x,z) &= \frac{2}{\pi}\int_0^\infty f(x,k)\sin kz\,\mathrm{d}k, \\ f(x,k) &= \int_0^\infty P(x,z)\sin kz\,\mathrm{d}z. \end{aligned} \tag{63}$$

Integration by parts of the P_{zz} term in (17) yields an extra term: the equation for f is now

$$\begin{aligned} (x^2-1)f_{xx} &+ \frac{(1+x^2)}{x} f_x - (x^2-1)k^2 f \\ &= g_s(x,k) - (x^2-1)kP_c, \end{aligned} \tag{64}$$

where the suffix s is put on g as a reminder that it is a Fourier sine integral:

$$g_s = \int_0^\infty S\frac{\mathrm{d}S}{\mathrm{d}P}\sin kz\,\mathrm{d}z. \tag{65}$$

The expression (63) for P is valid in the domain $z > 0$; in $z < 0$ it represents $-P$. The Fourier back-transform of a function with a discontinuity will yield the arithmetic mean of the values on either side of the discontinuity. Thus at $z = 0$, (63) yields $P = 0$, which is indeed the mean of P_c and $-P_c$. Because of the presence of P_c on the right, the solution of (64) will show the ‘Gibbs phenomenon’ (e.g. Jeffreys & Jeffreys 1972).

The solution of equation (64) may be written

$$f = P_c\left(\frac{1}{k} + F(x,k)\right) \tag{66}$$

with F satisfying

$$(x^2-1)F_{xx} + \frac{(1+x^2)}{x}F_x - (x^2-1)k^2F = \frac{g_s(x,k)}{P_c}. \tag{67}$$

As an illustration, consider the Michel form (32) for $S(P)$, for which (17) has the known solution (31), valid in both inner and outer domains. We quote the known Fourier cosine transformation (AS, 9.6.25)

$$\int_0^\infty \frac{\cos kz}{(x^2+z^2)^{\nu+1/2}}\,\mathrm{d}z = \pi^{1/2}\left(\frac{k}{2x}\right)^\nu \frac{\mathrm{K}_\nu(kx)}{\Gamma(\nu+1/2)}. \tag{68}$$

Integration by parts of the case $\nu = 1$ then yields for F defined in (66)

$$F(x,k) = -x\mathrm{K}_1(kx), \quad g_s = 2kx^2\mathrm{K}_0(kx), \tag{69}$$

and (67) is seen to be satisfied.

One can perform a similar JWKB asymptotic analysis for this Fourier sine transform. When applied to the Michel solution, the method merely confirms that the condition $A_k = 0$, ensuring that the solution behaves well all the way to infinity, does indeed yield $F(x,k) \simeq -(\pi x/2k)^{1/2}\mathrm{e}^{-kx}$, the asymptotic form of F as given by (69). One can therefore expect generalised dissipation-free, Michel-type solutions to exist in the outer domain: with the change in the equatorial boundary condition to $B_z = 0$ — as discussed above, implying a local breakdown in the force-free condition — there is apparently no longer an automatic bar to satisfying appropriate conditions both at infinity and at the l-c.

8 Conclusion

As noted by CKF, when the field lines beyond the l-c are quasi-radial, the γ-values of the particles as given by (27) remain moderate. In analogous non-relativistic problems, one thinks of field lines achieving such a structure when the kinetic energy density of the flow dominates over the magnetic. Without such an outward pull, an equatorially pinching, quasi-radial field is likely to be unstable against reconnection, spontaneously converting into a structure with field lines closing across the equator. In the relativistic problem, however, quasi-MHD flow along such a field

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beyond the l-c does then lead to high γ-values (MRWW; FM; Beskin, Gurevich & Istomin 1993). My guess is that in a realistic model, the field lines will neither run parallel to the equator as in CKF nor cross it nearly normally as in MS. High-γ plasma will flow along the field into an equatorial sheet, and will continue its (probably unsteady) outflow, crossing the field through a dynamically driven macro-resistivity. It is not clear whether the dissipative processes will be confined to the equatorial domain, or whether there will be also a volume dissipative domain as in MS. The present work is a preliminary to this still more formidable problem, with its as yet uncertain observable consequences.

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