A result in queueing theory

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In a single-server queueing system, subject to the queue
discipline 'First come first served', the equilibrium
distribution function of the waiting time of a server depends on
the distribution of the random variable (u) which is the
difference between the service time and the inter-arrival time.
If in two queueing systems u's are equivalent in distribution,
the waiting times are also equivalent in distribution (known
result). It has been shown in this note that equivalence in
waiting time distributions does not necessarily imply equivalence
in distributions of u's . The proof is heuristic. This result
has useful practical implications.

1. Introduction

Let us consider a single-server queueing system in which W_n is the
waiting time of nth customer, measured from his time of arrival in the
queue to the time his service starts, s_n the service time, and t_n the
time interval between arrivals of the nth and $(n+1)$th customers. Let
it be assumed that the queue discipline is 'First come first served'. If
the waiting room has an infinite capacity and there is no constraint on the
waiting time of a customer, then W_{n+1} is related to W_n in the following
manner:

(1.1) $$W_{n+1} = \left(W_n + u_n\right)^+ ,$$

where $u_n = s_n - t_n$. If both s_n and t_n are random variables, u_n is
also a random variable, and the process W , given by (1.1), is an embedded
Markov process (Lindley [3]). Let $F(y) = \text{prob}(W_n \leq y)$ be the
equilibrium distribution function of W , and $G(x) = \text{prob}(u_n \leq x)$ the
distribution function of u ; then we can derive the distribution function
of W from that of u from the following integral equation [3]:

$$(1.2) \qquad\qquad F(y) = -\int_0^\infty F(t)dG(y-t) , \quad y \in (0, \infty) .$$

The above is an integral equation of Wiener-Hopf type, and it has been
shown that if $G(\cdot)$ is continuous over the domain $(-\infty, \infty)$, then for a
specified distribution function $G(\cdot)$, we obtain $F(y)$ in a unique
manner. If, therefore, there are two queueing systems in which
distribution functions of u are $G_1(\cdot)$ and $G_2(\cdot)$, and distribution
functions of W are $F_1(\cdot)$ and $F_2(\cdot)$, then

(1.3) $G_1(x) = G_2(x)$ for all $x \in (-\infty, \infty)$

                            implies $F_1(y) = F_2(y)$ for all $y \in [0, \infty)$.

We state (1.3) in the form of Lemma 1.1 below and give an alternative
proof.

   LEMMA 1.1. $u^{(1)} \sim u^{(2)}$ *implies* $W^{(1)} \sim W^{(2)}$ *where* $u^{(i)}$ *is the* u
for the system i , $W^{(i)}$ *the waiting time for the system* i $(i = 1, 2)$,
and '$\sim$' *implies 'equivalence in distribution functions'.*

   We can also prove this lemma from Smith's [4] paper. Let $U(\alpha)$ and
$W(\alpha)$ be the Laplace transforms of u and W respectively. Then Smith
derives

$$(1.4) \qquad\qquad 1 - U(\alpha) = \alpha\varphi^+(\alpha)/\varphi^-(\alpha) ,$$

where $\varphi(\alpha)$ is analytic and free from zeros in the right half plane
$\text{re}(\alpha) > -\gamma$, and $\varphi^-(\alpha)$ is analytic and free from zeros in the left half
plane $\text{re}(\alpha) < \gamma$. Then he proves that

$$(1.5) \qquad\qquad W(\alpha) = \varphi^+(0)/\varphi^+(\alpha) .$$

Let $1 - U_1(\alpha) = \alpha\varphi_1^+(\alpha)/\varphi_1^-(\alpha)$, $1 - U_2(\alpha) = \alpha\varphi_2^+(\alpha)/\varphi_2^-(\alpha)$. Then

$U_1(\alpha) = U_2(\alpha)$, if $\varphi_1^+(\alpha) = \varphi_2^+(\alpha)$ and $\varphi_1^-(\alpha) = \varphi_2^-(\alpha)$. Hence $W_1(\alpha) = W_2(\alpha)$.

2. Basic result

We prove the following theorem.

THEOREM 2.1. *Let there be two queueing systems in which* $u^{(1)} = s^{(1)} - t^{(1)}$ *and* $u^{(2)} = s^{(2)} - t^{(2)}$, *and* $W^{(1)}$ *and* $W^{(2)}$ *are waiting times. While equivalence in distribution functions of* $u^{(1)}$ *and* $u^{(2)}$ *ensures equivalence in distribution functions of* $W^{(1)}$ *and* $W^{(2)}$, *equivalence in distribution functions of* $W^{(1)}$ *and* $W^{(2)}$ *does not necessarily ensure equivalence in distribution functions of* $u^{(1)}$ *and* $u^{(2)}$.

Proof. The fact that in any two systems $u^{(1)} \sim u^{(2)}$ implies $W^{(1)} \sim W^{(2)}$ is proved in Lemma 1.1.

The second part of the theorem, namely that $W^{(1)}$ and $W^{(2)}$ can be equivalent in distribution functions without $u^{(1)}$ and $u^{(2)}$ being equivalent in distribution functions is proved by citing an example. It has been proved by Smith [4] that in a GI/M/1 system (with a general arrival distribution function and exponential service distribution function) the waiting time follows an exponential distribution function. From this, Ghosal ([2], Theorem 3.3 proved that for any GI/M/1 system, an M/M/1 isomorph (with respect to waiting time distribution functions) exists: also the service time in two such systems are the same, but the inter-arrival distribution function is general in GI/M/1 (system A_1) and exponential in the M/M/1 isomorph (system A_2). For a definition of *restricted isomorph* see [3]: for example, if two systems have the same waiting distribution function but their idle time (of server) distributions functions are different, one is an isomorph of the other in the restricted sense.) Thus, we have u_1 (for system A_1) and u_2 (for system A_2) such that

$$u_1 = s_1 - t_1 \; , \quad u_2 = s_2 - t_2$$

where $s_1 \sim s_2$ but t_1 and t_2 are not equivalent in distribution functions. Hence u_1 and u_2 are not equivalent in distribution functions. Thus, Theorem 2.1 follows. In Section 3 below we discuss the conditions under which Theorem 2.1 is valid under more general situations.

3.   Interpretation of Theorem 2.1

Let $W(\alpha)$ be the Laplace transform of the waiting time distribution function, and $U(\alpha)$ the Laplace transform of U. Then, from Lindley [3] and Ghosal ([1], equation (1.34)),

(3.1) $W(\alpha)[U(\alpha)-1] = \eta^+(\alpha) - F(0)$,

where $\eta^+(\alpha)$ is the Laplace transform of the idle-time of server (I) , leaving the concentration at zero $\big(\mathrm{pr}(I = 0) = \big(1-F(0)\big)\big)$; $F(0) = \mathrm{prob}(W = 0) = \mathrm{prob}(I > 0)$. In deriving (3.1), we assume a distribution function $F^*(y)$ which is related to $F(y)$, the waiting time distribution function of W , as follows:

$$F^*(y) = F(y) \quad \text{on} \quad y \in [0, \infty) \; ,$$
$$= \mathrm{prob}(-I \le y) = \mathrm{prob}(I \ge -y) \quad \text{on} \quad y \in (-\infty, 0) \; .$$

We have

$$\eta(\alpha) = \int_{-\infty}^{0} \exp(-\alpha y)dF^*(y) = 1 - F(0) + \eta^+(\alpha) \; .$$

In the two systems A_1 and A_2 referred to in Theorem 2.1, we get, since $W_1 \sim W_2$ $\big($so that $W_1(\alpha) = W_2(\alpha) = W(\alpha)$, and $F_1(0) = F_2(0)\big)$,

(3.2) $[U_1(\alpha)-1]/[U_2(\alpha)-1] = [\eta_1(\alpha)-1]/[\eta_2(\alpha)-1]$,

or

(3.3) $[\eta_1(\alpha)-\eta_2(\alpha)] = W(\alpha)[U_1(\alpha)-U_2(\alpha)]$.

When Theorem 2.1 holds, the relation (3.2) or (3.3) holds.

4.   Concluding remarks

It is possible that Theorem 2.1 holds true for a large number of pairs

of Gl/G/l systems. The condition for its validity is (3.2) or (3.3) which
leads to possibilities of good simulation experiments. Theorem 2.1
provides a basic relation between two single-server queueing systems one
of which is an isomorph of the other in the restricted sense - in other
words, two are equivalent in respect of any one or two but not all of the
output elements - here they are equivalent in respect of waiting time
distribution functions; see [2] for more details of properties of
isomorphic queues.

References

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