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UNSTABLE SETS, HETEROCLINIC ORBITS AND GENERIC QUASI-CONVERGENCE FOR ESSENTIALLY STRONGLY ORDER-PRESERVING SEMIFLOWS

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Abstract We present conditions guaranteeing the existence of non-trivial unstable sets for compact invariant sets in semiflows with certain compactness conditions, and then establish the existence of such unstable sets for an unstable equilibrium or a minimal compact invariant set, not containing equilibria, in an essentially strongly order-preserving semiflow. By appealing to the limit-set dichotomy for essentially strongly order-preserving semiflows, we prove the existence of an orbit connection from an equilibrium to a minimal compact invariant set, not consisting of equilibria. As an application, we establish a new generic convergence principle for essentially strongly order-preserving semiflows with certain compactness conditions.

Keywords: essentially strongly order-preserving semiflow; invariant set; unstable set

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1. Introduction

In [**9**], Matano proved that, in a strongly order-preserving semiflow defined on an ordered Banach space, an unstable equilibrium always has a non-trivial unstable set. More details about various applications of this result can be found in [**9**, **10**]. Such a result was later improved by Poláčik [**11**]. More precisely, he presented conditions guaranteeing the existence of non-trivial unstable sets for compact invariant sets in compact semiflows on a metric space. By checking these conditions, he proved the existence of such unstable sets for a non-trivial minimal set and an unstable equilibrium in a compact strongly order-preserving semiflow on an ordered metric space, from which the existence of a heteroclinic orbit connection from a minimal set to an equilibrium was then derived. For strongly increasing discrete-time semiflows on ordered Banach spaces, Dancer and Hess [**1**] established the existence of a heteroclinic orbit connection between related equilibria. By applying the existence theorem of a heteroclinic orbit connection between related equilibria in [**1**], many convergence results have been obtained for monotone systems (see, for example, [**1**, **6**–**8**, **16**, **17**]). For the theory and applications of the monotone dynamical system, we refer the reader to [**5**, **13**, **18**, **20**] and references therein.

However, the above-mentioned results still have their own drawbacks, and thus require further study. One reason is that the strong compactness assumptions on the semiflow make the above-mentioned results awkward to apply to delay differential equations, since the compactness of the functions in the right-hand side of delay differential equations cannot generally guarantee the compactness of the induced semiflows (see [**2**]). Secondly, the abstract results of Poláčik [**11**] fail for essentially strongly order-preserving semiflows and it thus becomes necessary to improve the main results of [**11**] in order that they can be applied to essentially strongly order-preserving semiflows. Additionally, the existence of a heteroclinic orbit connection from a non-trivial minimal set to an equilibrium in essentially strongly order-preserving semiflows leads to convenient applications of the above-mentioned results to quasi-monotone functional differential equations without the *ignition* assumption. Finally, we also hope that the existence theorem of a heteroclinic orbit in [**11**] can be used to prove some generic convergence principles.

In the present paper, motivated by the above work and discussion, we will consider the semiflows with conditional compactness defined on an ordered metric space. The organization of the rest of this paper is as follows. In § 2, we provide sufficient conditions guaranteeing the existence of a non-trivial unstable set for a compact invariant set in semiflows with conditional compactness on a metric space. As an application, we prove the existence of such an unstable set for an isolated compact invariant set in semiflows with certain compactness conditions. In § 3, applying the results in § 2 to essentially strongly order-preserving semiflows with certain compactness conditions, we show that both a non-trivial minimal set and an unstable equilibrium have non-trivial unstable sets, which proves the existence of a heteroclinic orbit connection from a non-trivial minimal set to an equilibrium by appealing to the limit-set dichotomy of Yi and Huang [**19**]. In § 4, as an application of the existence theorem of a heteroclinic orbit, we establish a new generic convergence principle for a class of essentially strongly order-preserving semiflows with certain compactness conditions. On the one hand, our principle avoids the ω-compactness conditions (see [**13**–**15**, **20**]) and so does [**4**], but we do not require the lattice conditions in [**4**]. On the other hand, like [**19**], our principle is suitable for applications to delay differential equations without the *ignition* assumption. Recall that in order to ensure that the delay differential equation generates a semiflow with the strongly order-preserving property, Smith [**12**] introduced the *ignition* assumption and pointed out that the proper choice of the delay is important in order to guarantee this assumption. An advantage of the essentially strongly order-preserving property introduced by Yi and Huang [**19**] is that it does not require a delicate choice of the state space and the technical ignition assumption required in the classical work. Thus, one can more conveniently apply the above results of essentially strongly order-preserving semiflows generated by the systems of quasi-monotone delay differential equations without the ignition assumption.

2. Preliminary results

The aim of this section is to investigate the existence of non-trivial unstable sets for compact invariant sets in semiflows with certain compactness conditions. We start with some notation and definitions that will be used throughout the paper.

Let $R^1 = (-\infty, \infty)$, $R^1_+ = [0, \infty)$, and let (X, d) denote a metric space. We use the notation $\bar{A}$ for the closure of a subset A of X. By $\operatorname{dist}(x, A)$ we denote the distance from a point x to a set A: $\operatorname{dist}(x, A) = \inf\{d(x, y) : y \in A\}$. Recall that a continuous semiflow on X is a continuous map $\Phi : X \times R^1_+ \to X$ with $\Phi_t(x) \equiv \Phi(x, t)$ satisfying $\Phi_0(x) = x$ for all $x \in X$ and $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$ for all $x \in X$ and $t, s \in R^1_+$.

We write $O^+(x) = \{\Phi_t(x) : t \in R^1_+\}$ for the positive orbit of a point $x \in X$. The positive orbit $O^+(H)$ of a set $H \subseteq X$ is the union of the positive orbits of all elements of H. The ω-limit set of a point $x \in X$ is defined by $\omega(x) = \bigcap_{t \in R^1_+} \overline{O(\Phi_t(x))}$. Replacing x by $H \subseteq X$ in this definition, we obtain the definition of the ω-limit set of H. Let $E = \{x \in X : \Phi_t(x) = x,\ t \geqslant 0\}$ be the set of equilibria of Φ. By a negative orbit we mean a function $v : (-\infty, 0] \to X$ such that $\Phi_t(v(s)) = \Phi(v(t+s))$ for all $t \geqslant 0 \geqslant s$ with $t + s \leqslant 0$. If $v(0) = x$, we say that v is a negative orbit of x. For a negative orbit $v(\cdot)$, the α-limit set $\alpha(v(\cdot))$ of $v(\cdot)$ is the set of all limit points of $v(t)$ as $t \to -\infty$. A set $K \subseteq X$ is called positively invariant if $\Phi_t(K) \subseteq K$ for all $t \in R^1_+$. K is called negatively invariant if any element of K has a negative orbit which takes values in K. If $K \neq \emptyset$ is both negatively and positively invariant, it is an invariant set. Let $K \subseteq X$ be a compact invariant set. We say that K has a non-trivial unstable set if there exists a pre-compact negative orbit whose ω-limit set is contained in K.

Let $H \subseteq G$ be two subsets of X, and let $H^G_t = \{x \in H : \Phi_s(x) \in G \text{ for all } s \in [0, t]\}$. We define the positive orbit of H relative to G by

$$O^+_G(H) = \{\Phi_t(x) : x \in H,\ t \in R^1_+ \text{ and } \Phi_s(x) \in G \text{ for all } s \in [0, t]\} = \bigcup_{t \in R^1_+} \Phi_t(H^G_t).$$

Clearly, if G is open, then $O^+_G(H)$ is just the positive orbit of H for the restricted semiflow $\Phi_t|_G$, $t \geqslant 0$. If $H \subseteq G$, and G is closed, we define the ω-limit set of H relative to G by

$$\begin{aligned} \omega_G(H) = \{y \in X : {} & \text{there are two sequences } x_n \in H \text{ and } t_n \to +\infty \text{ such that} \\ & \Phi_s(x_n) \in G \text{ for all } s \in [0, t_n] \text{ and } \Phi_{t_n}(x_n) \to y\}. \end{aligned}$$

Clearly, $\omega_G(H) \subseteq G$ and $\omega_G(H) = \bigcap_{t \in R^1_+} O^+_G(\Phi_t(H^G_t))$.

We assume the following compactness condition on the semiflow.

(C) For any bounded set $G \subseteq X$, there exists a $\delta > 0$ such that $\Phi_\delta(G^G_\delta)$ is pre-compact.

Remark 2.1. Obviously, the compactness condition (C) we use here is weaker than that used by Poláčik [**11**]. In this paper, the compactness condition (C) is equivalent to requiring that the semiflow Φ is conditionally compact in the sense of [**2**, **3**]. Note that [**2**, Theorem 3.6.1] implies a semiflow generated by delay differential equations with the conditional compactness property. As a result, our main results shall be more suitable for applications to delay differential equations.

Lemma 2.2. *Let $H \subseteq G \subseteq X$. If G is a bounded set, then there exists a $\delta > 0$ such that $O^+_G(\Phi_\delta(H^G_\delta))$ is pre-compact.*

Proof. Note that the compactness condition (C) implies that there exists a $\delta > 0$ such that $\Phi_\delta(G^G_\delta)$ is pre-compact. Also, we have

$$O^+_G(\Phi_\delta(H^G_\delta)) = \bigcup_{t\geqslant\delta} \Phi_t(H^G_t) = \Phi_\delta\biggl(\bigcup_{t\geqslant\delta} \Phi_{t-\delta}(H^G_t)\biggr).$$

Since $\Phi_{t-\delta}(H^G_t) \subseteq G^G_\delta$ for all $t \geqslant \delta$, it follows that $\bigcup_{t\geqslant\delta} \Phi_{t-\delta}(H^G_t) \subseteq G^G_\delta$, and hence $O^+_G(\Phi_\delta(H^G_\delta)) \subseteq \Phi_\delta(G^G_\delta)$. Therefore, $O^+_G(\Phi_\delta(H^G_\delta))$ is pre-compact. This completes the proof. □

Lemma 2.3. *Let $G \subseteq X$ be closed and let $H \subseteq G$. If $O^+_G(\Phi_\delta(H_\delta))$ is pre-compact for some $\delta \geqslant 0$, then $\omega_G(H)$ is compact negatively invariant. Moreover, $\operatorname{dist}(\Phi_t H_t, \omega_G(H)) \to 0$ as $t \to \infty$.*

Proof. Lemma 2.3 follows easily by applying [**11**, Lemma 2.1 and Remark 2.2]. □

Remark 2.4. Assume that $H \subseteq G \subseteq X$ and G is bounded and closed. If the compactness condition (C) is satisfied, then the conclusions of Lemma 2.3 follow from Lemma 2.2.

Proposition 2.5. *Let $K \subseteq X$ be a compact positively invariant set. Assume that there exist a neighbourhood U of K, a set $M \subseteq \bar{U}$ and a constant $t_0 \geqslant 0$ with the following properties:*

(i) *$K \subseteq M$;*

(ii) *there exists a sequence $y_n \in M \setminus K$ such that $\Phi_{t_0}(y_n) \notin K$ and $\operatorname{dist}(y_n, K) \to 0$ as $n \to \infty$;*

(iii) *If $y \in M \setminus K$, then either $\Phi_{t_0}(y) \in K$ or there exists $t_1 > 0$ such that $\Phi_{t_1}(y) \in \partial U$ and $\Phi_t(y) \in M$ for all $t \in [0, t_1)$.*

(iv) *if $H \subseteq \bar{M} \cap U$ is compact and invariant, then $H \subseteq K$.*

Then there exists a negative orbit $v(\cdot)$ with pre-compact image such that $v(t) \in M \setminus K$ for all $t \geqslant 0$ and $\alpha(v(\cdot)) \subset K$.

Proof. Choose a bounded open set V such that $K \subseteq V \subseteq \bar{V} \subseteq U$ and let $H = M \cap V$. From Remark 2.4, we see that $\omega_{\bar{V}}(H)$ is compact and negatively invariant. By (iii), we have $O^+_{\bar{V}}(\Phi_{t_0}(H^{\bar{V}}_{t_0})) \subseteq M$, and hence $\omega_{\bar{V}}(H) \subseteq \bar{M} \cap \bar{V}$. We show next that $\omega_{\bar{V}}(H) \setminus K \neq \emptyset$. Consider the sequence $y_n \in M \setminus K$ as in (ii). By compactness of K and the fact that $\operatorname{dist}(y_n, K) \to 0$ as $n \to \infty$, passing to a subsequence, we may assume that $\lim_{n\to\infty} y_n = y$. Since $\Phi_{t_0}(y_n) \notin K$, by (iii) and $V \subseteq U$, there exists $t_n > 0$ such that $\Phi_{t_n}(y_n) \in \partial V$ and $\Phi_t(y_n) \in V$ for all $0 \leqslant t < t_n$. It follows from $y \in K$, the invariance of K and the continuity of Φ that for any $T > 0$ there exists $n_0 > 1$ such that $\Phi_t(y_n) \in V$ for $0 \leqslant t \leqslant T$ and $n > n_0$. Hence, $t_n \to +\infty$ as $n \to \infty$. Again since

$$\Phi_{t_n}(y_n) = \Phi_\delta(\Phi_{t_n-\delta}(y_n)) \in \Phi_\delta(\bar{V}^{\bar{V}}_\delta)$$

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for all sufficiently large n, we can find a convergent subsequence of the sequence $\{\Phi_{t_n}(y_n)\}_{n\geqslant 1}$ such that $\Phi_{t_{n_k}}(y_{n_k}) \to z$ as $n_k \to \infty$. By definition of $\omega_{\bar V}(H)$, we have $z \in \omega_{\bar V}(H)$. From $z \in \partial V$, it follows that $z \notin K$, that is, $\omega_{\bar V}(H) \setminus K \neq \emptyset$. Now choose $x \in \omega_{\bar V}(H) \setminus K$. Compactness and negative invariance of $\omega_{\bar V}(H)$ imply that x has a pre-compact negative orbit $v(\cdot)$ such that $v(t) \in \omega_{\bar V}(H) \subseteq \bar M \cap \bar V$ for all $t \leqslant 0$. Since $x = v(0) \notin K$ and K is positively invariant, we have $v(t) \notin K$ for all $t \leqslant 0$. In view of

$$\alpha(v(\cdot)) \subseteq \omega_{\bar V}(H) \subseteq \bar M \cap \bar V \subseteq \bar M \cap U,$$

by (iv) we obtain $\alpha(v(\cdot)) \subseteq K$. This completes the proof. □

Proposition 2.6. *Let the hypotheses (i)–(iii) of Proposition 2.5 be satisfied and let M be closed. Then the result of Proposition 2.5 holds.*

Proof. We now show that the hypothesis (iv) of Proposition 2.5 is satisfied. Assume that $H \subset \bar M \cap U = M \cap U$ is a compact invariant set and $H \setminus K \neq \emptyset$. Choose $x \in H \setminus K$. By (iii) and invariance of H, we have $\Phi_{t_0}(x) \in K$. Since x is arbitrary, it follows that $H = \Phi_{t_0}(H) \subseteq K$, which yields a contradiction. Hence, (iv) of Proposition 2.5 is proved to be satisfied. Therefore, Proposition 2.6 follows by applying Proposition 2.5. □

Remark 2.7. If $t_0 = 0$, then Propositions 2.5 and 2.6 with the compactness condition (C) improve Theorems 1 and 1′ of [**11**], respectively.

Now recall that a compact invariant set K is called an isolated compact invariant set if it is a maximal compact invariant set in some neighbourhood of K.

Definition 2.8. A compact invariant set K is called stable if for every neighbourhood U of K, there exists a neighbourhood V_U of K such that $O(V_U) \subseteq U$. A compact invariant set K is called unstable or instable if K is not stable, that is, there exist a neighbourhood U of K and a sequence $\{x_n\}_{n\geqslant 1}$ of U such that $\operatorname{dist}(x_n, K) \to 0$ as $n \to +\infty$ and $O^+(x_n) \setminus U \neq \emptyset$ for $n \geqslant 1$.

Corollary 2.9. *Let* (C) *hold. Let $K \subseteq X$ be an isolated compact invariant set. If K is unstable, then there exists a negative orbit $v(\cdot)$ with pre-compact image disjoint from K such that $\alpha(v(\cdot)) \subseteq K$.*

Proof. Since K is an isolated compact invariant set, there exists a bounded open neighbourhood V of K such that any compact invariant set $\tilde K \subseteq V$ is contained in K. By Definition 2.8 and instability of K, there exist a neighbourhood U of K and a sequence $\{x_n\}_{n\geqslant 1}$ of U such that $\operatorname{dist}(x_n, K) \to 0$ and $O^+(x_n) \setminus U \neq \emptyset$. Let $W = U \cap V$ and $M = (\bigcup_{n\geqslant 1} O^+_{\bar W}(x_n)) \bigcup K$. Then the hypotheses of Proposition 2.5 are satisfied if $t_0 = 0$. This completes the proof. □

As an application of Corollary 2.9, we can obtain the following results by arguing as in the proof of [**11**, Proposition 3.2].

Corollary 2.10. *Let* (C) *hold. Assume that the α-limit set of any pre-compact negative orbit consists of equilibria. Then, for any unstable isolated equilibrium e, there exists a pre-compact negative orbit $v(\cdot)$ such that $v(t) \neq e$ for all $t \leqslant 0$ and $\alpha(v(\cdot)) = \{e\}$.*

3. Existence of heteroclinic orbits and instability for essentially strongly order-preserving semiflows

Assume that X is an ordered metric space with a metric d and a closed partial order relation $\leqslant$. Let Φ be an order-preserving semiflow on X, that is, whenever $x, y \in X$ with $x \leqslant y$, we have $\Phi_t(x) \leqslant \Phi_t(y)$ for all $t \geqslant 0$. For any $x, y \in X$ and some constant $t_0 \geqslant 0$, we write $x \prec_{t_0} y$ if $x \leqslant y$ and $\Phi_{t_0}(x) < \Phi_{t_0}(y)$. We shall write '$\prec$' for '$\prec_{t_0}$' when no confusion results.

Given a set $K \subseteq X$, we denote $X_+(K) = \{y \in X : y \geqslant x \text{ for some } x \in K\}$ and $X_-(K) = \{y \in X : y \leqslant x \text{ for some } x \in K\}$. In particular, for $x \in X$, $X_+(x) = \{y \in X : y \geqslant x\}$ and $X_-(x) = \{y \in X : y \leqslant x\}$.

The following definition can be found in [**19**].

Definition 3.1. The semiflow Φ is said to be essentially strongly order-preserving if Φ is order-preserving and for any $x, y \in X$ with $x \prec y$, there exist open sets U and V, and some constant $T_0 \geqslant 0$ such that $x \in U$, $y \in V$ and $\Phi_{T_0}(U) \leqslant \Phi_{T_0}(V)$.

In the following, we always assume that Φ is an essentially strongly order-preserving semiflow whose orbits have compact closure in X.

The following three propositions come from Corollary 3.1, Lemma 3.1 and Theorem 3.2 in [**19**], respectively.

Proposition 3.2. *If $z \in X$, $x \in \omega(z)$ and $x \leqslant \omega(z)$ ($\omega(z) \leqslant x$), then $\omega(z) = \{x\}$.*

Proposition 3.3. *Let K and H be two compact subsets of X satisfying $K \prec H$. Then there are two open sets U and V, $K \subset U$, $H \subset V$, and $T_0 \geqslant 0$, $\varepsilon > 0$ such that*

$$\Phi_{T_0+s}(U) \leqslant \Phi_{T_0}(V) \quad \text{and} \quad \Phi_{T_0}(U) \leqslant \Phi_{T_0+s}(V), \quad 0 \leqslant s \leqslant \varepsilon.$$

Proposition 3.4 (limit-set dichotomy). *Let $x, y \in X$ satisfy $x \prec y$. Then one of the following holds:*

(i) *$\omega(x) < \omega(y)$;*

(ii) *$\omega(x) = \omega(y) \subset E$; moreover, $\lim_{t\to\infty} d(\Phi_t(x), \Phi_t(y)) = 0$.*

Definition 3.5. An equilibrium $e \in E$ is said to be stable from above (below) if it is stable for the restricted semiflow on $X_+(e)$ ($X_-(e)$). Otherwise the equilibrium point is said to be unstable from above (below).

Hereafter, we always assume that the compactness condition (C) holds.

Lemma 3.6. *Let $e \in E$ be unstable from above. Then there exists a neighbourhood U of e with the following property: for any $x \in X_+(e) \cap U$, either $\Phi_{t_0}(x) = e$ or there exists an $s_x > 0$ such that $\Phi_{s_x}(x) \in \partial U$.*

Proof. Since $e \in E$ is unstable from above, there exist an $\varepsilon_0 > 0$ and sequence $x_n \in X_+(e)$, $t_n \in (0, +\infty)$ such that $x_n \to e$, $d(\Phi_{t_n}(x_n), e) = \varepsilon_0$, and $d(\Phi_t(x_n), e) < \varepsilon_0$ for all $t \in [0, t_n)$. By continuity of Φ, we have $t_n \to +\infty$. Without loss of generality, we may assume that $t_n > 2t_0$. Let $N = \{\Phi_{t_n - t_0}(x_n) : n \geqslant 1\}$. Then it is obvious that $e < N$

and $e \leqslant \bar{N}$. We show that $\bar{N}$ is compact. Indeed, let $G = \{x \in X : d(x, e) \leqslant \varepsilon_0\}$. Then, by the compactness condition (C), there exists a $\delta > 0$ such that $\Phi_\delta(G_\delta^G)$ is pre-compact. Since $t_n - t_0 > \delta$ for all sufficiently large n, it follows that

$$\Phi_{t_n - t_0}(x_n) = \Phi_\delta(\Phi_{t_n - t_0 - \delta}(x_n)) \in \Phi_\delta(G_\delta^G)$$

for all sufficiently large n, and hence $\bar{N}$ is compact. Next, we show that $e \prec \bar{N}$. Assume, for the sake of contradiction, that there exists $z \in \bar{N}$ such that $\Phi_{t_0}(z) = e$. Then there exists a subsequence t_{n_k} such that $\Phi_{t_{n_k} - t_0}(x_{n_k}) \to z$ as $n_k \to \infty$. Continuity of $\Phi_{t_0}(\cdot)$ implies that $\Phi_{t_{n_k}}(x_{n_k}) \to \Phi_{t_0}(z) = e$, which contradicts the statement that $\mathrm{dist}(\Phi_{t_n}(x_n), e) = \varepsilon_0 > 0$ for all $n \geqslant 1$. Since $e \prec \bar{N}$, by Proposition 3.3, there exist an open neighbourhood V of e and $T_1 > 0$ such that $\Phi_{T_1}(V) \leqslant \Phi_{T_1}(\bar{N})$. We claim that $\mathrm{dist}(\Phi_{T_1}(\bar{N}), e) > 0$. Otherwise we have $\Phi_{T_1}(\bar{N}) = e$ and hence $\Phi_{T_1}(V \cap X_+(e)) = e$. This means that the equilibrium e is stable from above, which yields a contradiction and the above claim is therefore established. Let

$$\delta_0 = \tfrac{1}{2} \min\{\mathrm{dist}(\Phi_{T_1}(\bar{N}), e), \varepsilon_0, \mathrm{dist}(\bar{N}, e)\} > 0, \quad U = V \cap B(e, \delta_0).$$

Then U is an open neighbourhood of e. We shall show that such a U has the property required in the conclusion. Assume that $x \in U \cap X_+(e)$. If $\Phi_{t_0}(x) = e$, then the proof is complete. In what follows, we assume that $\Phi_{t_0}(x) \neq e$. We now show that there exists $s_x > 0$ such that $\Phi_{s_x}(x) \in \partial U$. Otherwise we have $O(x) \subset U$. Clearly, $e \prec x$. Since Φ is an essentially strongly order-preserving semiflow, there exist an open neighbourhood W of e and $T_2 > 0$ such that $\Phi_{T_2}(W) \leqslant \Phi_{T_2}(x)$. Since $x_n \to e$, there exists $n_0 > 0$ such that $\Phi_{T_2}(x_n) \leqslant \Phi_{T_2}(x)$ for all $n \geqslant n_0$. It follows from $t_n \to \infty$ and monotonicity of Φ that there exists $n_1 > n_0$ such that

$$\Phi_{T_1}(\Phi_{t_n - t_0}(x_n)) = \Phi_{t_n + T_1 - t_0}(x_n) \leqslant \Phi_{t_n + T_1 - t_0}(x) = \Phi_{T_1}(\Phi_{t_n - t_0}(x))$$

for all $n > n_1$. Since $O(x) \subseteq U \subseteq V$, we obtain $\Phi_{t_n - t_0}(x) \in V$. Thus, from $\Phi_{T_1}(V) \leqslant \Phi_{T_1}(\bar{N})$, we have $\Phi_{T_1 + t_n - t_0}(x) = \Phi_{T_1 + t_n - t_0}(x_n) \in \Phi_{T_1}(\bar{N})$. Hence,

$$\mathrm{dist}(\Phi_{T_1 + t_n - t_0}(x), e) \geqslant \mathrm{dist}(\Phi_{T_1}(\bar{N}), e) \geqslant 2\delta_0.$$

Therefore, the choice of U implies that $\Phi_{T_1 + t_n - t_0}(x) \notin U$ for all $n > n_2$. This completes the proof. □

Theorem 3.7. *Let $e \in E$ be unstable from above. Then there exists a negative orbit $v(\cdot)$ with pre-compact image such that $v(t) > e$ for all $t \leqslant 0$ and $v(t) \to e$ as $t \to -\infty$.*

Proof. Let U be as in Lemma 3.6, let $K = \{e\}$ and $M = X_+(e) \cap \bar{U}$. The theorem follows immediately from Lemma 3.6 and Proposition 2.6. □

Remark 3.8. A similar conclusion to that of Theorem 3.7 holds if $e \in E$ is unstable from below.

Definition 3.9. A point $x \in X$ is said to be essentially ordered from below (respectively, above) if in every neighbourhood of x there exist y and z such that $z \prec y \prec x$ ($x \prec y \prec z$). X is said to be essentially ordered from below (respectively, above) if every point $x \in X$ is essentially ordered from below (respectively, above). X is called essentially ordered if every point $x \in X$ is essentially ordered from either below or above. Hence, if X is essentially ordered from below or above, then X must be essentially ordered.

Remark 3.10. Let $C = C([-r, 0], R^n)$ denote the Banach space of all continuous mappings $\varphi : [-r, 0] \to R^n$ and $C_+ = C([-r, 0], R^n)$. Then (C, C_+) is an ordered Banach space. For any $\varphi, \psi \in C$, $\varphi \leqslant \psi$ if and only if $\psi - \varphi \in C_+$, $\varphi < \psi$ if and only if $\varphi \leqslant \psi$ and $\varphi \neq \psi$, $\varphi \ll \psi$ if and only if $\psi - \varphi \in \operatorname{Int} C_+$. For every open subset X of C and for any $x \in X$ there exists a sequence $\{x_n\}$ such that $x_n \ll x_{n+1}$ and $x_n \to x$ as $n \to \infty$. Similarly, for any $x \in X$ there exists a sequence $\{x_n\}$ such that $x_n \gg x_{n+1}$ and $x_n \to x$ as $n \to \infty$. According to [**19**, Theorem 2.1], we know that '$\gg$' (respectively, '$\ll$') implies '$\succ$' (respectively, '$\prec$') if a system of delay differential equations is essentially cooperative and irreducible (without ignition assumption) in the sense of [**19**]. So every open subset X of C is essentially ordered.

Lemma 3.11. *Let K be a minimal compact invariant set which contains a point essentially ordered from above and which is not an equilibrium. If U is an open neighbourhood of K, then there exists $y \in U \cap X_+(K)$ satisfying $\Phi_{t_0}(y) \notin K$.*

Proof. Assume that $x \in K$ is essentially ordered from above. Then there exists $y \in U$ such that $x \prec y$. We shall show that $\Phi_{t_0}(y) \notin K$. Otherwise there exists $\tilde{y} \in K$ such that $\tilde{y} = \Phi_{t_0}(y)$. By the invariance of K, there exists $\tilde{x} \in K$ such that $\Phi_{t_0}(y) = \Phi_{t_0}(\tilde{x})$. It follows from $x \prec y$ that there exist $T_1 > t_0$ and an open neighbourhood V_x of x such that $\Phi_{T_1}(V_x) \leqslant \Phi_{T_1}(y) = \Phi_{T_1}(\tilde{x})$. Since $x \in K = \omega(\tilde{x})$, there exist $T_2 > 0$ and $\tau > 0$ such that $\Phi_{T_1+s}(\tilde{x}) \leqslant \Phi_{T_1}(\tilde{x})$ for all $s \in [T_2, T_2 + \tau]$. Applying [**13**, Theorem 1.2.1], we have $\omega(\tilde{x}) = \{e\}$ for some $e \in E$, which contradicts the statement that $\omega(\tilde{x}) = K$ is not an equilibrium. Therefore, y satisfies the hypotheses of Lemma 3.11. This completes the proof. $\square$

Theorem 3.12. *Let K be a minimal compact invariant set which contains a point essentially ordered from above and which is not an equilibrium. Then there exists a negative orbit $v(\cdot)$ with pre-compact image such that $v(t) \in X_+(K) \setminus K$ for all $t \leqslant 0$ and $\alpha(v(\cdot)) = K$.*

Proof. Let $V = \{x \in X : x \geqslant K\}$. Then the set V is closed. We shall show that $V \cap K = \emptyset$. Suppose, by way of contradiction, that there exists $x \in K$ such that $x \geqslant K$. Then, by Proposition 3.2, we have $K = \{x\} \subseteq E$, which contradicts the statement that $K \cap E = \emptyset$. Choose a bounded open set $U \subseteq \bar{U} \subseteq X \setminus V$ such that $K \subseteq U$. We claim that if $y \in M \setminus K$, then either $\Phi_{t_0}(y) \in K$ or there exists $t_1 > 0$ such that $\Phi_{t_1}(y) \in \partial U$ and $\Phi_t(y) \in M$ for all $t \in [0, t_1)$. Suppose that this is not the case. Then there exists $y \in U \cap X_+(K)$ such that $\Phi_{t_0}(y) \notin K$ and $O(y) \subseteq U$. Since $y \in X_+(K)$, there exists $x \in K$ such that $y > x$. By invariance of K, we have $\Phi_{t_0}(y) \neq \Phi_{t_0}(x)$ and hence $x \prec y$. It follows from $\omega(x) = K$, $K \cap E = \emptyset$ and Proposition 3.4 that $\omega(y) > \omega(x) = K$ and so

$\omega(y) \subseteq V$. Now, since $O(y) \subseteq U$, we have $\omega(y) \subseteq \bar{U} \subseteq X \setminus V$, which yields a contradiction. Let U be as above. Choose $M = \bar{U} \cap X_+(K)$. Then Theorem 3.12 follows immediately from Proposition 2.6 and Lemma 3.11. $\square$

Theorem 3.13. *Assume that $K < H$ are two minimal compact invariant sets such that there is no equilibrium e satisfying $K < e < H$. Further assume that K contains a point essentially ordered from above and is not an equilibrium. Then $H = \{e\}$ for some equilibrium e and there is a connecting orbit from K to H. The image of this connecting orbit lies in $D \equiv X_+(K) \cap X_-(H)$. An analogous statement holds if H contains a point essentially ordered from below and is not an equilibrium.*

Proof. By assumption of K and Theorem 3.12, there exists a negative orbit $v(\cdot) \in X_+(K) \setminus K$ such that $\alpha(v(\cdot)) = K$. Next we prove that $v(t) \in X_-(H)$. Assume that $p \in K$. Then there exists a sequence $t_n \to -\infty$ such that $v(t_n) \to p$. Since $p \prec H$, by Proposition 3.3, there exist an open neighbourhood V of p and $T_1 > 0$ such that $\Phi_{T_1}(V) \leqslant H$. Now, since $v(t_n) \to p \in V$, there exists $n_0 > 0$ such that $v(t_n) \in V$ for all $n \geqslant n_0$. Hence, $\Phi_{T_1}(v(t_n)) \leqslant H$ for all $n \geqslant n_0$. By invariance of H, we have $v(t) \in X^-(H)$ for all $t \leqslant 0$. Let $v(t) = \Phi_t(v(0))$ for all $t \geqslant 0$. Clearly, the entire orbit $v(\cdot)$ is contained in D. Since $v(-t_0) \in X_+(K) \setminus K$, there exists $q \in K$ such that $v(-t_0) > q$. The fact that $\Phi_{t_0}(v(-t_0)) = v(0) \notin K$ implies that $q \prec v(-t_0)$. Since Φ is essentially strongly order-preserving, there exist $T_2 > 0$ and a neighbourhood V_q of q such that $\Phi_{T_2}(v(-t_0)) \geqslant \Phi_{T_2}(V_q)$. From $\alpha(v(\cdot)) = K$ and $q \in V_q \cap K$, it follows that there exist $T_3 > 0$ and $\tau > 0$ such that $\Phi_{T_2}(v(-t_0)) \geqslant \Phi_{T_2}(v(-T_3 + s))$ for all $s \in [0, \tau]$, that is, $v(T_2 - t_0) \geqslant v(T_2 - T_3 + s)$ for all $s \in [0, \tau]$. By [**13**, Theorem 1.2.1], there exists $e \in E$ such that $v(t) \to e$ as $t \to \infty$. We show next that $H = \{e\}$. Otherwise we have $e \notin H$. As D is closed, it follows that $e \in D$. Thus, there exist $p' \in K$ and $q' \in H$ such that $p' \leqslant e \leqslant q'$. Since $K \cap E = \emptyset$ and $e \notin H$, by Proposition 3.4, we get $K = \omega(p') < e < \omega(p') = H$, a contradiction. This completes the proof. $\square$

4. Generic convergence for essentially strongly order-preserving semiflows

In this section, we establish a new generic convergence principle by combining the obtained results in the previous section and the limit-set dichotomy for essentially strongly order-preserving semiflows.

In what follows, by $Q = \{x \in X : \omega(x) \subseteq E\}$ and $C = \{x \in X : \omega(x) \text{ is a singleton set}\}$ we denote the sets of quasi-convergent points and convergent points, respectively.

We make the following assumption on the space X.

(X) If $a, b \in X$ satisfying $a < b$, then $[a, b]$ is a bounded set in X.

Remark 4.1. If X is a subset of a normal ordered Banach space, then X satisfies the assumption (X).

Remark 4.2. If the assumption (X) holds and A, B are two compact sets in X, then the set $\{x \in X : A \leqslant x \leqslant B\}$ is bounded in X.

Lemma 4.3. *Let* (X) *hold and let* X *be essentially ordered. If* $x \in X \setminus Q$, *then* $x \in \overline{\operatorname{Int} C}$.

Proof. Without loss of generality, we may assume that X is essentially ordered from above. Let $A = \{y \in X : \omega(x) \leqslant y\}$. We show that $A \cap E \neq \emptyset$. Otherwise we have $A \cap E = \emptyset$. As X is essentially ordered from above, it follows that there exist $y, z \in X$ such that $x \prec y \prec z$. Since $A \cap E = \emptyset$ and $x \notin Q$, by Proposition 3.4, we obtain $\omega(x) < \omega(y) < \omega(z)$. Hence, there exist two minimal compact invariant sets K and H such that $K \subseteq \omega(z)$, $H \subseteq \omega(y)$, $K \cap E = \emptyset$ and $H \cap E = \emptyset$, because $\omega(z)$ and $\omega(y)$ are compact invariant sets. By Lemma 3.11 and Theorem 3.13, there exists $e \in E$ such that $H = \{e\}$, which yields a contradiction. Therefore, $A \cap E \neq \emptyset$, that is, $B \equiv \{e \in E : \omega(x) \leqslant e\} \neq \emptyset$. Choose $f \in B$. By assumptions (X) and (C), the set $D \equiv \{e \in B : e \leqslant f\}$ is compact in X. By Zorn's lemma, D must have a minimal element p. If $p \in \omega(x)$, then Proposition 3.2 implies that $\omega(x) = \{p\}$: a contradiction. If $p \notin \omega(x)$, then $\omega(x) < p$. By Proposition 3.3, there exist an open set $\tilde{U} \subseteq X$ and $\tilde{T}_0 > 0$ such that $\omega(x) \subseteq \tilde{U}$ and $\Phi_{\tilde{T}_0}(\tilde{U}) \leqslant p$. Again by definition of $\omega(x)$, there exists $\tilde{T}_1 > 0$ such that $\Phi_{\tilde{T}_1}(x) \in \tilde{U}$. Let $\tilde{V} = (\Phi_{\tilde{T}_1})^{-1}(\tilde{U})$ and $\tilde{T}_2 = \tilde{T}_0 + \tilde{T}_1$. Then $\Phi_{\tilde{T}_2}(\tilde{V}) \leqslant p$ and $\tilde{V}$ is an open neighbourhood of x. Hence, we have $\omega(\tilde{y}) \leqslant p$ for all $\tilde{y} \in \tilde{V}$. Assume that $V \subseteq \tilde{V}$ is an open neighbourhood of x. Below we claim that there exists $y \in V$ such that $x \prec y$ and $\omega(y) = \{p\}$. If this claim is not true, then for any $y \in V$ we have $\omega(y) < p$ whenever $x \prec y$. Since X is essentially ordered from above, there exist $x', x'' \in V$ such that $x \prec x' \prec x''$. It follows from Proposition 3.4, $x \notin Q$ and the choice of p that $\omega(x) < \omega(x') < \omega(x'')$,

$$\omega(x') \cap E = \emptyset, \quad \omega(x'') \cap E = \emptyset, \quad \{h \in X : \omega(x') < h < \omega(x'')\} \cap E = \emptyset.$$

Applying Theorem 3.13 yields that $\omega(x'')$ is a singleton set. This is a contradiction and proves the above claim. Let y be defined as in the above claim. As X is essentially ordered from above, it follows that there exists $z \in V$ such that $y \prec z$. Thus, there exist $\tilde{T}_3 > 0$ and an open neighbourhood $V_z \subseteq V$ of z such that $\Phi_{\tilde{T}_3}(y) \leqslant \Phi_{\tilde{T}_3}(V_z)$. Applying Proposition 3.4 again, we have either $\omega(y) < \omega(u)$ for all $u \in V_z$ or $\omega(y) = \omega(u)$ for all $u \in V_z$. Since $V_z \subseteq V \subseteq \tilde{V}$ implies that $\omega(u) \leqslant p$ for all $u \in V_z$, it follows that $\omega(u) = \{p\}$ for all $u \in V_z$. Hence, $x \in \overline{\operatorname{Int} C}$. This completes the proof. □

Applying Lemma 4.3, we can obtain the following result.

Theorem 4.4. *Let* (X) *hold and let* X *be essentially ordered. Then* $X \setminus Q \subseteq \overline{\operatorname{Int} C}$. *Hence,* $\operatorname{Int} Q$ *is dense.*

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