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ON MEROMORPHIC FUNCTIONS OF ONE COMPLEX VARIABLE HAVING ALGEBRAIC LAURENT COEFFICIENTS

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Dedicated to Professor Theodor Schneider
on the occasion of his seventieth birthday

We study the set of points at which two algebraically independent meromorphic functions have algebraic coefficients in their Laurent expansions. After a survey of the present knowledge in this field, we obtain two general transcendence criteria which sharpen previous results of Straus, Schneider and Lang. As a corollary, we give a new proof, based on Gel'fond's method, of some of Siegel's results on E-functions.

Let f_1, f_2 be two functions meromorphic on $\mathbb{C}$, of finite order ρ_1, ρ_2, which are algebraically independent. We denote by $B(f_1, f_2)$ the set of points w in $\mathbb{C}$ such that the Laurent expansions of f_1 and f_2 at w have algebraic coefficients. When $f_1(z) = z$ and $f_2 = f$ is a transcendental meromorphic function of order ρ, we denote this set by $A(f)$.

It is well known (see [12], §36 for similar constructions) that one can construct an entire transcendental function f of order 0 such that for any algebraic number α, the Taylor coefficients of f at α belong to $\mathbb{Q}(\alpha)$; thus $A(f)$ can be the set $\overline{\mathbb{Q}}$ of all algebraic numbers.

However in certain circumstances one can expect that $B(f_1, f_2)$ is a finite set of cardinality at most $\rho_1 + \rho_2$. We obtain some results in

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this direction under further arithmetical assumptions on the Laurent coefficients, involving the growth of their heights.

Although we limit our study to meromorphic functions on the complex plane, it is worth mentioning that our results can be extended to the case where the functions under consideration are meromorphic except at a finite set of points (not only the point at infinity), assuming their orders at these essential singularities are finite (see [3], Proposition 1 and §5). Multidimensional analogues of our results can also be obtained (see [20], and the end of this paper).

We start with a short historical survey of both problems $A(f)$ and $B(f_1, f_2)$. Further references are given in [2], [7] and [8].

It will be convenient to introduce the following two notations, where m is a rational integer:

if m is non negative, D^m denotes the mth iterate of the derivative operator $D = d/dz$;

if f is a function which is meromorphic at a point z_0 , then $\frac{1}{m!}\Delta^m f(z_0)$ denotes the coefficient of $(z-z_0)^m$ in the Laurent expansion of f at z_0 .

1. Historical survey

1. PROBLEM $A(f)$

Pólya was the first to study the connections between the growth of an entire function and the arithmetic nature of its Taylor coefficients at the origin. In particular, he gave a lower bound for $|f|_R = \max_{|z|=R} |f(z)|$ when f is a transcendental entire function such that all the numbers $D^m f(0)$, $m \in \mathbb{N}$, are rational integers [13].

By a completely different method (which has been extended to the case of several variables by Gross [9]), Straus [19] extended Pólya's estimate to the case where all numbers $D^m f(h)$ (for $m \in \mathbb{N}$, $h = 0, 1, \ldots, k-1$) are rational integers. Straus proved that such a function is of order at least k , where the order ρ of an entire function is defined by

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$$\rho = \limsup_{R\to+\infty} \frac{\operatorname{Log}\operatorname{Log}|f|_R}{\operatorname{Log} R} .$$

Moreover, in the case $\rho = k$, Straus gave a lower bound for the type of f. We now enunciate the most far reaching generalization obtained by Straus (disregarding type).

THEOREM 1.1 (Straus [19], Theorem 4). *Let f be a transcendental entire function of finite order ρ. Let K be a number field of degree δ over $\mathbb{Q}$, and let s_1, t_1 be two non negative real numbers. Denote by S_1 the set of points w of K such that all the numbers $D^m f(w)$ (with $m \in \mathbb{N}$) are in K, and such that*

$$\limsup_{m\to+\infty} \frac{\operatorname{Log}\boxed{D^m f(w)}}{m \operatorname{Log} m} \le s_1 \tag{1.1.1}$$

and

$$\limsup_{m\to+\infty} \frac{\operatorname{Log} d_m}{m \operatorname{Log} m} \le t_1 \tag{1.1.2}$$

where d_m is the least positive integer such that all the numbers $d_m D^\mu f(w)$ (for $0 \le \mu \le m$) are algebraic integers.

Then S_1 is finite, and $\operatorname{card} S_1 \le \rho\delta(1+t_1) + \rho(\delta-1)s_1$.

As usual, we have assumed that the algebraic closure of $\mathbb{Q}$ is embedded into $\mathbb{C}$, and we have denoted by $\boxed{\alpha}$ the maximum of the absolute values of the conjugates of an algebraic number α.

Under a slightly stronger hypothesis on the denominators (see Remark 2.2 below), we shall improve the conclusion of Theorem 1.1 to $\operatorname{card} S_1 \le \rho(1+\delta t_1) + \rho(\delta-1)s_1$ (*cf.* Theorem A below). But, as we shall see, Straus' theorem is the single result requiring such a weak assumption on denominators. Unfortunately, his method can be applied only to an entire transcendental function at algebraic points, and has not yet been generalised either to a meromorphic transcendental function or to the general problem $B(f_1, f_2)$.

2. PROBLEM $B(f_1, f_2)$

In 1949, Schneider obtained a very powerful result concerning algebraically independent meromorphic functions of finite order. Recall that a meromorphic function in $\mathbb{C}$ is said to be of finite order at most ρ if it can be expressed as a quotient of two entire functions of order at most ρ (Nevanlinna's theory shows that this definition makes sense). Then:

THEOREM 1.2 (Schneider [15], Satz III). *Let $\{\zeta_n\}_{n\geq 1}$ be a sequence of complex numbers. For $n \geq 1$, we denote by $z_0^{(n)}, \ldots, z_{k_n}^{(n)}$ the distinct elements of the set $\{\zeta_1, \ldots, \zeta_n\}$, and, for $0 \leq \varkappa \leq k_n$, let $m_\varkappa^{(n)} + 1$ be the multiplicity of $z_\varkappa^{(n)}$ in this set, so that*

$$\sum_{\varkappa=0}^{k_n} \left(m_\varkappa^{(n)}+1\right) = n$$

and

$$\prod_{\nu=1}^{n} \left(X-\zeta_\nu\right) = \prod_{\varkappa=0}^{k_n} \left(X-z_\varkappa^{(n)}\right)^{m_\varkappa^{(n)}+1},$$

for all $n \geq 1$. We then define

$$m^{(n)} = \max_{0\leq\varkappa\leq k_n} m_\varkappa^{(n)},$$

$$r_n = \max_{1\leq\nu\leq n} |\zeta_\nu| = \max_{0\leq\varkappa\leq k_n} \left|z_\varkappa^{(n)}\right|,$$

and

$$\alpha = \liminf_{n\to+\infty} \left((\operatorname{Log} n)/(\operatorname{Log} r_n)\right),$$

and we assume that

$$m^{(n)} \leq n/\operatorname{Log} n \quad \textit{for all} \quad n \geq 1 .$$

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Let K *be a number field, and* $f_1, \ldots, f_l$ *be* l *algebraically independant meromorphic functions of order at most* $\rho_1, \ldots, \rho_l$ *respectively, such that all the numbers* $D^\mu f_j\left(z_\varkappa^{(n)}\right)$, *for* $0 \le \mu \le m_\varkappa^{(n)}$, $0 \le \varkappa \le k_n$, $n \ge 1$, $1 \le j \le l$ *are defined and belong to* K. *For* $n \ge 1$, $0 \le \varkappa \le k_n$ *and* $1 \le j \le l$, *we denote by* $d_j\left(z_\varkappa^{(n)}\right)$ *a common denominator of the algebraic numbers* $\left\{D^\mu f_j\left(z_\varkappa^{(n)}\right);\ 0 \le \mu \le m_\varkappa^{(n)}\right\}$.

Finally, for $1 \le j \le l$, *let* h_j *be an entire function of order at most* ρ_j, *such that* $h_j f_j$ *is entire (hence of order at most* ρ_j*), and* $h_j(\zeta_n) \ne 0$ *for all* $n \ge 1$. *We assume that*

$$\limsup_{n\to+\infty} \frac{1}{\operatorname{Log} n} \operatorname{Log}\operatorname{Log} \max_{0\le\varkappa\le k_n} \left|h_j\left(z_\varkappa^{(n)}\right)\right|^{-1} \le \frac{\rho_j}{a}$$

and

$$\limsup_{n\to+\infty} \frac{1}{\operatorname{Log} n} \operatorname{Log}\operatorname{Log} \max_{\substack{0\le\varkappa\le k_n \\ 0\le\mu\le m_\varkappa^{(n)}}} \left\{\left|\overline{D^\mu f_j\left(z_\varkappa^{(n)}\right)}\right|;\ d_j\left(z_\varkappa^{(n)}\right)\right\} \le \frac{\rho_j}{a}.$$

Then $\rho_1 + \ldots + \rho_l \le (l-1)a$.

In spite of its technical nature, this result should be better known. It is the first general criterion of transcendence which contains the theorem of Hermite-Lindemann, as well as Schneider's solution of Hilbert's seventh problem (take $f_1(z) = z$, $f_2(z) = \alpha^z$, and $m^{(n)} = 0$ for all n, so that, in this case, no derivative is involved). Gel'fond's solution is not contained in Theorem 1.2 (indeed, the hypothesis $m^{(n)} \le n/\operatorname{Log} n$ implies that the sequence $\{\zeta_n\}$ contains infinitely many distinct points). This fact led Schneider to give another version of his theorem.

THEOREM 1.3 (Schneider [*16*], Satz 12). *Let* f_1, f_2 *be two algebraically independent meromorphic functions of order at most* ρ. *Let* K *be a number field of degree* δ *over* $\mathbb{Q}$ *and let* s *be a non negative*

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real number. Denote by S_2 *the set of complex points* w *such that, for* $j = 1, 2$ *, and all* $m \in \mathbb{N}$ *, the numbers* $D^m f_j(w)$ *are defined, belong to* K *and satisfy*

$$(1.3.1) \qquad \limsup_{m\to+\infty} \frac{\operatorname{Log} \overline{|D^m f_j(w)|}}{m \operatorname{Log} m} \le s \, ,$$

$$(1.3.2) \qquad q_w^{m+1} D^m f_j(w) \ \textit{are algebraic integers,}$$

where q_w *denotes a positive rational integer depending only on* w *,* f_1 *and* f_2 *.*

Then S_2 *is finite, and* $\operatorname{card} S_2 \le (2\rho+1)\big(\delta(2s+1)-s+(\tfrac{1}{2})\big)$.

2. Variations with the degrees

Now let f be a transcendental meromorphic function of finite order ρ . Let $S(f)$ be the set of w in $\mathbb{C}$ where f is analytic and such that $D^m f(w) \in \mathbb{Z}$ for all m in $\mathbb{N}$. From Straus' result, we deduce that the intersection of $S(f)$ with any number field K has at most $[K : \mathbb{Q}]\rho$ elements if f is entire. In the general case where f is meromorphic, Theorem 1.3 still implies that $K \cap S(f)$ is finite (with a slightly weaker bound for its cardinality). Using a development of Schneider's method which involves a refined version of Siegel's lemma due to Mignotte, it is possible to show that the set of w in $S(f)$ which are algebraic of degree less than or equal to d_0 is finite with at most $d_0\rho$ elements,

$$\sum_{d=1}^{d_0} \operatorname{card}\big(\overline{\mathbb{Q}}_d \cap S(f)\big) \le d_0\rho \, ,$$

if $\overline{\mathbb{Q}}_d$ is the set of algebraic numbers of degree d . In [1], the sharper inequality

$$\sum_{d\ge 1} \frac{1}{d} \operatorname{card}\big(\overline{\mathbb{Q}}_d \cap S(f)\big) \le \rho \, ,$$

was proved. Using his method of conjugate auxiliary functions, Choodnovsky [4] succeeded in proving that $\overline{\mathbb{Q}} \cap S(f)$ is finite and $\operatorname{Card} \overline{\mathbb{Q}} \cap S(f) \le \rho$.

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THEOREM 2.1 (Choodnovsky [4]). *Let* f *be a transcendental meromorphic function of order* ρ . *The set of algebraic numbers* α *such that* $D^m f(\alpha)$ *is defined and belongs to* $\mathbb{Z}$ *for all* $m \in \mathbb{N}$ *is finite with at most* ρ *elements.*

A simpler proof of this result has been derived by Reyssat [14]. Further discussion of this topic and its generalization to several variables can be found in [5] and [6].

We now present some general results concerning the dependence on the degrees in problems $A(f)$ and $B(f_1, f_2)$. We recall the notation introduced in the beginning of this paper.

1. Problem $A(f)$

THEOREM A. *Let* f *be a transcendental meromorphic function of order at most* ρ *and let* S_1 *be a subset of the field of algebraic numbers. To each element* w *of* S_1 , *we associate a number field* K_w , *and two non negative real numbers* s_w, t_w . *We denote the degree of* K_w *over* $\mathbb{Q}$ *by* δ_w *if* K_w *is not totally imaginary, by* $2\delta_w$ *otherwise.*

Assume that for all w *belonging to* S_1 , *and all rational integers* m , *the numbers* w *and* $\Delta^m f(w)$ *are elements of* K_w , *and, for* $\Lambda \in \mathbb{N}$, *denote by* $d_{w,m,\Lambda}$ *the least common denominator of the algebraic numbers* $\{\Delta^\mu f^\lambda(w);\ \mu \le m,\ 0 \le \lambda \le \Lambda\}$. *Suppose further that*

$$\text{(2.A.1)} \qquad \limsup_{m\to+\infty} \frac{\operatorname{Log} \boxed{\Delta^m f(w)}}{m \operatorname{Log} m} \le s_w$$

and

$$\text{(2.A.2)} \qquad \text{for all } \Lambda \in \mathbb{N}, \quad \limsup_{m\to+\infty} \frac{\operatorname{Log} d_{w,m,\Lambda}}{m \operatorname{Log} m} \le t_w .$$

Then

$$\sum_{w\in S_1} \frac{1}{(\delta_w - 1)s_w + \delta_w t_w + 1} \le \rho .$$

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REMARK 2.0. The following obvious consequence should be kept in mind while reading this paper: under the hypothesis of Theorem A, if there exist an algebraic number ω such that $(\delta_\omega-1)s_\omega + \delta_\omega t_\omega + 1 = 1/\rho$, then the set S_1 is finite (and reduced to $\{\omega\}$).

We now turn to the computation of s_ω, t_ω in special cases.

REMARK 2.1. Assume that for all $m \in \mathbb{Z}$ and $\omega \in S_1$, the numbers $\Delta^m f(\omega)$ are in an imaginary quadratic extension of $\mathbb{Q}$. Then, hypothesis (2.A.1) is satisfied with

$$s_\omega = 1 - (1/\rho) \quad \text{if } f \text{ is entire,}$$

$$s_\omega = 1 \qquad \text{in the general case}$$

(see [20], Lemma 5.3, and [1], Appendix).

REMARK 2.2. As can easily be checked with the help of Leibnitz formula, Straus' hypothesis

$$\limsup_{m\to+\infty} \frac{\operatorname{Log} d_{\omega,m,1}}{m \operatorname{Log} m} \le t_1$$

implies

$$\limsup_{m\to+\infty} \frac{\operatorname{Log} d_{\omega,m,\Lambda}}{m \operatorname{Log} m} \le \Lambda t_1 .$$

In particular, if $t_1 = 0$, then (2.A.2) is satisfied with $t_\omega = 0$. Note that for $\omega = 0$, the assumption $t_1 = 0$ is the condition imposed on the denominators of the Taylor coefficients of Siegel's E-functions (see [18], Condition 3, p. 33).

REMARK 2.3. Here is another situation where t_ω is easily bounded: we assume (*cf.* [1], Condition H^3_ω , p. 2) that (2.A.2 *bis*) there exists a non negative integer ν_ω , and a sequence of positive integers $q_m = q_m(\omega)$ such that, for all $m \in \mathbb{N}$, $d_{\omega,m,1}$ divides $q_m(m!)^{\nu_\omega}$, and

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$\limsup_{m\to+\infty} \frac{\operatorname{Log} q_m}{m}$ is finite.

Then we deduce from Leibnitz formula that (2.A.2) is satisfied with $t_w \leq \nu_w$. Note that the assumption $\nu_0 = 0$ is the condition on the denominators of the Taylor coefficients of E-functions in Lang's definition ([11], Condition E2, p. 76).

Of course, hypothesis (2.A.2) is satisfied with $t_w = 0$ if the numbers $D^m f(w)$ are algebraic integers. As a corollary one obtains the transcendency of such numbers as π, Log 2, $\operatorname{Log}(p/q)$.

Finally, we note that the method of conjugate auxiliary functions [4] enables us to replace the hypothesis $w \in K_w$ in Theorem A by $w \in \overline{\mathbb{Q}}$ (see Theorem 2.2 below for a similar situation).

2. Problem $B(f_1, f_2)$

The assumption on denominators in the study of the general problem are, as we shall see, slightly stronger than in the preceding theorem.

THEOREM B. *Let f_1, f_2 be two $\mathbb{Q}$-algebraically independant meromorphic functions, of order at most ρ_1, ρ_2 respectively, and let S_2 be a set of distinct complex numbers. To each element w of S_2, we associate a number field K_w, two non negative real numbers $s_1(w), s_2(w)$, and three non negative integers q_w, ν'_w, ν''_w. We denote the degree of K_w over $\mathbb{Q}$ by δ_w if K_w is not totally imaginary, by $2\delta_w$ otherwise. Assume that, for all w belonging to S_2, and all rational integers m, the numbers $\Delta^m f_1(w)$ and $\Delta^m f_2(w)$ are elements of K_w. Suppose further that, for $j = 1, 2$,*

$$\text{(2.B.1)} \qquad \limsup_{m\to+\infty} \frac{\operatorname{Log}\overline{|\Delta^m f_j(w)|}}{m \operatorname{Log} m} \leq s_j(w)$$

and

(2.B.2) *the numbers* $q_w^{m+1}[(\nu_w' m)!]^{\nu_w''}\Delta^m f_j(w)$ *are algebraic integers.*

Then

$$\sum_{w\in S_2} \frac{1}{\delta_w \nu_w'\nu_w''+1+(\delta_w-1)s_w'} \leq \rho_1 + \rho_2 ,$$

where $s_w' = \max\{1; \max_{j=1,2} s_j(w)\}$. *Moreover, if* $s_j(w) = 1 - (1/\rho_j)$ *for all* $w \in S_2$ *and* $j = 1, 2$ *, then*

$$\sum_{w\in S_2} \frac{1}{\delta_w(\rho_1+\rho_2)(\nu_w'\nu_w''+1)-\delta_w+1} \leq 1 .$$

REMARK 2.4. Lemmas 6 and 7 of [*20*] show that the set of functions satisfying (2.B.1) and (2.B.2) at w is a K_w-algebra. More specifically, Leibnitz formula implies that under the assumption (2.B.2), Condition (2.1.2) is satisfied with $t_w \leq \nu_w'\nu_w''$. Note that (2.B.2) is a generalisation of hypothesis (1.3.2), where $\nu_w'' = 0$.

Once again one can study the particular case of Gaussian integers and entire functions. For instance the transcendency of e^π follows from Remarks 2.0 and 2.1 and the consideration of the function $f_1(z) = \exp z$.

In the special case where $f_1(z)$ admits a law of addition over $\mathbb{Q}$ in the sense of [5], that is $f_1(z)$ is either z or $\exp z$, or else $p(z)$, where p is a Weierstrass elliptic function with rational invariants, a stronger result can be proved by means of the method of [*4*]. The result is essentially due to Choodnovsky (see [5], Theorem 10).

THEOREM 2.2. *If* $f_2(z)$ *is* z , $\exp z$ *, or* $p(z)$ *with* $g_2, g_3 \in \mathbb{Q}$ *, then it is possible to replace in Theorem* B *the assumption* $\Delta^m f_1(w) \in K_w$ *by* $\Delta^m f_1(w) \in \overline{\mathbb{Q}}$ *for all* $m \in \mathbb{Z}$.

3. Algebraic differential equations

Theorems A and B above deal with "small" subsets of $A(f)$ and

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$B(f_1, f_2)$. In order to describe larger subsets, we shall now restrict our study to solutions of differential equations of a certain type. The results of this section are taken from [1] and [2].

1. Problem $B(f_1, f_2)$

We first note that the K_w-algebra of functions satisfying (2.B.1) and (2.B.2) at w is mapped into itself by differentiation. However, if we impose that the bound ν'_w is fixed, this will remain true if and only if $\nu''_w = 0$. Conversely, Lang proved (see Lemma 3.2 below) that finitely generated algebras of analytic functions stable under differentiation provide examples of functions satisfying (2.B.2) with $\nu''_w = 0$.

Schneider had already noted this fact in the special case where the algebra generated by the derivatives of a given function is finitely generated, a situation from which he could deduce a lot of transcendence results (see [16], Chapter II, §4). But later on, for his study of the exponential map on group varieties, Lang needed the more general system of differential equation

$$Df_j = P_j(f_1, \ldots, f_l) \quad (j = 1, \ldots, l) ,$$

where the P_j's are polynomials with algebraic coefficients. He proved the following related transcendence result.

THEOREM 3.1 (Lang [10], [11]). *Let K be a number field, and let $f_1, \ldots, f_l$ be l meromorphic functions of order at most ρ . Assume that f_1 and f_2 are algebraically independent over K , and that the derivation D maps the algebra $K[f_1, \ldots, f_l]$ into itself. Denote by Σ_1 the set of complex numbers w such that $f_1, \ldots, f_l$ simultaneously take values in K at w .*

Then Σ_1 is finite and has at most $20\rho[K : \mathbb{Q}]$ elements.

The following lemma explains the connection between the differential hypothesis and Theorem 1.3.

LEMMA 3.2. *Assume the hypothesis of Theorem 3.1 are fulfilled, and*

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let w *be a complex number such that* $K_w = K(f_1(w), \ldots, f_l(w))$ *is a number field. Then, for* $j = 1, \ldots, l$ *, the function* f_j *satisfies Condition* (2.B.2) *of Theorem* B *with* $\nu''_w = 0$ *(that is Condition* 1.3.2 *of Theorem* 1.3*), and Condition* (2.B.1) *with* $s_w = 1$.

For a proof of this result, see [11] (Chapter III, Lemma 1), and [1] (Proposition 2 *bis*), where explicit computations can be found. Applying Theorem B we therefore obtain:

THEOREM 3.3 ([2], Theorem 1). *Assume the hypotheses of Theorem* 3.1 *are fulfilled, and, for any positive integer* δ *, denote by* $\Sigma(\delta)$ *the set of complex numbers* w *such that* $K(f_1(w), \ldots, f_l(w))$ *is an algebraic extension of* K *of degree* δ *. Then*

$$\sum_{\delta=1}^{+\infty} \frac{\operatorname{card} \Sigma(\delta)}{\delta} \leq (\rho_1+\rho_2)[K : \mathbb{Q}] .$$

A further result concerning the case $K = \mathbb{Q}$ when f_1 satisfies a law of addition has been given by Choodnovsky ([5], Theorem 5).

2. Problem $A(f)$

So far we have discussed polynomial differential equations. Suppose now that the differential system $Df_j = P_j(f_1, \ldots, f_l)$ is satisfied with rational functions P_j in $K(X_1, \ldots, X_l)$, and let Q denote a common denominator of the P_i's . Then Theorem 3.3 holds with the additional condition that the numbers w are not zeroes of $Q(f_1(z), \ldots, f_l(z))$ - consider the functions $f_1, \ldots, f_l$, and $f_{l+1} = 1/Q(f_1, \ldots, f_l)$.

In fact, this additional condition cannot be omitted, as can be seen from the following example, taken from [21]. Denote by $\wp$ a Weierstrass elliptic function with algebraic invariants g_2, g_3 , and by σ, ζ the corresponding sigma and zeta functions. Let u be a point which is not a pole of $\wp$ and such that $\wp(u) \in \overline{\mathbb{Q}}$, and let K be a number field containing $g_2, g_3, \wp(u), \wp'(u)$. We consider the functions

$$f_1(z) = \wp(z) , \quad f_2(z) = \frac{\sigma(z-u)}{\sigma(z)\sigma(u)} e^{z\zeta(u)} , \quad f_3(z) = \wp'(z) .$$

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Then we have the following system of differential equations:

$$f_1' = f_3 \ , \quad f_2'(z) = \tfrac{1}{2}\,\frac{f_3(z)+\wp'(u)}{f_1(z)-\wp(u)}\,f_2(z) \ , \quad f_3' = 6f_1^2 - \frac{g_2}{2} \ ;$$

and for all periods ω of $\wp$, the three functions take values in K at the points $u + \omega$. It should be noted however that the derivative f_2' of f_2 takes transcendental values at these points (*cf.* [21], §3). We now discuss a situation related to the case of regular points of linear differential equations where the analysis can be taken further. Namely:

THEOREM 3.4 ([1], Theorem I). *Let* $m \in \mathbb{N}$, $P \in \mathbb{Q}[X_1, X_2]$, *and assume that the differential equation* $z^m Dy = P(z, y)$ *has a solution* f *regular at the origin. Suppose further that* f *is a transcendental meromorphic function of order* $\rho \le 1$ *satisfying hypothesis* (2.A.2 bis) *of Remark* 2.3 *at* $w = 0$, *with* $K_0 = \mathbb{Q}$, $\nu_0 \le (1/\rho) - 1$. *Then, if* α *is a non zero algebraic number which is not a pole of* f , *the number* $f(\alpha)$ *is transcendental.*

Proof. Assume that $f(\alpha)$ is algebraic, and let δ_α denote the degree of $\mathbb{Q}(\alpha, f(\alpha))$ over $\mathbb{Q}$. Applying Lemma 3.2 to the functions $f_1(z) = z$, $f_2(z) = f(z)$, $f_3(z) = 1/z$, we conclude that f satisfies hypotheses (2.A.1) and (2.A.2) of Theorem A at $w = \alpha$, with $s_\alpha = 1$, $t_\alpha = 0$ (for this last assertion see Remark 2.2). On the other hand, the assumptions of Theorem 3.4, together with Remarks 2.1 and 2.3 imply that these hypotheses are satisfied at $w = 0$, with $s_0 = 1$, $t_0 \le (1/\rho) - 1$. Therefore it follows from Theorem A that

$$\frac{1}{(1/\rho)-1+1} + \frac{1}{\delta_\alpha} \le \rho \ ;$$

this inequality provides the desired contradiction.

As explained in [1], the ideas discussed above can be used to recover in a very simple way some of Siegel's results on E-functions. For instance, Theorem A, applied to the logarithmic derivative of the Bessel function of order 0 yields the transcendency of the continued fraction [1, 2, 3, ...] (see [2], §4). Theorem B, on the other hand, implies another well known result of Siegel, namely, the zeroes of the incomplete gamma function

$$\Gamma_i(\lambda) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!(n+\lambda)}$$

are irrational (see [1], I, §3).

To prove this assertion consider a complex number $\lambda \neq -1, -2, \ldots$. The differential equation

$$zy' + (\lambda - z)y = \lambda$$

admits a solution F_λ regular at the origin where its Taylor expansion is given by $F_\lambda(0) = 1$, and, for $m \geq 1$, by

$$D^m F_\lambda(0) = m!/[(\lambda+1) \ldots (\lambda+m)] \ .$$

Hence F_λ is an entire function of order 1 . On the other hand, Satz 5 of [18] implies that F_λ satisfies (2.A.2 *bis*) at $w = 0$, with $v_0 = 0$, if λ is a rational number.

Now Γ_i is the analytic continuation of the function

$$\lambda \to \int_0^1 t^{\lambda-1} e^{-t} dt \ ,$$

while F_λ is given, for positive real values of λ , by

$$F_\lambda(z) = \lambda e^z \int_0^1 t^{\lambda-1} e^{-tz} dt \ .$$

Thus $\Gamma_i(\lambda) = (1/\lambda e) F_\lambda(1)$. But Theorem 3.4 implies that $F_\lambda(1)$ is transcendental if λ is rational, and the claim follows.

4. The proofs

We now turn to the proof of Theorems A and B. The ideas and notations of [20] enable us to give a single proof for both theorems: in the case of Theorem A we put $\nu = 1$, $f_1 = f$, $\rho_1 = \rho$, $S = S_1$; in the case of Theorem B we put $\nu = 2$, $\rho = \rho_1 + \rho_2$, $S = S_2$, and

$$s_\omega = s'_\omega \quad \text{if} \quad \max_{j=1,2} s_j(\omega) \geq 1 ,$$

$$s_\omega = 1 - \min_{j=1,2} \frac{\bigl(1-s_j(\omega)\bigr)\rho_j}{\rho} \quad \text{otherwise;}$$

$$t_\omega = \nu'_\omega \nu''_\omega \quad \text{(see Remark 2.6).}$$

Let $\omega_1, \ldots, \omega_p$ be p distinct elements of S, and, for $i = 1, \ldots, p$, set $c_i = (\delta_{\omega_i} - 1)s_{\omega_i}$, $\gamma_i = \delta_{\omega_i} t_{\omega_i}$. We shall prove that the inequality

$$(I) \qquad \sum_{i=1}^{p} \frac{1}{c_i+\gamma_i+1+\varepsilon} \leq \rho$$

holds for any positive real number ε. Hence the family $\left\{\frac{1}{c_\omega+\gamma_\omega+1}\right\}_{\omega\in S}$ is summable, and its sum is bounded by ρ. This implies the conclusions of Theorems A and B.

We first note that since we are trying to obtain non strict inequalities, there is no restriction in assuming that the orders of the functions are taken in the "strict" meaning of [*20*], §2. With this convention the global estimates of [*20*], §4, 3rd step (computation of $|\Phi|_R$) are still valid. On the other hand, the arithmetical computations will of course be the same as the ones of [*20*] (Lemma 4.1). In fact, our proof resembles that of [*20*] so closely that we shall limit ourselves to a sketch of the constructions, and stress the arguments only when they differ from [*20*].

From now on $\omega_1, \ldots, \omega_p$ are fixed. We choose a sufficiently large integer Λ, and we denote by $\varepsilon_1, \ldots, \varepsilon_{10}$ positive numbers depending on Λ, and which tend towards 0 as Λ tends to infinity. Using the notation stated above (notice that this is consistent with those of [*20*]), we then consider a sufficiently large integer N, and we define Λ_j, L, and $\varphi_{\lambda,l}$ as in [*20*], §4, with $n = 1$.

We shall describe two ways of proving inequality (I). As we shall see, the second one is more suitable to generalisation to the case of n variables.

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α. FIRST METHOD (*cf.* [1], [2])

1st Step. There exist rational integers $p(\lambda, l)$, not all zero, bounded in absolute value by $\exp\left(\varepsilon_1 N \operatorname{Log} N\right)$, such that the function

$$F = \sum_{\lambda} \sum_{l} p(\lambda, l)\varphi_{\lambda,l}$$

has a zero of order at least N at each point $w_1, \ldots, w_p$.

Indeed, this amounts to solving a linear system with coefficients in the composition of the number fields K_{w_i}. We note that if f, g are meromorphic at z_0 and if n is a rational integer, then

$$\Delta^n(fg)\left(z_0\right) = \sum_{h\in\mathbb{Z}} \binom{n}{h}\Delta^h f\left(z_0\right)\Delta^{n-h} g\left(z_0\right) .$$

Therefore the estimates of §3 of [20] are still valid in the present context, and Siegel's lemma furnishes the desired solution (note, however, that the number of equations here depends on the Λ_j's).

2nd Step. For each $i = 1, \ldots, p$, the point w_i is a zero of F of order greater than or equal to N. We denote this order by m_i, and we set $\xi_i = D^{m_i} F\left(w_i\right)$. Then

$$\operatorname{Log}\left|\xi_i\right| > -\left(c_i+\gamma_i+\varepsilon_2\right)m_i \operatorname{Log} m_i .$$

This is an application of the usual size inequality in the number field K_{w_i}, together with the estimates of [20].

3rd Step. The analytic nature of this step makes it easier (and sharper) than in the many variables situation of [20]. We claim that, for all $i = 1, \ldots, p$,

$$\operatorname{Log}\left|\xi_i\right| \le \left(1+\varepsilon_3\right)m_i \operatorname{Log} m_i - \frac{m_1+\ldots+m_p}{\rho} \operatorname{Log} m_i .$$

We fix an integer i in the range $1, \ldots, p$, and we first note that, since we are in the case of one variable, the denominator h_j of the meromorphic function f_j (see [20], §4) can be assumed, for $j = 1, \ldots, \nu$, to be non zero at w_i. The entire function $\Phi = F \prod_{j=1} h_j^{\Lambda_j}$ then satisfies $D^{m_i}\Phi(w_i) = \xi_i \prod_{j=1}^{\nu} h_j^{\Lambda_j}(w_i) \neq 0$, and has order at least m_k at each point w_k, for $k = 1, \ldots, p$. Hence Φ has at least $m_1 + \ldots + m_p$ zeroes (counted with their multiplicities) in the disk $|z| < r$, where $r = \max_{k=1,\ldots,p} (|w_k|+1)$. Using the estimates of [20] we see that the one variable version of Schwarz lemma, applied to the function Φ on a disk of radius $R = m_i^{1/\rho}$, and Cauchy's inequalities at w_i, imply the desired upper bound for $|\xi_i|$.

4th Step. Combining the preceding inequalities, we obtain, for all $i = 1, \ldots, p$,

$$m_1 + \ldots + m_p \leq \rho(c_i+\gamma_i+1+\varepsilon_4)m_i .$$

A term to term summation yields

$$\sum_{i=1}^{p} \frac{1}{c_i+\gamma_i+1+\varepsilon_4} \leq \rho ,$$

which is inequality *(1)*, with $\varepsilon = \varepsilon_4$.

β. SECOND METHOD

The idea of this method consists in weighting the orders of the zeroes $w_1, \ldots, w_p$. The integral part of a positive real number x will be denoted by $[x]$.

1st Step. Let $N_i = \left[\frac{N}{c_i+\gamma_i+1}\right]$ for $i = 1, \ldots, p$. There exist rational integers $p(\lambda, l)$, not all zero, bounded in absolute value by $\exp(\varepsilon_5 N \operatorname{Log} N)$, such that the function $F = \sum_\lambda \sum_l p(\lambda, l)\varphi_{\lambda,l}$ has a zero of order at least N_i at each point w_i, for $i = 1, \ldots, p$.

2nd Step. There exists a minimal integer $M \geq N$ such that, for each

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$i = 1, \ldots, p$, and all non negative integers $k_i < \left[\frac{M}{c_i+\gamma_i+1}\right]$, we have $D^{k_i}F(w_i) = 0$. Let $w = w_{i_0}$ be one of the points w_i at which F has a zero of order $m = m_{i_0} = \left[\frac{M}{c_{i_0}+\gamma_{i_0}+1}\right]$, and set $\xi = D^m F(w)$. Then

$$\operatorname{Log}|\xi| \geq -(c_{i_0}+\gamma_{i_0}+\varepsilon_6)m \operatorname{Log} m .$$

3rd Step. Follow the 3rd step of the preceding method, replacing i by i_0 . The definition of M shows that the function Φ has here at least $\sum_{i=1}^{p} \left[\frac{M}{c_i+\gamma_i+1}\right]$ zeroes (counted with their multiplicities) in the disk $|z| < r$. Applying Schwarz' lemma on a disk of radius $m^{1/\rho}$, we obtain

$$\operatorname{Log}|\xi| \leq (1+\varepsilon_7)m \operatorname{Log} m - \frac{1}{\rho}\left(\sum_{i=1}^{p} \frac{M}{c_i+\gamma_i+1+\varepsilon_8}\right) \operatorname{Log} m .$$

4th Step. A comparison of the preceding inequalities yields

$$M \sum_{i=1}^{p} \frac{1}{c_i+\gamma_i+1+\varepsilon_8} \leq \rho(c_{i_0}+\gamma_{i_0}+1+\varepsilon_9)m .$$

Recalling the definition of m , and dividing both terms by M , we obtain

$$\sum_{i=1}^{p} \frac{1}{c_i+\gamma_i+1+\varepsilon_8} \leq \rho(1+\varepsilon_{10}) ,$$

from which inequality *(1)* follows.

γ. A REMARK ON THE n-DIMENSIONAL CASE

Using the latter method, together with the arguments of [*20*] and [*21*], Chapter 7, it is not difficult to extend the previous results to meromorphic functions in $\mathbb{C}^n$. For the sake of brevity, we shall merely state the corresponding generalization of Theorem A; now, S_1 is a set of algebraic points in $\mathbb{C}^n$ and, for all m in $\mathbb{N}^n$, D^m denotes the partial

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derivative $\partial^{|m|}/\left(\partial^{m_1} z_1 \ldots d^{m_n} z_n\right)$. The conclusion is then the following:

For any finite subset S of S_1 , and any positive real number M , there exists a non zero polynomial P_M in $\mathbb{C}|z_1, \ldots, z_n|$, of degree at most $(M+n-1)\rho$, such that each element w of S is a zero of P_M of order at least $\left[\frac{M}{1+(\delta_w-1)s_w+\delta_w t_w}\right]$.

This result obviously implies Theorems A of [20] and of the present paper.

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