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ON AN INEQUALITY OF PEANO[1]

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Let f be a real valued function on an open subset of R^2. It is assumed that f satisfies Carathéodory's conditions: $f(t, x)$ is continuous in x for each t, Lebesgue measurable in t for each x and there is a locally integrable function $m(t)$ such that $|f(t, x)| \leq m(t)$ uniformly in x. A proof will be given of the following theorem.

THEOREM. *Suppose $u(t)$ is a real valued function on $[t_0, t_0+a)$ such that $f(t,u(t))$ is Lebesgue measurable and locally integrable on $[t_0, t_0+a)$,*

$$u(t_2)-u(t_1) \leq \int_{t_1}^{t_2} f(s, u(s))\, ds \tag{1}$$

for each $t_1, t_2 \in [t_0, t_0+a)$, $t_1 < t_2$, and

$$u(t_0) \leq v_0, \qquad |v_0| < +\infty.$$

Then there exists a solution $v(t)$ (in the sense of Carathéodory) of

$$v' = f(t, v), \qquad v(t_0) = v_0 \tag{2}$$

such that

$$u(t) \leq v(t) \tag{3}$$

on the intersection of the domains of $u(t)$ and $v(t)$.

COROLLARY. *Suppose $u(t)$ is a finite valued, essentially bounded measurable function on $[t_0, t_0+a)$ such that $\bar{D}u(t) < +\infty$ for all t and*

$$\bar{D}u(t) \leq f(t, u(t)), \quad \text{a.e.} \quad [t_0, t_0+a). \tag{1'}$$

Then $u(t_0) \leq v_0$ implies the existence of a solution of (2) *such that* (3) *holds.*

It follows from essential boundedness and measurability of $u(t)$ that $f(t, u(t))$ is integrable (cf. [8, p. 92, Lemma 2.1]). The other conditions of the corollary imply that (1) holds (cf. [4, p. 267, Lemma 2]).

General results for differential inequalities of this type were obtained by Peano [6] and Perron [7] in the case that f and u are continuous on their domains and (1′) holds everywhere. When f satisfies the Carathéodory conditions such results

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have been obtained for the inequality (1′) by Cafiero [1] when u is absolutely continuous locally on $[t_0, t_0+a)$, and by Olech and Opial [5] when u is continuous and locally of bounded variation with nonincreasing singular part. Lakshmikantham [3] has stated a similar result for a more restrictive inequality than (1) in which the left-hand side of (1) is replaced by its absolute value. Although that theorem as stated is true the proof in [3] is incorrect since it is implicitly assumed there that a differential inequality holds everywhere which in fact only holds almost everywhere. All of the results mentioned above are implied by the present theorem; it is not assumed here that u is continuous. The proof is a modified version of the proof of the Peano result given in the book of Kamke [2, pp. 82–86].

The following lemma will be used.

LEMMA. *Let $\theta(\lambda, t)$ be a real valued function defined for $\lambda \in K$, a subset of R^2, and $t \in J$, an interval in R. Suppose that θ is continuous in λ for each t and measurable in t for each λ. Then*

$$\omega(t) = \sup\{\theta(\lambda, t) : \lambda \in K\}$$

is measurable.

Proof of the lemma. Let K_1 be a countable subset of K dense in K and

$$\omega_1(t) = \sup\{\theta(\lambda, t) : \lambda \in K_1\}$$

Clearly $\omega_1(t)$ is measurable and $\omega_1(t) \leq \omega(t)$. It is asserted that $\omega_1(t) = \omega(t)$ and hence $\omega(t)$ is measurable.

To prove this assertion, suppose $\omega_1(t_0) < \omega(t_0)$. By the definition of ω there exists $\lambda_0 \in K$ such that $\omega_1(t_0) < \theta(\lambda_0, t_0)$. But $\theta(\lambda, t_0)$ is continuous on K and hence there is a point $\lambda_1 \in K_1$ sufficiently close to λ_0 to ensure $\omega_1(t_0) < \theta(\lambda_1, t_0)$ contradicting the definition of ω_1.

Proof of the theorem. Let $\omega(t, \varepsilon) = \varepsilon + \sup\{|f(t, x) - f(t, y)| : x \in K, |x-y| < \varepsilon\}$ where K is a compact neighbourhood of v_0. Then for each $\varepsilon > 0$, ω is measurable in t by the lemma, $|\omega(t, \varepsilon)| \leq 2m(t) + \varepsilon$ and $\lim_{\varepsilon \to 0+} \omega(t, \varepsilon) = 0$ for each t. There exists α, $0 < \alpha < a$, such that for all sufficiently small $\varepsilon > 0$ solutions $v_\varepsilon(t)$ to

$$v' = f(t, v) + \omega(t, \varepsilon), \qquad v_\varepsilon(t_0) = v_0 \tag{4}$$

exist on $[t_0, t_0+\alpha]$ and $v_\varepsilon(t) \in K$. It suffices to show that $u(t) \leq v_\varepsilon(t)$; then, by the Arzelà-Ascoli theorem, a subsequence $\{v_{\varepsilon(k)}\}$ of $\{v_\varepsilon\}$ converges uniformly on $[t_0, t_0+\alpha]$ to a solution v of (2) for which (3) holds.

Suppose $u(\tau) > v_\varepsilon(\tau)$ for some $\tau \in (t_0, t_0+\alpha)$. Let $t_1 = \sup\{t < \tau : u(t) \leq v_\varepsilon(t)\}$. It follows from (1) that u is left lower semicontinuous on $(t_0, t_0+\alpha]$ and right upper semicontinuous on $[t_0, t_0+\alpha)$. Hence $t_1 < \tau$. If $t_1 = t_0$ then

$$0 \geq u(t_1) - v_\varepsilon(t_1),$$

by hypothesis, and if $t_1 > t_0$

$$0 \geq \liminf_{t \to t_1 -} \{u(t) - v_\varepsilon(t)\} \geq u(t_1) - v_\varepsilon(t_1);$$

also

$$0 \leq \limsup_{t \to t_1 +} \{u(t) - v_\varepsilon(t)\} \leq u(t_1) - v_\varepsilon(t_1)$$

so that

$$u(t_1) = v_\varepsilon(t_1). \tag{5}$$

Therefore, since $u(t) > v_\varepsilon(t)$, $t \in (t_1, \tau]$, u is also right lower semicontinuous at t_1, i.e. u is right continuous at t_1. From this it follows that there exists a point $t_2 \in (t_1, \tau]$ such that

$$u(t_2) > v_\varepsilon(t_2) \tag{6}$$

and $|u(t) - u(t_1)| < \varepsilon/2$, $|v_\varepsilon(t) - u(t_1)| < \varepsilon/2$ if $t \in [t_1, t_2]$ so that

$$|u(t) - v_\varepsilon(t)| < \varepsilon, \qquad t \in [t_1, t_2]. \tag{7}$$

Hence

$$u(t_2) - v_\varepsilon(t_2) \leq \int_{t_1}^{t_2} \{f(s, u(s)) - f(s, v_\varepsilon(s)) - \omega(s, \varepsilon)\}\, ds$$

from (1), (4), and (5). Therefore, by (7),

$$u(t_2) - v_\varepsilon(t_2) \leq \int_{t_1}^{t_2} \{\omega(s, \varepsilon) - \varepsilon - \omega(s, \varepsilon)\}\, ds$$

$$= -\varepsilon\, |t_2 - t_1| < 0$$

contradicting (6) so that $u(t) \leq v_\varepsilon(t)$ for each $t \in [t_0, t_0 + \alpha]$.

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