Canad. Math. Bull. Vol. **16** (1), 1973

# ON AN INEQUALITY OF PEANO[1]

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Let $f$ be a real valued function on an open subset of $R^2$. It is assumed that $f$ satisfies Carathéodory's conditions: $f(t, x)$ is continuous in $x$ for each $t$, Lebesgue measurable in $t$ for each $x$ and there is a locally integrable function $m(t)$ such that $|f(t, x)| \leq m(t)$ uniformly in $x$. A proof will be given of the following theorem.

THEOREM. *Suppose $u(t)$ is a real valued function on $[t_0, t_0+a)$ such that $f(t,u(t))$ is Lebesgue measurable and locally integrable on $[t_0, t_0+a)$,*

$$u(t_2)-u(t_1) \leq \int_{t_1}^{t_2} f(s, u(s))\, ds \tag{1}$$

*for each $t_1, t_2 \in [t_0, t_0+a)$, $t_1 < t_2$, and*

$$u(t_0) \leq v_0, \qquad |v_0| < +\infty.$$

*Then there exists a solution $v(t)$ (in the sense of Carathéodory) of*

$$v' = f(t, v), \qquad v(t_0) = v_0 \tag{2}$$

*such that*

$$u(t) \leq v(t) \tag{3}$$

*on the intersection of the domains of $u(t)$ and $v(t)$.*

COROLLARY. *Suppose $u(t)$ is a finite valued, essentially bounded measurable function on $[t_0, t_0+a)$ such that $\bar{D}u(t) < +\infty$ for all $t$ and*

$$\bar{D}u(t) \leq f(t, u(t)), \quad \text{a.e.} \quad [t_0, t_0+a). \tag{1$'$}$$

*Then $u(t_0) \leq v_0$ implies the existence of a solution of* (2) *such that* (3) *holds.*

It follows from essential boundedness and measurability of $u(t)$ that $f(t, u(t))$ is integrable (cf. [8, p. 92, Lemma 2.1]). The other conditions of the corollary imply that (1) holds (cf. [4, p. 267, Lemma 2]).

General results for differential inequalities of this type were obtained by Peano [6] and Perron [7] in the case that $f$ and $u$ are continuous on their domains and (1′) holds everywhere. When $f$ satisfies the Carathéodory conditions such results

Received by the editors May 11, 1971 and, in revised form, July 8, 1971.

(1) This work was supported by Defence Research Board Grant DRB-9540-28.



https://doi.org/10.4153/CMB-1973-016-8 Published online by Cambridge University Press

have been obtained for the inequality (1′) by Cafiero [1] when $u$ is absolutely continuous locally on $[t_0, t_0+a)$, and by Olech and Opial [5] when $u$ is continuous and locally of bounded variation with nonincreasing singular part. Lakshmikantham [3] has stated a similar result for a more restrictive inequality than (1) in which the left-hand side of (1) is replaced by its absolute value. Although that theorem as stated is true the proof in [3] is incorrect since it is implicitly assumed there that a differential inequality holds everywhere which in fact only holds almost everywhere. All of the results mentioned above are implied by the present theorem; it is not assumed here that $u$ is continuous. The proof is a modified version of the proof of the Peano result given in the book of Kamke [2, pp. 82–86].

The following lemma will be used.

LEMMA. *Let $\theta(\lambda, t)$ be a real valued function defined for $\lambda \in K$, a subset of $R^2$, and $t \in J$, an interval in $R$. Suppose that $\theta$ is continuous in $\lambda$ for each $t$ and measurable in $t$ for each $\lambda$. Then*

$$\omega(t) = \sup\{\theta(\lambda, t) : \lambda \in K\}$$

*is measurable.*

**Proof of the lemma.** Let $K_1$ be a countable subset of $K$ dense in $K$ and

$$\omega_1(t) = \sup\{\theta(\lambda, t) : \lambda \in K_1\}$$

Clearly $\omega_1(t)$ is measurable and $\omega_1(t) \le \omega(t)$. It is asserted that $\omega_1(t) = \omega(t)$ and hence $\omega(t)$ is measurable.

To prove this assertion, suppose $\omega_1(t_0) < \omega(t_0)$. By the definition of $\omega$ there exists $\lambda_0 \in K$ such that $\omega_1(t_0) < \theta(\lambda_0, t_0)$. But $\theta(\lambda, t_0)$ is continuous on $K$ and hence there is a point $\lambda_1 \in K_1$ sufficiently close to $\lambda_0$ to ensure $\omega_1(t_0) < \theta(\lambda_1, t_0)$ contradicting the definition of $\omega_1$.

**Proof of the theorem.** Let $\omega(t, \varepsilon) = \varepsilon + \sup\{|f(t, x) - f(t, y)| : x \in K, |x-y| < \varepsilon\}$ where $K$ is a compact neighbourhood of $v_0$. Then for each $\varepsilon > 0$, $\omega$ is measurable in $t$ by the lemma, $|\omega(t, \varepsilon)| \le 2m(t) + \varepsilon$ and $\lim_{\varepsilon \to 0+} \omega(t, \varepsilon) = 0$ for each $t$. There exists $\alpha$, $0 < \alpha < a$, such that for all sufficiently small $\varepsilon > 0$ solutions $v_\varepsilon(t)$ to

$$v' = f(t, v) + \omega(t, \varepsilon), \qquad v_\varepsilon(t_0) = v_0 \tag{4}$$

exist on $[t_0, t_0+\alpha]$ and $v_\varepsilon(t) \in K$. It suffices to show that $u(t) \le v_\varepsilon(t)$; then, by the Arzelà-Ascoli theorem, a subsequence $\{v_{\varepsilon(k)}\}$ of $\{v_\varepsilon\}$ converges uniformly on $[t_0, t_0+\alpha]$ to a solution $v$ of (2) for which (3) holds.

Suppose $u(\tau) > v_\varepsilon(\tau)$ for some $\tau \in (t_0, t_0+\alpha)$. Let $t_1 = \sup\{t < \tau : u(t) \le v_\varepsilon(t)\}$. It follows from (1) that $u$ is left lower semicontinuous on $(t_0, t_0+\alpha]$ and right upper semicontinuous on $[t_0, t_0+\alpha)$. Hence $t_1 < \tau$. If $t_1 = t_0$ then

$$0 \ge u(t_1) - v_\varepsilon(t_1),$$

by hypothesis, and if $t_1 > t_0$

$$0 \geq \liminf_{t\to t_1-}\{u(t)-v_\varepsilon(t)\} \geq u(t_1)-v_\varepsilon(t_1);$$

also

$$0 \leq \limsup_{t\to t_1+}\{u(t)-v_\varepsilon(t)\} \leq u(t_1)-v_\varepsilon(t_1)$$

so that

$$u(t_1) = v_\varepsilon(t_1). \tag{5}$$

Therefore, since $u(t) > v_\varepsilon(t)$, $t \in (t_1, \tau]$, $u$ is also right lower semicontinuous at $t_1$, i.e. $u$ is right continuous at $t_1$. From this it follows that there exists a point $t_2 \in (t_1, \tau]$ such that

$$u(t_2) > v_\varepsilon(t_2) \tag{6}$$

and $|u(t)-u(t_1)| < \varepsilon/2$, $|v_\varepsilon(t)-u(t_1)| < \varepsilon/2$ if $t \in [t_1, t_2]$ so that

$$|u(t)-v_\varepsilon(t)| < \varepsilon, \qquad t \in [t_1, t_2]. \tag{7}$$

Hence

$$u(t_2)-v_\varepsilon(t_2) \leq \int_{t_1}^{t_2} \{f(s, u(s))-f(s, v_\varepsilon(s))-\omega(s, \varepsilon)\}\, ds$$

from (1), (4), and (5). Therefore, by (7),

$$\begin{aligned} u(t_2)-v_\varepsilon(t_2) &\leq \int_{t_1}^{t_2} \{\omega(s, \varepsilon)-\varepsilon-\omega(s, \varepsilon)\}\, ds \\ &= -\varepsilon\, |t_2-t_1| < 0 \end{aligned}$$

contradicting (6) so that $u(t) \leq v_\varepsilon(t)$ for each $t \in [t_0, t_0+\alpha]$.

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