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Dear Editor,

Sharp bounds for winning probabilities in the competitive rank selection problem

1. Introduction

In this problem two players A and B observe sequentially n uniquely rankable options. All arrival orders of ranks are supposed to be equally likely (probability = $1/n!$ each) and A and B have to select one option each. The decision must be based on relative ranks only (no-information game) and A has the priority of choice.

Let $p(n, k)$ be the probability that player A will choose a better rank than player B, given that neither A nor B has stopped (selected an option) before step k. We call $p(n, k)$ the winning probability of A at step k in a n-options game.

Note that $p(n, n)$ is not defined, because if A has not yet stopped on $\{1, 2, \cdots, n-1\}$ then A must select option n and thus B must have stopped earlier.

Enns and Ferenstein [2], who studied this problem as 'the horse game', pointed out already that the $p(n, k)$ are not monotone. Therefore the proof of the existence of $\lim_{n\to\infty} p(n, k(n))$ is not easy (this question will be studied in a more technical paper; see also Enns *et al.* [3]). The corresponding question for the full information game has been completed by Chen *et al.* [1].

Another interesting question is: what is the range of $p(n, k)$ for different n and k? Numerical evidence (already obtained by Enns and Ferenstein) suggest that 1/2 is a lower bound and 3/4 is an upper bound. We now present an elementary probabilistic proof that these values are indeed the sharp uniform bounds. (We formulate our results in terms of $q(n, k) = 1 - p(n, k)$.)

2. Results

Theorem 2.1. *Let* $q(n, k) = 1 - p(n, k)$. *Then* $1/4 \le q(n, k) \le 1/2$ *for all* n, $1 \le k \le n - 1$.

Proof. The step $k = n - 1$ is special in the sense that if A does not stop then B must stop. Therefore A must stop at option number $n - 1$ if $P(A \text{ wins at step } n-1) > 1/2$ and may stop if $P(A \text{ wins at step } n-1) = 1/2$ (but must refuse otherwise). Therefore it is easy to see that $q(n, n-1) \downarrow 1/4$ as $n \to \infty$. Thus $q(n, k) \ge 1/4$ for $k = n - 1$, i.e. for $k + 1 = n$.

Our proof is based on backwards induction. Suppose that

$$q(n, m) \ge 1/4, \qquad k + 1 \le m \le n. \tag{1}$$

We now show that $q(n, k) \ge 1/4$. Let

$$A_k = \{A \text{ accepts option number } k\}$$
$$W(A) = \{A \text{ wins the game }\}$$

and let B_k and $W(B)$ denote the corresponding events for B. Since A_k and B_k are mutually exclusive we have $P(A_k \cup B_k) = P(A_k) + P(B_k)$. Also, clearly, $P(W(B)) = 1 - P(W(A))$.

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Thus we can write

$$q(n,k) = P\left(A_k \cap W(B) \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right) + P\left(B_k \cap W(B) \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right) + P\left(\bar{A}_k \cap \bar{B}_k \cap W(B) \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right), \tag{2}$$

where $C_j = A_j \cup B_j$ and where $\bar{E}$ denotes the complement of E.

We look first at the last term. If both A and B refuse k then both players pass on to step $k+1$. In this case B will win, under optimal play, with probability $q(n, k+1)$, i.e. by the induction hypothesis (1), with probability 1/4 at least. Therefore

$$P\left(\bar{A}_k \cap \bar{B}_k \cap W(B) \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right) \geq \frac{1}{4} P\left(\bar{A}_k \cap \bar{B}_k \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right). \tag{3}$$

Secondly, if A does not accept k, then B has the choice of either stopping at step k or else passing on to step $k+1$. Optimal behaviour forces B to accept k only if this yields a winning probability strictly greater than $q(n, k+1)$, i.e. only if

$$P\left(W(B) \mid B_k \cap \bigcap_{j=1}^{k-1} \bar{C}_j\right) > q(n, k+1) \geq 1/4 \tag{4}$$

and to refuse k if the reverse strict equality $<$ holds. Thus the second term of (2) yields

$$P\left(B_k \cap W(B) \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right) \geq q(n,k+1) P\left(B_k \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right) \geq \frac{1}{4} P\left(B_k \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right). \tag{5}$$

Now, since $A_k \cup B_k \cup (\bar{A}_k \cap \bar{B}_k)$ is the certain event and since (3) and (5) holds, it suffices from (2) to show that

$$P\left(A_k \cap W(B) \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right) \geq \frac{1}{4} P\left(A_k \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right). \tag{6}$$

We note first that, as in the case $k = n-1$, A would act suboptimally if A accepted k unless

$$\binom{k}{r} \Big/ \binom{n}{r} \geq \frac{1}{2}$$

and that $p(n,k) \geq 1/2$ for all $1 \leq k \leq n-1$.

Indeed A can use any strategy B can use (at least) and optimal play must therefore yield a winning probability of 1/2 at least. On the other hand, A must accept if

$$\binom{k}{r} \Big/ \binom{n}{r} > \frac{1}{2}$$

because otherwise B would accept and win with this probability, which again would contradict A's optimal behaviour. Therefore A accepts k under optimal play only if the relative rank r of k satisfies the inequality (see also [2])

$$\binom{k}{r} \Big/ \binom{n}{r} \geq \frac{1}{2}.$$

A wins in this case with this probability

$$\binom{k}{r} \Big/ \binom{n}{r}.$$

Consequently, since all relative ranks are equally likely (Rényi [4]), and since $P(W(A)) = 1 - P(W(B))$,

$$P\left(A_k \cap W(B) \mid \bigcap_{j=1}^{k-1} \bar{C}_j\right) = \frac{1}{k} \sum_{r=1}^{s} \left(1 - \binom{k}{r} \Big/ \binom{n}{r}\right),$$

where

$$s = \sup\left\{r \in \mathbb{N} : \binom{k}{r} \Big/ \binom{n}{r} \geq \frac{1}{2}\right\}.$$

If $s = 0$ then $A_k = \varnothing$, by definition, and nothing remains to be shown. Therefore let $s \geq 1$. We now show that

$$b(n, k, r) = \binom{k}{r} \Big/ \binom{n}{r}$$

is, for all $1 \leq k < n$ and $1 \leq r \leq k$, a convex function of r. Note that

$$b(n, k, r) = \frac{k(k-1)\cdots(k-r+1)}{n(n-1)\cdots(n-r+1)}$$

so that

$$b(n, k, r+1) = b(n, k, r)\frac{k-r}{n-r}.$$

To prove convexity it suffices to show that

$$b(n, k, r+2) + b(n, k, r) \geq 2b(n, k, r+1).$$

But since $1 \leq r \leq k < n$ we can write $k = cn$, $r = dn$ for some $0 < d \leq c < 1$. The validity of the preceding inequality follows then, after straightforward simplifications, from

$$\text{sign}\left\{\frac{(n+1-cn)(1-c)}{n(1-d)^2 + 1 - d}\right\} > 0.$$

Therefore the $b(n, k, r)$ are (strictly) convex in $1 \leq r \leq k$ for all $n \geq k$.

Now let

$$a(s) := \sum_{r=1}^{s} b(n,k,r),$$
$$b(s) := \sum_{r=1}^{s} (1 - b(n,k,r)) = s - a(s).$$

By Rényi's theorem on relative ranks the kth observation has relative rank $r \le k$ with probability $1/k$ (independently of preceding observations). Conditioned on the event that neither A nor B have stopped before k, $a(s)/k$ is thus the probability that A stops on k and wins and $b(s)/k$ the probability that A stops and B wins.

Therefore, to show inequality (6), it suffices to show that

$$\frac{b(s)}{a(s)+b(s)} = \frac{b(s)}{s} \ge \frac{1}{4}, \tag{7}$$

or equivalently, that $b(s) \ge s/4$.

Now,

$$b(s) = s - \sum_{r=1}^{s} b(n,k,r)$$
$$\ge s - \sum_{r=1}^{s} \frac{b(n,k,1) + b(n,k,s+1)}{2} \tag{8}$$
$$\ge s - \sum_{r=1}^{s} \frac{1+\frac{1}{2}}{2} = \tfrac{1}{4}s, \tag{9}$$

where the inequality (8) follows from the convexity of the $b(n,k,r)$ and (9) from the inequality $b(n,k,s+1) < \frac{1}{2} \le b(n,k,s) \le b(n,k,1) \le 1$. This proves (7) which implies (6), and thus the proof is complete.

Corollary 2.1. *The bounds* $1/4 \le q(n,k) \le 1/2$ *are sharp.*

Proof. Since $q(n, n-1) \downarrow 1/4$ as $n \to \infty$ the lower bound is sharp. Since $p(n,k) \ge 1/2$ for all $1 \le k \le n-1$ we have $q(n,k) \le 1/2$ for all $1 \le k \le n-1$, and so 1/2 is an upper bound. This bound is sharp too since $p(2,1) = q(2,1) = 1/2$.

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