Canad. Math. Bull. Vol. 17 (5), 1975

ON THE COMPLETE REGULARITY OF SOME CATEGORY SPACES

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1. **Introduction.** A *category space* is a measure space which is also a topological space, the measure and the topology being related by 'a set is measurable iff it has the Baire property' and 'a set is null iff it is nowhere dense' [4]. We considered some category spaces in [3]; now we show that if a null set is deleted from the space, then the topology can be taken to be completely regular. The essential part of the construction consists of obtaining a suitable refinement of the original sequential covering class and using the consequent strong upper density function to define the required topology. Then the complete regularity follows much as in [1].

The notation and definitions are as in [2], [3]: (X, ρ) is a metric space, τ a gauge on C, a sequential covering class of closed sets, and ϕ is the metric outer measure defined by C and τ.

We assume that $\phi(X)$ is finite, that the regularity conditions given in [2], [3] hold, and that C does not contain any singleton sets.

As we saw in [3], the strong upper density function D defined by C, τ can be used to construct a topology on X: the closure of $A \subseteq X$ is

$$A \cup \{x \mid D(A, x) > 0\}.$$

We will refer to this topology as the D topology to distinguish it from the metric topology. Sets which have the Baire property with respect to the D topology are ϕ-measurable and conversely, and the D-nowhere-dense sets are ϕ-null. Furthermore, if $D(X, x) > 0$ for all $x \in X$, then the ϕ-null sets are D-nowhere-dense. Since $\{x \mid D(X, x) = 0\}$ is ϕ-null [2, theorem 6], its deletion from X would not affect the measure-theoretic properties of X; thus we can suppose that X, with ϕ and the D topology, is a category space.

2. **The D' topology**. For each positive integer n, let $\{I(i, n)\}$ be a sequence of sets from C such that:

$$X = \bigcup_i I(i, n);$$

$$d(I(i, n)) < 1/n \quad \text{for all } i;$$

$$\sum_i \tau(I(i, n)) \leq \phi(X) + 1/n;$$

Received by the editors September 22, 1972.
* Research supported in part by a fellowship from the City of London Polytechnic.

(d is the diameter given by the metric ρ). Choose $m(n)$ so that

$$\sum_{i=m(n)+1}^{\infty} \tau(I(i, n)) < (\tfrac{1}{2})^n.$$

Let

$$A(p) = \bigcap_{n=p}^{\infty} \bigcup_{i=1}^{m(n)} I(i, n)$$

and

$$Z = \bigcup_{p=1}^{\infty} A(p).$$

Also, let W be the class

$$W = \{I(i, n), i = 1, 2, 3, \ldots, m(n); n = 1, 2, 3, \ldots\}.$$

The set Z is sequentially covered by W (although the sets in W need not be subsets of Z), and thus W and τ define a metric outer measure α on Z in the usual way.

THEOREM 1. $\phi(X-Z)=0$ *and, for all* $A\subseteq Z$, $\phi(A)=\alpha(A)$.

Proof. Since

$$X-Z \subseteq \bigcup_{n=p}^{\infty} \bigcup_{i=m(n)+1}^{\infty} I(i, n),$$

$\phi(X-Z)=0$. We prove the second part of the theorem in several steps.

(i) Since $W\subseteq C$, $\phi(A)\leq\alpha(A)$ for all $A\subseteq Z$.

(ii) For $n\geq p$ we have

$$A(p) \subseteq \bigcup_{i=1}^{m(n)} I(i, n)$$

and thus the infimum of all sums of the type $\sum_i \tau(J_i)$ where $J_i \in W$ and $d(J_i)< 1/n$ for all i, and $A(p)\subseteq \bigcup_i J_i$, cannot exceed $\phi(X)+1/n$. Therefore $\alpha(A(p))\leq \phi(X)$ and so $\alpha(Z)\leq\phi(X)$. This, with (i), proves that $\alpha(Z)=\phi(X)$.

(iii) Let A be α-measurable. Then

$$\begin{aligned}\alpha(A)+\alpha(Z-A) &= \alpha(Z)\\ &= \phi(X) = \phi(Z)\\ &\leq \phi(A)+\phi(Z-A)\end{aligned}$$

and so, by (i), we have $\alpha(A)=\phi(A)$.

(iv) Let $A\subseteq Z$ and let B be a Borel subset of Z which is a ϕ-measurable cover for A. There is such a set B because A has a ϕ-measurable Borel cover and because Z is Borel. Then B is α-measurable and so $\alpha(B)=\phi(B)$. Thus

$$\phi(A) = \phi(B) = \alpha(B) \geq \alpha(A)$$

and therefore $\phi(A)=\alpha(A)$.

This concludes the proof of theorem 1.

Now, for $x \in Z$ and $A \subseteq Z$, let

$$D'(A, x) = \lim_{\varepsilon \to 0^+} \sup \frac{\phi(A \cap I)}{\tau(I)}$$

where the supremum is taken for all $I \in W$ such that $x \in I$ and $d(I) < \varepsilon$. Since the sets in W need not be subsets of Z, D' is not, strictly speaking, a strong upper density function as defined in [2], but the proof of [2, Theorem 6] still applies, and so

$$D'(A, x) \geq 1$$

ϕ-almost-everywhere in A. Also, $D'(A, x) \leq D(A, x)$ since $W \subseteq C$, and thus

$$D'(A, x) = 0$$

ϕ-almost-everywhere in $Z - A$ iff A is ϕ-measurable [2, theorems 3, 7, 8]. Therefore D' does define a topology on Z as in [3]. We will call it the D' topology. It has the properties noted in the introduction—if $\{x \mid D'(Z, x) = 0\}$ is deleted from Z, the result is a category space.

Next we prove a type of Lusin-Menchoff theorem.

THEOREM 2. *Let A be a closed subset of Z and B a ϕ-measurable subset of Z such that $A \subseteq i(B)$. Then, there is a closed subset F of Z such that*

$$A \subseteq i(F) \subseteq F \subseteq B.$$

(The set $i(C)$ is the interior of C with respect to the D' topology. It is $C \cap \{x \mid D'(\tilde{C}, x) = 0\}$.)

Proof. For each positive integer n, let

$$R_n = \{x \mid 1/n < \rho(x, A) \leq 1/n+1\} \cap B,$$

and assume that

$$B = A \cup \bigcup_n R_n.$$

Clearly, this assumption can be made without loss of generality. For each $\varepsilon > 0$ there are only a finite number of sets in W with diameter greater than ε, and so $\gamma(\varepsilon) > 0$, where we define $\gamma(\varepsilon)$ to be the infimum of all the numbers $\tau(I)$, where $d(I) > \varepsilon$ and $I \in W$.

Choose F_n to be a closed set such that $F_n \subseteq R_n$ and

$$\phi(R_n - F_n) < \gamma(1/n)/2^n.$$

The set F_n exists because $\phi(X)$ is finite and every subset of X has a Borel cover. Let

$$F = A \cup \bigcup_n F_n$$

and let $x \in A$. Clearly, F is closed, and it only remains to show that $D'(\tilde{F}, x) = 0$.

https://doi.org/10.4153/CMB-1974-118-2 Published online by Cambridge University Press

Let $x \in I \in W$, and suppose that I has a non-empty intersection with one of the sets $R_1, R_2, R_3, \ldots$. Suppose that R_n is the first such set that I intersects. Then

$$\begin{aligned}\phi(\tilde{F} \cap I) &= \phi(\tilde{B} \cap I)+\phi((B-F) \cap I)\\ &= \phi(\tilde{B} \cap I)+\sum_{m=n}^{\infty} \phi((R_m-F_m) \cap I)\\ &\le \phi(\tilde{B} \cap I)+\gamma(1/n)\cdot 2^{1-n}.\end{aligned}$$

Thus, in this case,

$$\frac{\phi(\tilde{F} \cap I)}{\tau(I)} \le \frac{\phi(\tilde{B} \cap I)}{\tau(I)}+2^{1-n},$$

since $d(I)>1/n$. In the other case,

$$\frac{\phi(\tilde{F} \cap I)}{\tau(I)} = \frac{\phi(\tilde{B} \cap I)}{\tau(I)}$$

Therefore, $D'(\tilde{F}, x)=0$ as required.

The complete regularity of the D' topology now follows, using a suitable modification of a lemma due to Zahorski [5]. Alternatively, a proof similar to that of Urysohn's lemma can be given: let $C\subseteq Z$ be D'-closed and let $x \in Z-C$. Let $A=\{x\}$, $B=\tilde{C}$ in theorem 2, and let $O_{1/2}=i(F)$, and $F=F_{1/2}$. Then

$$x \in O_{1/2} \subseteq \bar{O}_{1/2} \subseteq F_{1/2} \subseteq \tilde{C}.$$

by $\bar{O}_{1/2}$ we mean the closure of $O_{1/2}$ with respect to (Z, ρ). We proceed inductively, associating a D'-open set O_u and a metrically closed set F_u with each positive dyadic rational u, so that

$$x \in O_u = i(F_u) \subseteq \bar{O}_u \subseteq F_u \subseteq \tilde{C},$$

for all such u, and so that if $u<v$, then $F_u\subseteq O_v$. The crucial step in the argument is, to show that having defined F_u and F_v for $u<v$, then there is a closed set F such that

$$F_u \subseteq O = i(F) \subseteq \bar{O} \subseteq O_v = i(F_v).$$

This is done by applying theorem 2, with $A=F_u$, $B=F_v$. It is only important to note that $Cl[i(F)]\subseteq F$, since $\bar{F}\subseteq F$. Thus we obtain a function f which is 0 at x, 1 on C, and which is continuous relative to the D' topology on Z and the usual topology on the real numbers. But, in fact, we obtain more: by using the closed sets $\{F_v\}$, we see that the inverse image, under f, of an open set is a $\mathscr{F}_\sigma$ subset of Z, relative to (Z, ρ). That is, f is of Baire class one, or less. Also, we note that the only property of x we used was that it is a closed subset of Z disjoint from C. Thus we have proved:

THEOREM 3. *Let A be a closed subset of Z and let C be a D'-closed subset of Z which is disjoint from A. Then there is a real-valued function f from Z which has the* *following properties:*

(i) $f(a)=0$ *for all* $a\in A$;

(ii) $f(a)=1$ *for all* $a\in C$;

(iii) f *is continuous with respect to the* D' *topology on* Z *and the usual topology on the reals;*

(iv) f *is of Baire class one, or less, with respect to the metric topology on* Z *and the usual topology on the reals.*

It follows from the complete regularity of the D'-topology that this topology is the coarsest topology on Z for which the D'-continuous functions are continuous. Of course, this class of functions depends on Z, and so it is of interest to know if Z can be chosen to be the original space X.

THEOREM 4. *If* X *is compact with respect to the metric topology, and if the number* β *given in the regularity conditions can be chosen arbitrarily,* $\beta>1$, *then* Z *can be chosen to be* X.

Proof. For each positive integer n, choose $\beta>1$ so that $\beta\phi(X)\leq\phi(X)+1/n$, and choose $\sigma>0$ by assigning $\varepsilon=1/n$ in the regularity conditions. Let $\{I_i\}$ be a sequence of sets from C with union X, each of diameter less than σ, and such that

$$\sum_i \tau(I_i) < \phi(x)+\gamma$$

where

$$\gamma = \phi(X)(1/\beta-1)+1/\beta n.$$

Then

$$\begin{aligned} X &= \bigcup_i I_i \\ &= \bigcup_i \{x \mid \rho(x, I_i) < ad(I_i)\} \\ &= \bigcup_{i=1}^{m} \{x \mid \rho(x, I_i) < ad(I_i)\} \\ &= \bigcup_{i=1}^{m} I_i' \end{aligned}$$

for some integer m. Since $d(I_i')<1/n$ for all i, and

$$\begin{aligned} \sum_{=1}^{m} \tau(I_i') &\leq \beta\sum_{i=1}^{m} \tau(I_i) \\ &\leq \phi(X)+1/n, \end{aligned}$$

the sets $I_1', I_2', \ldots, I_m'$ can be taken as the sets $I(1, n), I(2, n), \ldots, I(m(n), n)$ in the construction of Z. Since their union is X, Z is X.

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