This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

A COMBINATORIAL THEOREM

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Let n be an arbitrary but fixed positive integer. Let T_n be the set of all monotone-increasing n-tuples of positive integers:

$$(1) \qquad (k_1, k_2, \ldots, k_n), \qquad 1 \le k_1 < k_2 < \ldots < k_n .$$

Define

$$(2) \qquad \phi(k_1, \ldots, k_n) = 1 + \sum_{i=1}^{n} \binom{k_i - 1}{i}$$

In this note we prove that ϕ is a 1-1 mapping from T_n onto $\{1, 2, 3, \ldots\}$.

The following formula is well known:

$$(3) \qquad \sum_{i=0}^{n} \binom{m-1+i}{i} = \binom{m+n}{m} ; \forall n \ge 0, \forall m \ge 1 .$$

It can be verified on two lines if we note that the case $n = 0$ is trivial for all $m \ge 1$. Assume (3) holds for $n = k$. Then for all $m \ge 1$:

$$\sum_{i=0}^{k+1} \binom{m-1+i}{i} = \binom{m+k}{k} + \binom{m+k}{k+1} = \binom{m+k+1}{k+1} .$$

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Formula (3) readily implies:

$$1+\binom{k_n-1}{n} \leq \phi(k_1,\ldots,k_n) \leq \phi(k_n-n+1, k_n-n+2,\ldots,k_n-1,k_n)$$

(4) $$= 1+\binom{k_n-n}{1}+\binom{k_n-n+1}{2}+\ldots+\binom{k_n-2}{n-1}+\binom{k_n-1}{n}$$

$$=\binom{k_n}{n}.$$

We write, as usual, $(k_1,\ldots,k_n) < (q_1,\ldots,q_n)$ provided there is an integer t $(1 \leq t \leq n)$ with

(5) $$k_t < q_t\,,\ k_i = q_i\ ;\ i = t+1,\ldots,n.$$

If $(k_1,\ldots,k_n) < (q_1,\ldots,q_n)$, then by (4) and (5)

$$\phi(k_1,\ldots,k_t) \leq \binom{k_t}{t} < 1+\binom{q_t-1}{t} \leq \phi(q_1,\ldots,q_t)$$

and hence $\phi(k_1,\ldots,k_n) < \phi(q_1,\ldots,q_n)$.

For fixed k_n, there are $\binom{k_n-1}{n-1}$ n-tuples (1), since the $k_1,\ldots,k_{n-1}$ are chosen from $1,2,\ldots,k_n-1$. Also, from (4), $\phi(k_1,\ldots,k_n)$ is one of the $\binom{k_n-1}{n-1}$ integers in the interval

(6) $$1+\binom{k_n-1}{n} \leq x \leq \binom{k_n}{n} = \binom{k_n-1}{n-1}+\binom{k_n-1}{n}.$$

Hence for fixed k_n, ϕ is a 1-1 mapping from the subset of T_n with k_n fixed onto the interval (6). Since any positive integer is contained in exactly one interval (6) for some k_n, our main result follows.

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