IDENTICAL RELATIONS IN LOOPS, I

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Dedicated to Professor B. H. Neumann

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The systematic study of identities in groups was begun by B. H. Neumann in 1937, [7]. Although many special loop identities have been extensively investigated, there is not, as yet, a general theory of identical relations in loops. We cannot hope that such a theory will be as rich in results as in the group case (except in the sense that loop theory includes group theory) but there are many interesting problems, including some which have no analogue in the associative case.

After preliminary remarks on varieties and free loops, we begin a classification of loop identities and loop varieties by proving the following loop version of the well-known group theorem, [7]. Any set of identical relations in a loop is equivalent to a collection of commutator-associator identities and an identity

$$x^n \cdot p(x) = 1$$

where $p(x)$ is a commutator-associator word (we write x^n for right-powers, ${}^n x$ for left-powers).

A non-trivial variety and the set of identities it satisfies is called *anti-associative* (*anti-finite*) if it contains no non-trivial associative (finite) loops. We construct examples of such varieties and give an example of a finite loop satisfying the anti-associative identity $x(x \cdot (1/x)) = 1$. Thus, although anti-finiteness clearly implies anti-associativity, the converse is not true. We prove that a variety of loops is anti-associative if and only if it satisfies an identity $x \cdot p(x) = 1$, where $p(x)$ is a commutator-associator word. An example is given of a collection of uncountably many anti-finite varieties of loops, each of which contains the set of identities

$${}^{n!}x = x^{n!+1} \qquad n = 3, 4, 5, \cdots$$

in the identities it satisfies. There are in fact, uncountably many anti-finite varieties defined by equationally complete sets of identities. The other atoms in the lattice of loop varieties are the varieties of abelian groups of prime exponent and a countable number of anti-associative but not anti-finite varieties.

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The general theory of loop varieties including properties of fully invariant subloops, products of loop varieties, nilpotent varieties, will be developed in forthcoming papers.

1. Free loops

A loop is usually defined as a set L, closed under a multiplication $x \cdot y$, containing a neutral element 1 such that $1 \cdot x = x \cdot 1 = x$ and such that for any a, b in L, there are unique solutions u, v to the equations $au = b, va = b$. We will find it necessary to use the alternative definition in terms of three binary operations $x \cdot y$ called *multiplication*, $x \backslash y$ called *left division* and x/y called *right division*. (See [2], [3].) A loop is a non-empty set L closed under these three operations and satisfying the following axioms for all x, y

$$x \cdot (x \backslash y) = y, \quad x \backslash (x \cdot y) = y, \quad (x/y) \cdot y = x, \quad (x \cdot y)/y = x, \quad x \backslash x = y/y$$

It is easy to verify that the further identities

$$x/(y \backslash x) = y, \quad (x/y) \backslash x = y, \quad 1 \cdot x = x \cdot 1 = x, \quad x/1 = 1 \backslash x = x$$

hold for all x, y, where 1 is the neutral element whose existence is guaranteed by the axiom $x \backslash x = y/y$. If we omit this axiom, the remaining axioms characterize the variety of quasigroups. In a loop or quasigroup defined in this way, $u = a \backslash b$, $v = b/a$ are the unique solutions of $au = b$, $va = b$.

A *loop word* in some set of elements S is an expression built up from the elements of $S \cup \{1\}$ and the three loop operations $(\cdot, \backslash, /)$. An equivalence relation is induced on the set of loop words in the elements of $S = \{g_1, g_2, g_3, \cdots\}$ by the loop axioms above, two words being considered equal if we can transform one into the other by a finite sequence of applications of these identities. We obtain, as in the usual construction of a free algebra the *free loop on the free set of generators* $g_1, g_2, g_3, \cdots$. The free loop on a finite set of n generators will be denoted by F_n and on a countably infinite set of generators by F. We will also write F for a free loop if it is immaterial whether it is finitely or infinitely generated. We refer to [3] for a detailed study of free loops.

If $w(x_1, x_2, x_3, \cdots)$ is a loop word in $x_1, x_2, x_3, \cdots$, we say that a loop L satisfies the *identity* $w(x_1, x_2, x_3, \cdots) = 1$ if for any choice of elements $a_1, a_2, a_3, \cdots$ in L, we have $w(a_1, a_2, a_3, \cdots) = 1$ in L. A *variety* of loops is the class of all loops satisfying some *defining set* of identities.

Let V be a non-trivial variety of loops. A loop L in V is a free V-loop on free generators $g_1, g_2, g_3, \cdots$ if any mapping of $g_1, g_2, g_3, \cdots$ into any V-loop M can be extended to a homomorphism of L into M. We write $F_n(V)$ for the free V-loop on n generators, $F(V)$ for the free V-loop on a countably infinite set of generators.

There is an explicit construction for a free loop on generators $g_1, g_2, g_3, \cdots$

in a variety V similar to that for an (absolutely) free loop. The resulting loop we call the free V-loop on the free set of generators $g_1, g_2, g_3, \cdots$. Such a construction, of course, is simply a special case of the construction of a free algebra in any variety of universal algebras. We assume without further comment the usual properties of a free algebra in a variety.

2. Commutator-associators

Let F be the free loop on generators $g_1, g_2, g_3, \cdots$ and A the free abelian group on the same number of generators $h_1, h_2, h_3, \cdots$. The kernel of the homomorphism $F \to A$ given by $g_i \to h_i$, $i = 1, 2, 3, \cdots$ is called the *commutator-associator* subloop of F, [1]. We will denote it by $[F, F]$ and call its elements *commutator-associator words* in $g_1, g_2, g_3, \cdots$. In general, by a *commutator element* in a loop L, we mean an element c such that for some x, y in L, one of the equations $c \cdot yx = xy$, $cy \cdot x = xy$, $yc \cdot x = xy$, $y \cdot cx = xy$, $y \cdot xc = xy$, $yx \cdot c = xy$ holds. That is c is one of xy/yx, $(xy/x)/y, \cdots, yx \backslash xy$, for some x, y in L. By an *associator element* in L, we mean an element a such that for some x, y, z in L, one of the twenty equations

$$a(x \cdot yz) = xy \cdot z, \quad ax \cdot yz = xy \cdot z, \cdots, \quad (x \cdot yz)a = xy \cdot z,$$
$$x \cdot yz = a(xy \cdot z), \cdots, \quad x \cdot yz = (xy \cdot z)a$$

holds. That is, a is one of $(xy \cdot z)/(x \cdot yz)$, $\{(xy \cdot z)/yz\}/x, \cdots, (x \cdot yz)\backslash(xy \cdot z)$, for some x, y, z in L.

The commutator and associator elements in a loop L generate a normal subloop $K = [L, L]$, which is the smallest normal subloop of L such that L/K is an abelian group, [1]. In the free loop F, generated by $g_1, g_2, g_3, \cdots$ we call the commutator and associator elements, *basic commutator words* and *basic associator words*. It will be necessary later to develop the beginnings of a calculus of commutator-associators (this has already been done by Bruck for some special varieties of loops, [2]) but here we need only some very simple properties of commutators and associators.

Let $w(g_1, g_2, g_3, \cdots)$ be a word in the free loop F. We define the *exponent* of w in g_i, denoted by $e_i(w)$ by

(i) $e_i(1) = 0$, $e_i(g_i) = 1$, $e\,(g_i) = 0, j \neq i$

(ii) if $w = u \cdot v$, then $e_i(w) = e_i(u) + e_i(v)$

(iii) if $w = u \backslash v$, then $e_i(w) = -e_i(u) + e_i(v)$

(iv) if $w = u/v$, then $e_i(w) = e_i(u) - e_i(v)$.

By direct consideration of elementary transformations of words in a free loop F, we see that if $u = v$ in F, then $e_i(u) = e_i(v)$, $i = 1, 2, 3, \cdots$. The mapping $w \to h_1^{e_1} h_2^{e_2} h_3^{e_3} \cdots$, (where we write e_i for $e_i(w)$) is simply the homomorphism of

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F onto the free abelian group A, described at the beginning of this section. We obtain from this the following very useful property.

THEOREM 2.1. *$w(g_1, g_2, g_3, \cdots)$ is a commutator-associator word in F if and only if $e_i(w) = 0$, $i = 1, 2, 3, \cdots$.*

3. A standard form for identities

If $w(g_1, g_2, g_3, \cdots)$ is a word in F, let w_i denote the image of w under the endomorphism

$$\varepsilon_i : g_i \to g_i, \quad g_j \to 1, \qquad j \neq i.$$

That is, $w_i = w(1, 1, \cdots, g_i, 1, \cdots, 1)$. We see that $e_i(w_i) = e_i(w)$ and $e_j(w_i) = 0$, $j \neq i$. We will write $w_1 w_2 \cdots w_t$ for the right-product $((\cdots((w_1 w_2)w_3)\cdots)w_t)$. Now $(w_1 w_2 \cdots w_t)\backslash w$ has exponent zero in each g_i and hence is in $[F, F]$. If we denote it by $c(g_1, g_2, \cdots, g_t)$, then

$$w(g_1, g_2, \cdots, g_t) = (w_1 w_2 \cdots w_t) \cdot c(g_1, g_2, \cdots, g_t)$$

and we have proved the following.

LEMMA 3.1. *An element in F may be written as the product of powers of generators and a commutator-associator word.*
(By a *power* of a loop element u, we mean an element in the subloop generated by u.)

We may restate this lemma as our first version of a standard form for loop identities.

THEOREM 3.1. *Any loop identity*

$$w(x_1, x_2, x_3, \cdots) = 1$$

is equivalent to a finite collection of identities $w_1(x) = 1$, $w_2(x) = 1$, $w_3(x) = 1, \cdots$ involving only one variable, and an identity $c(x_1, x_2, x_3, \cdots) = 1$ where c is a commutator-associator word.

PROOF. We have shown above that, in F,

$$w(g_1, g_2, g_3, \cdots) = \{w_1(g_1) \cdot w_2(g_2) \cdot w_3(g_3) \cdots\} \cdot c(g_1, g_2, g_3, \cdots)$$

where each $w_i(g_i) = w_i(1, 1, \cdots, g_i, 1, \cdots, 1)$ is a word involving at most one generator and where $c(g_1, g_2, g_3, \cdots)$ is a commutator-associator word. Hence, this equation is satisfied by any elements in any loop. Now, if $w(x_1, x_2, x_3, \cdots) = 1$ is satisfied in a loop L, putting all variables but x_i equal to 1, we see that $w_i(x_i) = 1$ is satisfied identically in L. Hence, $c(x_1, x_2, x_3, \cdots) = 1$ is also satisfied identically in L. Conversely, these identities imply $w = 1$.

We may now obtain more information about the identities in one variable which a loop may satisfy. Let F_1 be the free loop freely generated by g. Let $w(g)$ be an element of F_1. We define a right-power g^m to be 1 if $m = 0$, by $g^{m+1} = g^m \cdot g$ if $m \geqq 0$, and by $g^{m-1} = g^m/g$ if $m \leqq 0$. Let $w(g)$ have exponent m in g and let $p(g) = g^m \backslash w(g)$. Now $p(g)$ has exponent zero in g and so is in $[F_1, F_1]$. Hence,

$$w(g) = g^m \cdot p(g)$$

where g^m is a right-power and $p(g)$ is in $[F_1, F_1]$.

LEMMA 3.2. *Any loop identity $w(x) = 1$ in one variable is equivalent to an identity $x^m \cdot p(x) = 1$, where $p(x)$ is a commutator-associator word and x^m is a right-power.*

We may now combine the preceding results.

THEOREM 3.2. *Any loop identity $w(x_1, x_2, \cdots, x_t) = 1$ may be written in the form*

$$\prod_{i=1}^{t} \{x_i^{m_i} \cdot p_i(x_i)\} \cdot c(x_1, x_2, \cdots, x_t) = 1$$

where each $x_i^{m_i}$ is a right-power and each $p_i(x_i)$ and $c(x_1, x_2, \cdots, x_t)$ are commutator-associator words [2].

COROLLARY. *The loop identity $w(x_1, x_2, \cdots, x_t) = 1$ is equivalent to a finite set of one-variable identities $x^{m_i} \cdot p_i(x) = 1$, where $p_i(x)$ is a commutator-associator word, and an identity $c(x_1, x_2, \cdots, x_t) = 1$ where $c(x_1, x_2, \cdots, x_t)$ is a commutator-associator word.*

There is a slight modification we can make in the above results. Let $x^m \cdot p(x) = 1$ be an identity in one variable, with $m < 0$. Then $x^m \cdot p(x) = 1$ is equivalent to $x^{|m|} \cdot q(x) = 1$ where $q(x)$ is a commutator-associator word, since in F_1 $(\cdots(\{1/p(g)\} \cdot g)g) \cdots)g = g^{|m|} \cdot q(g)$ where $q(g)$ is in $[F_1, F_1]$. Hence, any loop identity in one variable is equivalent to an identity $x^m \cdot p(x) = 1$ where $p(x)$ is a commutator-associator word and $m \geqq 0$.

Let $g^m \cdot p(g)$, $g^n \cdot q(g)$ be two elements in the free loop F_1, on the free generator g, where $p(g)$, $q(g)$ belong to the commutator-associator subloop of F_1 and $0 < m \leqq n$. Then $n = am+b$ for non-negative integers a, b, with $b < m$. Let $r(g) = g^b \backslash [g^n \cdot q(g)/\{g^m \cdot p(g)\}^a]$. Since $r(g)$ has exponent zero in g, $r(g)$ is a commutator-associator word in F_1. The identities $x^m \cdot p(x) = 1$, $x^n \cdot q(x) = 1$ are equivalent to the identities $x^m \cdot p(x) = 1$, $x^b \cdot r(x) = 1$. We obtain immediately the finite case of the following lemma.

LEMMA 3.3. *Any finite or countably infinite set of identities of the form $x^{m_i} \cdot p_i(x) = 1$ is equivalent to a collection of commutator-associator identities*

[2] It is possible to compute explicit expressions for the $p_i(x_i)$ and $c(x_1, x_2, \cdots, x_t)$ in terms of basic commutators and associators.

$r_i(x) = 1$ *and an identity* $x^m \cdot p(x) = 1$ *where* $p(x)$ *is a commutator-associator word and* $m \geqq 0$.

For an infinite set of identities, we need only note that any infinite set of positive integers contains a finite subset whose g.c.d. is the same as that of the infinite set.

THEOREM 3.3. *A variety of loops may be defined by a collection of commutator-associator identities and a single identity of the form*

$$x^m \cdot p(x) = 1$$

where $p(x)$ *is a commutator-associator word (i.e.* $p(x) = 1$ *is an identity satisfied by all abelian groups).*

PROOF. Immediate from the preceding lemmas.

4. Anti-associative varieties

Let V be a non-trivial variety of loops. V may be defined by a collection cf commutator-associator identities and an identity $x^m \cdot p(x) = 1$ where $m \geqq 0$ and $p(x)$ is a commutator-associator word. If $m = 0$, all commutative groups belong to V and if $m > 1$, then a cyclic group of order m belongs to V. That is, V contains non-trivial groups. If $m = 1$, then V contains no non-trivial groups since a group satisfying $x \cdot p(x) = 1$ satisfies $x = 1$. We define a non-trivial variety of loops to be *anti-associative* if it contains no non-trivial groups.

THEOREM 4.1. *A variety is anti-associative if and only if it satisfies an identity of the form*

$$x \cdot p(x) = 1$$

where $p(x)$ *is a commutator-associator word.*

PROOF. If the variety satisfies $x \cdot p(x) = 1$, then it contains no non-trivial groups. On the other hand, if it can be defined by commutator-assocatior identities and $x^m \cdot p(x) = 1$, where $m = 0$ or $m > 1$, then the variety contains non-trivial groups.

One of the simplest anti-associative identities is $x \cdot \{x \cdot (1/x)\} = 1$. Below we give the multiplication table for a loop of order eight satisfying this identity. The construction of this loop is based upon the equivalence of the identity $x \cdot \{x \cdot (1/x)\} = 1$ with the property that, for any elements a, b, if $ab = 1$ in the loop, then $b(ba) = 1$.

a	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	6	1	7	8	4	5	3
3	3	4	2	1	6	5	8	7
4	4	3	5	8	1	7	6	2
5	5	7	8	6	2	1	3	4
6	6	8	4	3	7	2	1	5
7	7	5	6	2	3	8	4	1
8	8	1	7	5	4	3	2	6

In a finite loop of order n, the left and right multiplications L_x, R_x are permutations on a set of n elements and hence $L_x^{n!} = R_x^{n!} =$ the identity mapping. Hence, a finite loop of order n satisfies the identity

$$^{n!}x = x^{n!}.$$

Using this, we see that in the above loop of order 8, the identity $x \cdot \{x \cdot (1/x)\} = 1$ is equivalent to

$$^{40,318}x = x^{40,319}$$

Another simple anti-associative identity is $x \cdot xx = (xx \cdot x)x$. This can be written in the form $x \cdot p(x) = 1$ where $p(x) = \{(xx \cdot x)\backslash(x \cdot xx)\}\backslash 1$. The construction of an infinite loop satisfying this identity uses the embedding theorem proved in the next section and will be discussed there.

5. An embedding theorem

Let P be a partial loop consisting of a countably infinite set of elements (which we will denote by the positive integers) and a partial multiplication on P satisfying the following conditions.

(i) $1 \cdot i = i \cdot 1 = i$, $i = 1, 2, 3, \cdots$. That is, 1 is the neutral element.

(ii) If $j \neq k$ and $i \cdot j$, $i \cdot k$ are assigned values in P, then $i \cdot j \neq i \cdot k$.

(iii) If $j \neq k$ and $j \cdot i$, $k \cdot i$ are assigned values in P, then $j \cdot i \neq k \cdot i$.

Conditions (i), (ii), (iii) guarantee that P is a partial loop [3], [4].

(iv) If $i \neq 1$, then there are an infinite number of elements j in P such that $i \cdot j$ is not assigned a value and an infinite number of elements k in P such that $k \cdot i$ is not assigned a value.

(v) If $i \neq 1$, then there are, for all $i \cdot j$ which are assigned values, an infinite number of elements of P which are not included among these values and for all $k \cdot i$ which are assigned values, an infinite number of elements which are not included among these values.

(vi) No element in P occurs an infinite number of times as the value of products in P.

(vii) There are an infinite number of elements in P which do not occur as values of the products in P.

Although the above conditions on P seem rather stringent, they are flexible enough to enable us to construct partial loops satisfying special identities. First, however, we need the following theorem.

THEOREM 5.1. *The partial loop P can be embedded in a loop L on the same set of elements.*

PROOF. Write $P = P_0$. We will construct a sequence of partial loops $P_1, P_2, P_3, \cdots$ such that (i) $P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$, (ii) each P_n satisfied the seven conditions which P satisfies, (iii) $\bigcup P_n$ is a loop. We begin by ordering all ordered pairs (i, j) of elements of P in a sequence $s_1, s_2, s_3, \cdots$. Assume that P_n has been constructed and satisfies the seven conditions P satisfies. Let s_{n+1} be the ordered pair (i, j). If $i \cdot j$ is assigned a value in P_n and if solutions x, y of $x \cdot i = j$, $i \cdot y = j$ exist in P_n, then we put $P_{n+1} = P_n$. Otherwise, we define $i \cdot j$ in P_{n+1} to be the smallest positive integer which does not occur as a value of $i \cdot x$ or $y \cdot i$ for any x, y in P. Furthermore, if there is no solution x of $x \cdot i = j$ in P_n, we choose the smallest positive integer x in P such that $x \cdot i$ is not assigned a value in P_n and $x \cdot k \neq j$ for any k, and put $x \cdot i = j$ in P_{n+1}. Similarly, we introduce a solution y of $i \cdot y = j$ in P_{n+1} if one does not already exist in P_n. Clearly, $P_n \subseteq P_{n+1}$ and P_{n+1} satisfies the seven conditions imposed on P. Consider the union $\bigcup P_n = L$. In L, all products $i \cdot j$ are defined and for any i, j in P, there are unique solutions x, y of $x \cdot i = j$, $i \cdot y = j$. Hence, L is a loop and P is contained in L.

We illustrate the use of this embedding theorem by constructing an infinite loop satisfying the anti-associative identity $x \cdot xx = (xx \cdot x)x$. Let P be a partial loop with the positive integers as elements and with the following products defined (we use $(\circ)$ to denote ordinary multiplication of positive integers)

(i) $1 \cdot i = i \cdot 1 = i$, for $i = 2, 3, 4, \cdots$.

(ii) $i \cdot i =$ the i^{th} prime, p_i, in the sequence of primes $2, 3, 5, \cdots$.

(iii) $ii \cdot i = p_i \circ i$, $(ii \cdot i)i = p_i \circ i \circ i$, for $i = 2, 3, 4, \cdots$.

(iv) $i \cdot ii = p_i \circ i \circ i$, for $i = 2, 3, 4, \cdots$.

This partial loop satisfies the conditions for the embedding theorem to apply. Complete the partial loop to a loop L. Then L satisfies the identity

$$x \cdot xx = (xx \cdot x)x.$$

THEOREM 5.2. *There exists a non-trivial loop satisfying the anti-associative identity*

$$x \cdot xx = (xx \cdot x)x.$$

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6. Anti-finite varieties

By an *anti-finite* variety, we mean a non-trivial variety which contains no non-trivial finite loops. A set of identities is said to be anti-finite if it has infinite models but only trivial finite models. There do not exist anti-finite varieties of groups (every non-trivial group variety contains finite cyclic groups) and it is not immediately obvious that anti-finite loop varieties exist. Before demonstrating their existence by use of the embedding theorem in the preceding section we state an interesting but easily proved theorem.

THEOREM 6.1. *Anti-finiteness implies anti-associativity.*

PROOF. Let V be a non-trivial variety of loops which is not anti-associative, i.e. V contains a non-trivial group G. But then V contains finite groups. That is, V is not anti-finite.

THEOREM 6.2. *There exists an anti-finite variety of loops satisfying the identities*

$$^{n}x = x^{n+1}, \qquad n = 3, 4, 5, \cdots.$$

PROOF. We construct a partial loop satisfying these identities and then use Theorem 5.1 to embed the partial loop in a loop. Let $p_1, p_2, p_3, \cdots$ denote the sequence of primes $2, 3, 5, \cdots$. Let P be the partial loop with the positive integers $\{1, 2, 3, \cdots\}$ as elements, and partial multiplication

(i) $1 \cdot i = i \cdot 1$, for $i = 1, 2, 3, \cdots$.

(ii) $i^n = p_i^{[n-1]}$, for all $i, n \geqq 2$ where i^n is the n^{th} right power of i in P and $p^{[m]}$ denote the ordinary m^{th} power of the prime p.

(iii) $^{n}i = p_i^{[n]}$, for all $i \geqq 2$ and all $n > 2$.

This partial loop satisfies $^{n}x = x^{n+1}$, $n = 3, 4, 5, \cdots$ and it is easily checked that it satisfies the conditions for Theorem 5.1 to apply. Let L be the loop on $\{1, 2, 3, \cdots\}$ in which P is embedded by this construction. Then L satisfies $^{n}x = x^{n+1}$, $n = 3, 4, 5, \cdots$.

Hence, the variety defined by these identities is non-trivial. To show that it is anti-finite, we note first that a loop of order two does not satisfy these identities. A loop of order n satisfies $^{n!}x = x^{n!} = 1$. Thus a loop of order $n \geqq 2$ in the variety satisfies $^{n!}x = x^{n!+1}$ and $^{n!}x = x^{n!} = 1$. These identities imply $x = 1$, a contradiction. Hence, the variety contains no finite non-trivial loops.

It was shown in [5] that there are uncountably many varieties of loops. Another proof of this follows from the next lemma.

LEMMA 6.1. *The set of identities* $^{n}x = x^{n+1}$, $n = 3, 4, 5, \cdots$ *is independent.*

PROOF. By a trivial modification of the construction used in the proof of Theorem 6.2, we can construct a loop satisfying all but one of the identities $^{n}x = x^{n+1}$, $n = 3, 4, 5, \cdots$. We omit the details.

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The uncountably many subsets of the set of identities ${}^{n}x = x^{n+1}$, $n = 3, 4, 5, \cdots$ thus define uncountably many different varieties. However, a stronger result than this is true.

THEOREM 6.3. *There exist uncountably many subvarieties of the anti-finite variety defined by the set of identities*

$$ {}^{n!}x = x^{n!+1}, \qquad n = 3, 4, 5, \cdots. $$

PROOF. Let N be any set of positive integers containing the factorials $3!, 4!, 5!, \cdots$. Let V be a variety defined by the identities

$$ {}^{n}x = x^{n+1}, \qquad n \in N. $$

By the same argument as in the proof of Theorem 6.2, V is anti-finite. If N_1, N_2 are two different sets of positive integers each containing $3!, 4!, 5!, \cdots$ and V_1, V_2 the corresponding varieties, then $V_1 \neq V_2$ by Lemma 6.1. Since there are uncountably many sets of positive integers containing $3!, 4!, 5!, \cdots$, there are uncountably many anti-finite varieties of loops satisfying ${}^{n!}x = x^{n!+1}$, $n = 3, 4, 5, \cdots$.

7. Equationally complete varieties

A set of identities I is said to be *equationally complete* if it is consistent and has no consistent extension. That is, for any loop identity $w = 1$, either $w = 1$ is a consequence of I or the set of identities $I \cup \{w = 1\}$ is satisfied by only a trivial loop. Thus, an equationally complete variety, defined by an equationally complete set of identities, is a minimal variety, in the sense that its only proper subvariety is the trivial variety. We now refine the preceding construction of anti-finite varieties and combine it with some ideas of Kalicki [6] to obtain the following theorem.

THEOREM 7.1. *There exist uncountably many equationally complete anti-finite varieties of loops.*

PROOF. Let S be the set of even positive integers $\geqq 4$ and M any subset of S. Let $M' = S - M$. Let $V(M)$ be the variety defined by the set of identities I_M, where

$$ I_M: \quad \begin{aligned} {}^{n}x &= x^{n+1}, & n \in M \\ {}^{n}x &= x^{n}\ , & n \in M' \end{aligned} $$

LEMMA 7.1. *$V(M)$ is non-trivial.*

PROOF. We construct a partial loop P satisfying I_M, then use Theorem 5.1, to complete the partial loop. Let the elements of P be the positive integers. We will denote ordinary multiplication of integers by $x \circ y$ and ordinary powers of

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integers by $x^{[y]}$. Let $p_1, p_2, p_3, \cdots$ be the sequence of positive primes. We define the partial multiplication in P as follows.

(i) $1 \cdot i = i \cdot 1 = i$, for every i in P

(ii) $i^n = p_i^{[n-1]}$, for every i in P and every right power i^n,

$$n = 2, 3, 4, \cdots$$

That is, $i^2 = p_i$, $p_i^{[n-1]} \cdot i = p_i^{[n]}$, $n \geqq 2$

(iii) If n is odd, ${}^n i = i \circ p_i^{[n-1]}$, for every $i \geqq 2$ in P
If n is in M, ${}^n i = p_i^{[n]}$, for every $i \geqq 2$ in P
If n is in M', ${}^n i = p_i^{[n-1]}$, for every $i \geqq 2$ in P

These conditions assign a value to each left power of i.

It is easily checked that P satisfies the conditions for Theorem 5.1 to apply (in the i^{th} column of the table for P only powers of p_i occur, in the i^{th} row, either powers of p_i or i times a power of p_i). We embed P in a loop L and from the construction of P, we see that L satisfies I_M. Hence, $V(M)$ is non-trivial.

If M_1, M_2 are different subsets of S, then by Lemma 6.1, $V(M_1) \neq V(M_2)$. However, a much stronger result than this is true.

LEMMA 7.2. *If $M_1 \neq M_3$, then $V(M_1) \cap V(M_2)$ is the trivial variety.*

PROOF. Since $M_1 \neq M_2$, there is at least one positive integer n which is in one of M_1, M_2 and in the complement of the other. This means that any loop in $V(M_1) \cap V(M_2)$ will satisfy ${}^n x = x^{n+1}$ and ${}^n x = x^n$. But these identities imply $x = 1$. Hence, $V(M_1) \cap V(M_2)$ contains only trivial loops.

We are now in a position to prove Theorem 7.1. Any consistent set of identities has an equationally complete extension, or equivalently, any non-trivial variety contains an equationally complete subvariety. Let $V^*(M)$ be any equationally complete variety contained in $V(M)$, where M is any set of even positive integers $\geqq 4$ which contains all factorials $3!, 4!, 5!, \cdots$. By Theorem 6.3, $V(M)$ is anti-finite and hence $V^*(M)$ is anti-finite. Now there are uncountably many such choices for M and by Lemma 7.2, the corresponding equationally complete varieties are all distinct. This completes the proof.

LEMMA 7.3. *There exist equationally complete varieties which are anti-associative but not anti-finite.*

PROOF. In Section 4, we exhibit a finite loop satisfying the anti-associative identity $x(x \cdot (1/x)) = 1$. Let V be the variety defined by the identities which this loop satisfies. For $n \geqq 2$, the free loop $F_n(V)$ is finite and non-trivial. Let V^* be an equational completion of V. Then $F_n(V^*)$, is a homomorphic image of $F_n(V)$ and for $n \geqq 2$, is finite and non-trivial. Clearly, V^* is anti-associative.

The collection of all varieties of loops forms a lattice (actually a complete modular lattice) under the partial ordering of ordinary inclusion. Equationally complete varieties are the atoms in this lattice. The associative equationally complete varieties are varieties of abelian groups satisfying $x^p = 1$, p prime. If V is any variety of loops which contains a non-trivial group, then V includes one of these group atoms. If V is a variety which does not contain a non-trivial group, then V satisfies some anti-associative law and so do all of the atoms which are included in V. In other words, an equationally complete variety of loops in either a variety of abelian groups satisfying $x^p = 1$, p prime, or an anti-associative variety.

THEOREM 7.2. *The atoms in the lattice of loop varieties are as follows* (i) *the countably infinite collection of varieties of abelian groups, satisfying* $x^p = 1$, *p prime,* (ii) *countably many anti-associative varieties containing finite loops,* (iii) *uncountably many anti-finite varieties.*

The only point in the above theorem which we have not already covered is the number of anti-associative varieties which contain finite loops. Since distinct atoms have only the trivial loop in common and there are only countably infinitely many finite loops altogether, the number of anti-associative but not anti-finite atoms is at most χ_0. Such atoms do exist as we showed above. Whether there is a finite number of such atoms or, as is more likely, a countably infinite number, we do not know at present.

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